Product formulas for basic hypergeometric series by evaluations of Askey–Wilson polynomials

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Adopted set notations

- $\mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$;
- $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ which represent the sets of integers, real numbers and complex numbers respectively
- $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$
- Multiset notation: $\mathbf{a} := \{a_1, \ldots, a_n\}$, $a_k \in \mathbb{C}$, $n \in \mathbb{N}$, $1 \leq k \leq n$:

$$\{\pm\}a := \{a, -a\},$$

$$\mathrm{e}^{\{\pm\}i\theta} := \{\mathrm{e}^{i\theta}, \mathrm{e}^{-i\theta}\},$$

$$z^{\{\pm\}} := \{z, z^{-1}\},$$

$$z^{\{\pm\}2} := \{z^2, z^{-2}\},$$

$$\{\pm\}z^{\{\pm\}} := \{z, -z, z^{-1}, -z^{-1}\}.$$

Finite and infinite q-shifted factorials–building blocks

- The finite shifted and q-shifted factorials,

$$(a)_n := (a)(a+1)(a+2)\cdots(a+n-1)$$

$$(a;q)_n := (1-a)(1-qa)(1-q^2a)\cdots(1-q^{n-1}a) = \prod_{k=0}^{n-1}(1-aq^k)$$

- The infinite q–shifted factorial, $0<|q|<1$,

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

- Other adopted product notations:

$$(a_1,\ldots,a_r;q)_n := (a_1;q)_n\cdots(a_r;q)_n,$$

$$(a_1,\ldots,a_r;q)_\infty := (a_1;q)_\infty\cdots(a_r;q)_\infty,$$

Useful identities for q-shifted factorials

$$(a;q)_{n+1} = (1-a)(qa;q)_n,$$

$$(a;q)_{n-1} = \frac{(\frac{a}{q};q)_n}{(1-\frac{a}{q})},$$

$$\frac{(1-q^n a)}{(1-a)} = \frac{(qa;q)_n}{(a;q)_n},$$

$$(a;q^{\frac{1}{2}})_n = \begin{cases} (a, q^{\frac{1}{2}}a;q)_{\frac{n}{2}} & \text{if } n \text{ even},\\ (1-a)(q^{\frac{1}{2}}a, qa;q)_{\frac{n-1}{2}} & \text{if } n \text{ odd},\end{cases}$$

$$(a;q)_{2n} = (a, qa;q^2)_n = (\{\pm\}\sqrt{a}, \{\pm\}\sqrt{qa};q)_n.$$

Basic and generalized hypergeometric series

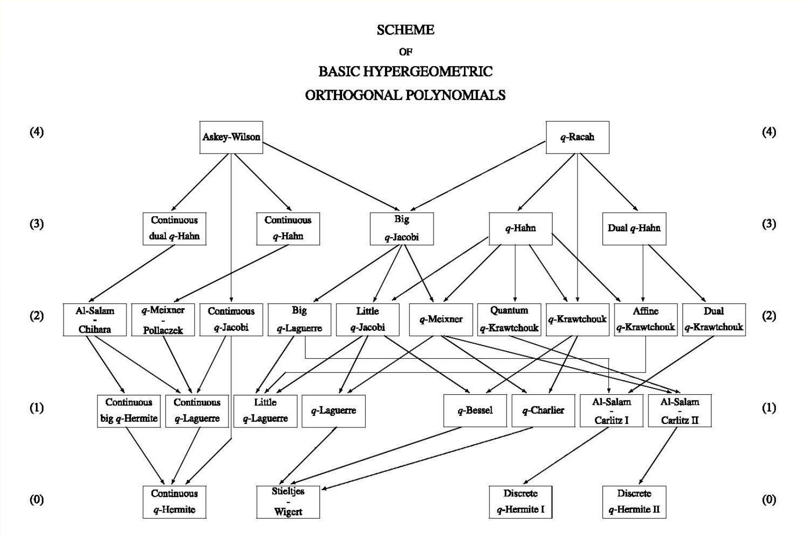
- Basic hypergeometric series (bhs):

$$ {}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right):=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}z^k $$

- If $(r,s)\mapsto(r+1,r)$ then the bhs is considered k-balanced if $b_1b_2\cdots b_r=q^k a_1a_2\cdots a_{r+1}$ (terminology due to Askey).
- Generalized hypergeometric series:

$$ {}_rF_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};z\right):=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_r)_k}{(b_1)_k\cdots(b_s)_k}\frac{z^k}{k!} $$

The q-Askey scheme of basic hypergeometric OPs



SCHEME
OF
BASIC HYPERGEOMETRIC
ORTHOGONAL POLYNOMIALS

Terminating balanced ${}_4\phi_3$ representations of the Askey–Wilson polynomials

Theorem

Let $n\in\mathbb{N}_0$, $x=\frac{1}{2}(z+z^{-1})$, *and* $q,z,a,b,c,d\in\mathbb{C}^*$. *Then*

$$\begin{aligned}
&p_n(x;a,b,c,d|q)\\
&\quad= a^{-n}(ab,ac,ad;q)_n\,{}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az^{\{\pm\}}\\ ab,ac,ad\end{matrix};q,q\right)\\
&\quad= q^{-\binom{n}{2}}(-a)^{-n}\frac{(\frac{abcd}{q};q)_{2n}(az^{\{\pm\}};q)_n}{(\frac{abcd}{q};q)_n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},\frac{q^{1-n}}{ab},\frac{q^{1-n}}{ac},\frac{q^{1-n}}{ad}\\ \frac{q^{2-2n}}{abcd},\frac{q^{1-n}}{a}z^{\{\pm\}}\end{matrix};q,q\right)\\
&\quad= z^n\Big(ab,\frac{c}{z},\frac{d}{z};q\Big)_n\,{}_4\phi_3\left(\begin{matrix}q^{-n},az,bz,\frac{q^{1-n}}{cd}\\ ab,q^{1-n}\frac{z}{c},q^{1-n}\frac{z}{d}\end{matrix};q,q\right).
\end{aligned}$$

The Ismail–Wilson generating function

In Equation (1.9) in Ismail & Wilson (1982) "Asymptotic and generating relations for the q-Jacobi and ${}_4\phi_3$ polynomials," Journal of Approximation Theory, **36**, 1, 43–54. They derived the following product generating function for Askey–Wilson polynomials. Let $x = \frac{1}{2}(z + z^{-1})$, $|tz^{\{\pm\}}| < 1$. Then

$$ {}_2\phi_1\left(\begin{matrix} az, bz \\ ab \end{matrix}; q, \frac{t}{z}\right) {}_2\phi_1\left(\begin{matrix} \frac{c}{z}, \frac{d}{z} \\ cd \end{matrix}; q, tz\right) = \sum_{n=0}^{\infty} \frac{t^n \, p_n(x; a, b, c, d|q)}{(q, ab, cd; q)_n}. $$

Consider $z \in \{a^{\{\pm\}}, b^{\{\pm\}}, c^{\{\pm\}}, d^{\{\pm\}}\}$. Then one can use the Askey–Wilson special value

$$ p_n[a; a, b, c, d|q] = a^{-n}(ab, ac, ad; q)_n. $$

Then one of the ${}_2\phi_1$'s becomes unity and the Ismail–Wilson generating function becomes one of the Heine transformations of a ${}_2\phi_1$ series.

Triple sum generalized Ismail–Wilson generating function

The following summation generalizes the Ismail–Wilson generating function with an extra parameter u:

$$
{}_3\phi_2\left(\begin{matrix}\frac{u}{t}, aw, bw\\ ab, uw\end{matrix}; q, \frac{t}{w}\right)\, {}_3\phi_2\left(\begin{matrix}\frac{u}{t}, \frac{c}{w}, \frac{d}{w}\\ cd, \frac{u}{w}\end{matrix}; q, tw\right)
$$

$$
= \frac{(\frac{u}{a}, \frac{u}{c}; q)_\infty}{(uw^{\{\pm\}}; q)_\infty} \sum_{n,k,l\geq 0} \frac{t^n u^{k+l}}{a^k c^l} \frac{(aw^{\{\pm\}}; q)_k (cw^{\{\pm\}}; q)_l \, p_n(x; q^k a, b, q^l c, d|q)}{(q;q)_k (q;q)_l (q;q)_n (ab;q)_{n+k} (cd;q)_{n+l}}.
$$

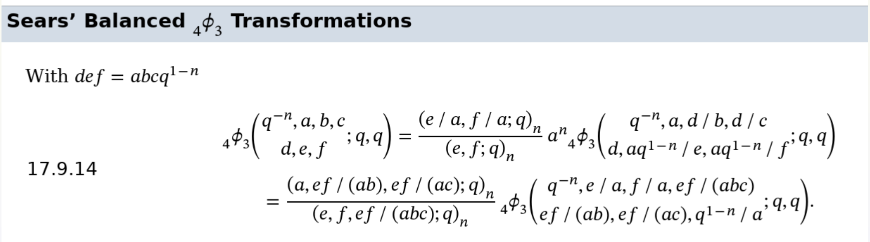
If $w = b$ or $w = d$ then one of the ${}_3\phi_2$'s become unity and we obtain

$$
{}_3\phi_2\left(\begin{matrix}\frac{u}{t}, ad, bd\\ ab, du\end{matrix}; q, \frac{t}{d}\right)
$$

$$
= \frac{(\frac{u}{a}; q)_\infty}{(du; q)_\infty} \sum_{n=0}^\infty \sum_{k=0}^\infty \frac{(\frac{a}{d}; q)_k (bd; q)_n (ad; q)_{n+k}}{(q;q)_k (q;q)_n (ab;q)_{n+k}} \left(\frac{t}{d}\right)^n \left(\frac{u}{a}\right)^k.
$$

Terminating balanced ${}_4\phi_3$ summations

- There are a plethora of terminating balanced ${}_4\phi_3$ summations in the literature.
- However, many of them should be considered equivalent.
- They satisfy the Sears transformation



Sears' Balanced ${}_4\phi_3$ Transformations

With $def = abcq^{1-n}$

17.9.14

$$ {}_4\phi_3\left({q^{-n},a,b,c \atop d,e,f};q,q\right)=\frac{(e/a,f/a;q)_n}{(e,f;q)_n}a^n\,{}_4\phi_3\left({q^{-n},a,d/b,d/c \atop d,aq^{1-n}/e,aq^{1-n}/f};q,q\right) $$

$$ =\frac{(a,ef/(ab),ef/(ac);q)_n}{(e,f,ef/(abc);q)_n}\,{}_4\phi_3\left({q^{-n},e/a,f/a,ef/(abc) \atop ef/(ab),ef/(ac),q^{1-n}/a};q,q\right). $$

- Through simple linear variable scaling they often can be seen to be equivalent to a specific Askey–Wilson polynomial evaluation.

Bailey's and Andrews' balanced terminating ${}_4\phi_3$ summation

The following summation of a balanced terminating ${}_4\phi_3$ was first derived by Bailey (1941) and is a consequence of a formula due to Jackson (1941) (see also Carlitz (1969) and G+R Exercise 2.6):

$${}_4\phi_3\left(\begin{matrix} q^{-n}, -\frac{q^{1-n}}{ab}, a, b \\ -ab, \frac{q^{1-n}}{a}, \frac{q^{1-n}}{b} \end{matrix}; q, q\right) = \begin{cases} \frac{(q,a^2,b^2;q^2)_{\frac{n}{2}}\,(ab;q)_n}{(a,b;q)_n(a^2b^2;q^2)_{\frac{n}{2}}} & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

The above equation is equivalent to Andrews' (1976) summation (Theorem 1 therein)

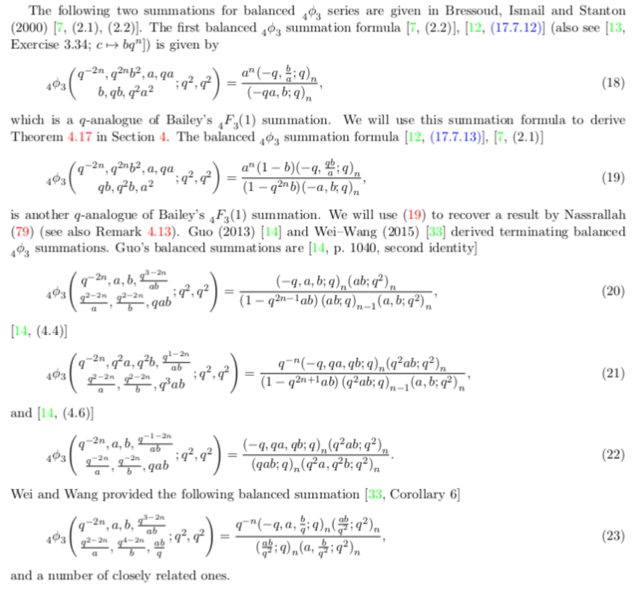
$${}_4\phi_3\left(\begin{matrix} q^{-n}, q^n a, \{\pm\}\sqrt{c} \\ \{\pm\}\sqrt{qa}, c \end{matrix}; q, q\right) = \begin{cases} \frac{c^{\frac{n}{2}}(q, \frac{qa}{c};q^2)_{\frac{n}{2}}}{(qa,qc;q^2)_{\frac{n}{2}}} & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

Andrews' (1976) alternative balanced ${}_4\phi_3$ summation

The following terminating balanced ${}_4\phi_3$ summation DLMF (17.7.11) due to Theorem 2 in Andrews (1976) "q-Analogues of the Watson and Whipple summations" SIMA **7**, is a terminating q-analogue of Whipple's ${}_3F_2$ sum

$$
{}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+1}, \{\pm\}a \\ -q, b, \frac{qa^2}{b} \end{matrix}; q, q\right)
= \begin{cases} \dfrac{a^n(\frac{q^2}{b}, \frac{qb}{a^2}; q^2)_{\frac{n}{2}}}{(qb, \frac{q^2a^2}{b}; q^2)_{\frac{n}{2}}} & \text{if } n \text{ even,} \\[2ex] \dfrac{q(1-\frac{b}{q})(1-\frac{a^2}{b})}{(1-b)(1-\frac{qa^2}{b})} \dfrac{(-a)^{n-1}(\frac{q^3}{b}, \frac{q^2b}{a^2}; q^2)_{\frac{n-1}{2}}}{(q^2b, \frac{q^3a^2}{b}; q^2)_{\frac{n-1}{2}}} & \text{if } n \text{ odd.} \end{cases}
$$

Summations due to Bressoud, Ismail and Stanton (2000)



The following two summations for balanced ${}_4\phi_3$ series are given in Bressoud, Ismail and Stanton (2000) [7, (2.1), (2.2)]. The first balanced ${}_4\phi_3$ summation formula [7, (2.2)], [12, (17.7.12)] (also see [13, Exercise 3.34; $c \mapsto bq^n$]) is given by

$$ {}_4\phi_3\left(\begin{matrix} q^{-2n}, q^{2n}b^2, a, qa \\ b, qb, q^2a^2 \end{matrix}; q^2, q^2\right) = \frac{a^n(-q, \frac{b}{a}; q)_n}{(-qa, b; q)_n}, \tag{18} $$

which is a q-analogue of Bailey's ${}_4F_3(1)$ summation. We will use this summation formula to derive Theorem 4.17 in Section 4. The balanced ${}_4\phi_3$ summation formula [12, (17.7.13)], [7, (2.1)]

$$ {}_4\phi_3\left(\begin{matrix} q^{-2n}, q^{2n}b^2, a, qa \\ qb, q^2b, a^2 \end{matrix}; q^2, q^2\right) = \frac{a^n(1-b)(-q, \frac{qb}{a}; q)_n}{(1-q^{2n}b)(-a, b; q)_n}, \tag{19} $$

is another q-analogue of Bailey's ${}_4F_3(1)$ summation. We will use (19) to recover a result by Nassrallah (79) (see also Remark 4.13). Guo (2013) [14] and Wei–Wang (2015) [33] derived terminating balanced ${}_4\phi_3$ summations. Guo's balanced summations are [14, p. 1040, second identity]

$$ {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, \frac{q^{3-2n}}{ab} \\ \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}, qab \end{matrix}; q^2, q^2\right) = \frac{(-q, a, b; q)_n (ab; q^2)_n}{(1-q^{2n-1}ab)\,(ab; q)_{n-1}(a, b; q^2)_n}, \tag{20} $$

[14, (4.4)]

$$ {}_4\phi_3\left(\begin{matrix} q^{-2n}, q^2a, q^2b, \frac{q^{1-2n}}{ab} \\ \frac{q^{2-2n}}{a}, \frac{q^{2-2n}}{b}, q^3ab \end{matrix}; q^2, q^2\right) = \frac{q^{-n}(-q, qa, qb; q)_n (q^2ab; q^2)_n}{(1-q^{2n+1}ab)\,(q^2ab; q)_{n-1}(a, b; q^2)_n}, \tag{21} $$

and [14, (4.6)]

$$ {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, \frac{q^{-1-2n}}{ab} \\ \frac{q^{-2n}}{a}, \frac{q^{-2n}}{b}, qab \end{matrix}; q^2, q^2\right) = \frac{(-q, qa, qb; q)_n (q^2ab; q^2)_n}{(qab; q)_n (q^2a, q^2b; q^2)_n}. \tag{22} $$

Wei and Wang provided the following balanced summation [33, Corollary 6]

$$ {}_4\phi_3\left(\begin{matrix} q^{-2n}, a, b, \frac{q^{3-2n}}{ab} \\ \frac{q^{2-2n}}{a}, \frac{q^{4-2n}}{b}, \frac{ab}{q} \end{matrix}; q^2, q^2\right) = \frac{q^{-n}(-q, a, \frac{b}{q}; q)_n (\frac{ab}{q^2}; q^2)_n}{(\frac{ab}{q^2}; q)_n (a, \frac{b}{q^2}; q^2)_n}, \tag{23} $$

and a number of closely related ones.

Bressoud, Ismail and Stanton (2000) (cont.)

Remark

On first appearance, one might consider (18)–(23) to be different. However, after replacing $q^2 \mapsto q$*, all the summations (18)–(23) should be considered to be equivalent when viewed as terminating balanced* ${}_4\phi_3$*'s which can be written in terms of Askey–Wilson polynomials. The summations (18) and (19) produce the same Askey–Wilson polynomial by replacing* $(a, b) \mapsto (q^{-\frac{1}{4}}a, ab)$ *and* $(a, b) \mapsto (q^{\frac{1}{4}}a, q^{\frac{1}{2}}ab)$ *respectively. The summations (20)–(23) produce the same Askey–Wilson polynomial by replacing* $(a, b) \mapsto (q^{-\frac{1}{4}}a, q^{-\frac{1}{4}}b)$*,* $(a, b) \mapsto (q^{\frac{1}{4}}a, q^{\frac{1}{4}}a)$*, and* $(a, b) \mapsto (q^{-\frac{1}{4}}a, q^{-\frac{1}{4}}b)$*,* $(a, b) \mapsto (q^{\frac{1}{4}}a, q^{\frac{3}{4}}b)$ *respectively.*

Computer search for k-balanced ${}_4\phi_3$ summations

- Schlosser made computer experiments and searched for new terminating k-balanced ${}_4\phi_3$ summations.
- Found 2 new 3-balanced ${}_4\phi_3$ summations. Example:

$$
{}_4\phi_3\left(\begin{matrix}\{\pm\}q^{-n},\{\pm\}\sqrt{c}\\ \{\pm\}\mathrm{i}\,q^{\frac{3}{2}-n},c\end{matrix};q,q\right)
= (-1)^n\frac{(1+q^{1-2n})}{(1+q)}\,\frac{(q;q^2)_{\lfloor\frac{n+1}{2}\rfloor}(-q^{1+2\lfloor\frac{n}{2}\rfloor}c;q^2)_{\lfloor\frac{n}{2}\rfloor}}{(qc;q^2)_{\lfloor\frac{n}{2}\rfloor}(-q^{1+2\lfloor\frac{n+1}{2}\rfloor};q^2)_{\lfloor\frac{n}{2}\rfloor}}.
$$

- Found 3 new 2-balanced summations. Example:

$$
{}_4\phi_3\left(\begin{matrix}q^{-n},q^na,\{\pm\}\sqrt{c}\\ \{\pm\}q\sqrt{a},c\end{matrix};q,q\right)
= (q^na)^n\left(\frac{q^{-2n}c}{a^2}\right)^{\lfloor\frac{n}{2}\rfloor}\frac{(1-a)}{(1-aq^{2n})}\,\frac{(q;q^2)_{\lfloor\frac{n+1}{2}\rfloor}(\frac{q^2a}{c};q^2)_{\lfloor\frac{n}{2}\rfloor}}{(a;q^2)_{\lfloor\frac{n+1}{2}\rfloor}(qc;q^2)_{\lfloor\frac{n}{2}\rfloor}}.
$$

New balanced summations

- We did a careful literature search and found 15 new 1-balanced (balanced) summations. However, many of them should be considered equivalent. We demonstrated the full equivalence of all the summations which were found and also in the literature.

11 q-quadratic special values for Askey–Wilson polynomials

Corresponding to the terminating balanced ${}_4\phi_3$ summations:

- 5 q-quadratic AW special values corresponding to $x=0$.
    - A1: $p_n(0;\{\pm\}\mathrm{i}a,\{\pm\}\mathrm{i}b|q)$ – Bailey (1941)
    - A2: $p_n(0;\{\pm\}\mathrm{i}a,\mathrm{i}b,-\mathrm{i}qb|q)$ – Cohl & Schlosser (2026)
    - A3: $p_n(0;\mathrm{i}a,-\mathrm{i}qa,\mathrm{i}b,-\mathrm{i}qb|q)$ – Cohl & Schlosser (2026)
    - A4: $p_n(0;\mathrm{i}a,\frac{\mathrm{i}q}{a},\mathrm{i}b,\frac{\mathrm{i}q}{b}|q)$ – Andrews (1976)
    - A5: $p_n(0;\{\pm\}\mathrm{i}a,\mathrm{i}b,-\mathrm{i}q^2b|q)$ – Cohl & Schlosser (2026)
- 3 q-quadratic AW special values corresponding to $x=\frac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12})$.
    - B1: $p_n(\frac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12});\{\pm\}\mathrm{i}a,\{\pm\}\mathrm{i}b|q)$ – Cohl & Schlosser (2026)
    - B2: $p_n(\frac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12});\{\pm\}\mathrm{i}a,\mathrm{i}b,-\mathrm{i}qb|q)$ – Cohl & Schlosser (2026)
    - B3: $p_n(\frac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12});\mathrm{i}a,-\mathrm{i}qa,\mathrm{i}b,-\mathrm{i}qb|q)$ – Cohl & Schlosser (2026)
- 2 q-quadratic AW special values: $x=\frac12(a+a^{-1})$, $x=\frac12(qa+(qa)^{-1})$
    - C1: $p_n(\frac12(a+\frac1a);\{\pm\}q^{\frac12},a,b|q)$ – Damjanovic
    - C2: $p_n(\frac12(qa+\frac1{qa});\{\pm\}q^{\frac12},a,b|q)$ – G+R (3.10.9)
- 1 q-quadratic AW special value corresponding to $x=\frac12(q^{\frac14}+q^{-\frac14})$.
    - D1: $p_n(\frac12(q^{\frac14}+q^{-\frac14});a,q^{\frac12}a,b,q^{\frac12}b|q)$ – Bressoud, Ismail & Stanton (2000)

5 q-quadratic AW special values $x=0$ (A1,A2)

Let $n\in\mathbb{N}_0$, $q,a,b\in\mathbb{C}^*$. Then

$$p_n(0;\{\pm\}\mathrm{i}a,\{\pm\}\mathrm{i}b|q)=\begin{cases}(-1)^{\frac{n}{2}}\dfrac{(q,a^2,b^2,\{\pm\}ab,\{\pm\}qab;q^2)_{\frac{n}{2}}}{(a^2b^2;q^2)_{\frac{n}{2}}} & \text{if } n \text{ even},\\ 0 & \text{if } n \text{ odd}.\end{cases}$$

$$p_n(0;\{\pm\}\mathrm{i}a,\mathrm{i}b,-\mathrm{i}qb|q)$$

$$=\begin{cases}(-1)^{\frac{n}{2}}\dfrac{(q,a^2,q^2b^2,\{\pm\}ab,\{\pm\}qab;q^2)_{\frac{n}{2}}}{(a^2b^2;q^2)_{\frac{n}{2}}} & \text{if } n \text{ even},\\ -\mathrm{i}(-1)^{\frac{n-1}{2}}(1-q)(1-a^2)b\dfrac{(q^3,q^2a^2,q^2b^2,\{\pm\}qab,\{\pm\}q^2ab;q^2)_{\frac{n-1}{2}}}{(q^2a^2b^2;q^2)_{\frac{n-1}{2}}} & \\ & \text{if } n \text{ odd}.\end{cases}$$

5 q-quadratic AW special values $x=0$ (A3,A4)

$$p_n(0;\mathrm{i}a,-\mathrm{i}qa,\mathrm{i}b,-\mathrm{i}qb|q)$$

$$=\begin{cases}(-1)^{\frac{n}{2}}\dfrac{(q,q^2a^2,q^2b^2,qab,-ab,q^2ab,-qab;q^2)_{\frac{n}{2}}}{(q^2a^2b^2;q^2)_{\frac{n}{2}}} & \text{if } n \text{ even},\\ -\mathrm{i}(-1)^{\frac{n-1}{2}}(1-q)(a+b)(1-qab) & \\ \qquad\times\dfrac{(q^3,q^2a^2,q^2b^2,q^2ab,-qab,q^3ab,-q^2ab;q^2)_{\frac{n-1}{2}}}{(q^2a^2b^2;q^2)_{\frac{n-1}{2}}} & \text{if } n \text{ odd}.\end{cases}$$

$$p_n(0;\mathrm{i}a,\tfrac{\mathrm{i}q}{a},\mathrm{i}b,\tfrac{\mathrm{i}q}{b}|q)$$

$$=\begin{cases}(-1)^{\frac{n}{2}}(-q,-q^2,-ab,-\frac{q^2}{ab},-\frac{qa}{b},-\frac{qb}{a};q^2)_{\frac{n}{2}} & \text{if } n \text{ even},\\ -\frac{\mathrm{i}q}{b}(-1)^{\frac{n-1}{2}}(1+q)(1+\frac{ab}{q})(1+\frac{b}{a}) & \\ \qquad\times(-q^2,-q^3,-qab,-\frac{q^3}{ab},-\frac{q^2a}{b},-\frac{q^2b}{a};q^2)_{\frac{n-1}{2}} & \text{if } n \text{ odd}.\end{cases}$$

5 q-quadratic AW special values $x=0$ (A5)

$$p_n(0;\{\pm\}\mathrm{i}a,\mathrm{i}b,-\mathrm{i}q^2b|q)$$

$$=\begin{cases}\dfrac{(-1)^{\frac{n}{2}}(a^2,q^2b^2,\{\pm\}ab,\{\pm\}qab;q^2)_{\frac{n}{2}}}{(1-q^2b^2)(1-a^2b^2)(q^2a^2b^2;q^2)_{\frac{n}{2}}}\\ \quad\times\left(\dfrac{(1-qb^2)(1-qa^2b^2)(q,q^3b^2,q^3a^2b^2;q)_{\frac{n}{2}}}{(qb^2,qa^2b^2;q^2)_{\frac{n}{2}}}\right.\\ \qquad\qquad\left.+\dfrac{qb^2(1-q)(1-\frac{a^2}{q})(q^3,qa^2;q^2)_{\frac{n}{2}}}{(\frac{a^2}{q};q^2)_{\frac{n}{2}}}\right) & \text{if } n \text{ even},\\ -\mathrm{i}(-1)^{\frac{n-1}{2}}b(1-q^2)(1-a^2)\\ \quad\times\dfrac{(q^3,q^2a^2,q^4b^2,\{\pm\}qab,\{\pm\}q^2ab;q^2)_{\frac{n-1}{2}}}{(q^2a^2b^2;q^2)_{\frac{n-1}{2}}} & \text{if } n \text{ odd}.\end{cases}$$

3 q-quadratic AW special values $x=\frac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12})$ (B1,B2)

$$p_n(\tfrac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12});\{\pm\}\mathrm{i}a,\{\pm\}\mathrm{i}b|q)$$

$$=q^{-\frac{n}{2}}\begin{cases}(-1)^{\frac{n}{2}}\dfrac{(q,qa^2,qb^2,\{\pm\}ab,\{\pm\}qab;q^2)_{\frac{n}{2}}}{(a^2b^2;q^2)_{\frac{n}{2}}} & \text{if } n \text{ even,}\\ -\mathrm{i}(-1)^{\frac{n-1}{2}}(1-q)(1-a^2b^2) & \\ \quad\times\dfrac{(q^3,qa^2,qb^2,\{\pm\}qab,\{\pm\}q^2ab;q^2)_{\frac{n-1}{2}}}{(a^2b^2;q^2)_{\frac{n-1}{2}}} & \text{if } n \text{ odd.}\end{cases}$$

$$p_n(\tfrac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12});\{\pm\}\mathrm{i}a,\mathrm{i}b,-\mathrm{i}qb|q)$$

$$=q^{-\frac{n}{2}}\begin{cases}(-1)^{\frac{n}{2}}\dfrac{(q,qa^2,q^{\frac52}b,qb^2,\{\pm\}ab,\{\pm\}qab;q^2)_{\frac{n}{2}}}{(q^{\frac12}b,a^2b^2;q^2)_{\frac{n}{2}}} & \text{if } n \text{ even,}\\ -\mathrm{i}(-1)^{\frac{n-1}{2}}(1-q)(1+q^{\frac12}b)(1-q^{\frac12}a^2b) & \\ \quad\times\dfrac{(q^3,qa^2,q^3b^2,\{\pm\}qab,\{\pm\}q^2ab,q^{\frac52}a^2b;q^2)_{\frac{n-1}{2}}}{(q^{\frac12}a^2b,q^2a^2b^2;q^2)_{\frac{n-1}{2}}} & \text{if } n \text{ odd.}\end{cases}$$

3 q-quadratic AW special values $x=\frac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12})$ (B3)

$$p_n\big(\tfrac{\mathrm{i}}{2}(q^{\frac12}-q^{-\frac12});\mathrm{i}a,-\mathrm{i}qa,\mathrm{i}b,-\mathrm{i}qb|q\big)$$

$$=\begin{cases}\dfrac{(-1)^{\frac{n}{2}}(q,qa^2,qb^2,-ab,\{\pm\}qab,q^2ab;q^2)_{\frac{n}{2}}}{q^{\frac{n}{2}+1}(1+ab)(1-\frac{q^{-\frac12}}{a})(q^2a^2b^2;q^2)_{\frac{n}{2}}} \\ \quad\times\left(\dfrac{(1-\frac{q^{\frac12}}{a})(1+qab)(q^{\frac52}b,-q^3ab;q^2)_{\frac{n}{2}}}{(q^{\frac12}b,-qab;q^2)_{\frac{n}{2}}} \right. \\ \qquad\qquad \left. -\dfrac{(1-q)(1+q^{\frac12}b)(q^3,q^3b^2;q^2)_{\frac{n}{2}}}{(q,qb^2;q^2)_{\frac{n}{2}}}\right) & \text{if } n \text{ even,} \\ -\mathrm{i}q^{-\frac{n}{2}}(-1)^{\frac{n-1}{2}}(1-q)(1+q^{\frac12}a)(1+q^{\frac12}b)(1-qab) \\ \quad\times\dfrac{(q^3,q^3a^2,q^3b^2,-qab,\{\pm\}q^2ab,q^3ab;q^2)_{\frac{n-1}{2}}}{(q^2a^2b^2;q^2)_{\frac{n-1}{2}}} & \text{if } n \text{ odd.}\end{cases}$$

2 q-quadratic AW special values (C1, C2) $x=\frac12(a+\frac1a)$, $x=\frac12(qa+\frac1{qa})$

$$p_n(\tfrac12(a+\tfrac1a);a,b,\{\pm\}q^{\frac12}|q)$$

$$=a^{-n}\begin{cases}(\{\pm\}q^{\frac12}a,\{\pm\}q^{\frac32}a,ab,qab;q^2)_{\frac n2} & \text{if } n \text{ even},\\ (1-qa^2)(1-ab)(\{\pm\}q^{\frac32}a,\{\pm\}q^{\frac52}a,qab,q^2ab;q^2)_{\frac{n-1}{2}} & \text{if } n \text{ odd}.\end{cases}$$

$$p_n(\tfrac12(qa+\tfrac1{qa});a,b,\{\pm\}q^{\frac12}|q)$$

$$=(qa)^{-n}\begin{cases}\dfrac{(\{\pm\}q^{\frac12}a,\{\pm\}q^{\frac32}a,ab,qab,\{\pm\}q^2\sqrt{ab};q^2)_{\frac n2}}{(\{\pm\}\sqrt{ab};q^2)_{\frac n2}} & \text{if } n \text{ even},\\ (1-qa^2)(1-q^2ab) & \\ \quad\times\dfrac{(\{\pm\}q^{\frac32}a,\{\pm\}q^{\frac52}a,qab,q^2ab,\{\pm\}q^3\sqrt{ab};q^2)_{\frac{n-1}{2}}}{(\{\pm\}q\sqrt{ab};q^2)_{\frac{n-1}{2}}} & \text{if } n \text{ odd}.\end{cases}$$

1 q-quadratic AW special value $x=\frac{1}{2}(q^{\frac14}+q^{-\frac14})$ (D1)

$$p_n(\tfrac12(q^{\frac14}+q^{-\frac14});a,q^{\frac12}a,b,q^{\frac12}b|q)$$
$$=q^{-\frac{n}{4}}\begin{cases}\dfrac{(-q^{\frac12},-q,q^{\frac14}a,q^{\frac34}a,q^{\frac14}b,q^{\frac34}b,ab,q^{\frac12}ab,qab,q^{\frac32}ab;q^2)_{\frac{n}{2}}}{(ab,q^{\frac12}ab;q)_{\frac{n}{2}}} & \text{if } n \text{ even},\\ (1+q^{\frac12})(1-q^{\frac14}a)(1-q^{\frac14}b)(1-q^{\frac12}ab) & \\ \quad\times\dfrac{(-q,-q^{\frac32},q^{\frac34}a,q^{\frac54}a,q^{\frac34}b,q^{\frac54}b,q^{\frac32}ab,q^2ab,q^{\frac52}ab;q)_{\frac{n-1}{2}}}{(q^{\frac12}ab;q)_{\frac{n-1}{2}}} & \text{if } n \text{ odd}.\end{cases}$$

q-quadratic product transformation from A1

In Schlosser (2018) "q-Analogues of two product formulas of hypergeometric functions by Bailey" [in Frontiers in Orthogonal Polynomials and q-Series (Z. Nashed and X. Li, eds.), World Scientific, 2018, pp. 445-449], Schlosser started from A1 with the Ismail–Wilson generating function to produce:

$$ {}_2\phi_1\left(\begin{matrix}\{\pm\}a\\ a^2\end{matrix};q,t\right)\,{}_2\phi_1\left(\begin{matrix}\{\pm\}b\\ b^2\end{matrix};q,-t\right) = {}_4\phi_3\left(\begin{matrix}\{\pm\}ab,\{\pm\}qab\\ qa^2,qb^2,a^2b^2\end{matrix};q^2,t^2\right), $$

However, this formula was already given by Jain and Srivastava Equation (4.9) in V.K. Jain and H.M. Srivastava, "q-Series identities and reducibility of basic double hypergeometric functions", Canad. J. Math. 38 (1986), 215-231. This is a q-analogue of

$$ {}_1F_1\left(\begin{matrix}a\\ 2a\end{matrix};z\right)\,{}_1F_1\left(\begin{matrix}b\\ 2b\end{matrix};-z\right) = {}_2F_3\left(\begin{matrix}\frac12(a+b),\frac12(a+b+1)\\ a+\frac12,b+\frac12,a+b\end{matrix};\frac14 z\right). $$

q-quadratic product transformation from A4

In Schlosser (2018) “q-Analogues of two product formulas of hypergeometric functions by Bailey” [in Frontiers in Orthogonal Polynomials and q-Series (Z. Nashed and X. Li, eds.), World Scientific, pp. 445-449], Schlosser started from A4 and produced:

$$ {}_2\phi_1\left(\begin{matrix} a,\frac{q}{a}\\ -q\end{matrix};q,z\right)\,{}_2\phi_1\left(\begin{matrix} b,\frac{q}{b}\\ -q\end{matrix};q,-z\right)$$

$$= {}_4\phi_3\left(\begin{matrix} ab,\frac{q^2}{ab},\frac{qa}{b},\frac{qb}{a}\\ -q^2,\{\pm\}q\end{matrix};q^2,z^2\right)+\frac{(b-a)(1-\frac{q}{ab})z}{1-q^2}\,{}_4\phi_3\left(\begin{matrix} qab,\frac{q^3}{ab},\frac{q^2a}{b},\frac{q^2b}{a}\\ -q^2,\{\pm\}q^3\end{matrix};q^2,z^2\right)$$

$$=\frac{1}{2\pi}\frac{(q,\{\pm\}a,\{\pm\}b,\{\pm\}\frac{q}{b};q)_\infty}{\vartheta(\{\pm\}f;q)(-q,ab,\frac{qa}{b};q)_\infty}$$

$$\times\int_{-\pi}^{\pi}\frac{(\{\pm\}\mathrm{i}f\frac{\sigma}{w},(\{\pm\}\mathrm{i}f,\mathrm{i}qa)\frac{w}{\sigma};q)_\infty}{(\{\pm\}\mathrm{i}\frac{\sigma}{w},(-\mathrm{i}a,\mathrm{i}b,\mathrm{i}\frac{q}{b})\frac{w}{\sigma};q)_\infty}\,{}_3\phi_2\left(\begin{matrix}ab,\frac{qa}{b},-\frac{\mathrm{i}q}{a}\frac{\sigma}{w}\\ -q,\mathrm{i}qa\frac{w}{\sigma}\end{matrix};q,\mathrm{i}z\frac{w}{\sigma}\right)\mathrm{d}\psi.$$

q-quadratic product transformation from A2

$$
\begin{aligned}
&{}_2\phi_1\left({-a,-b \atop -ab};q,t\right)\,{}_2\phi_1\left({-a,-qb \atop -qab};q,-t\right)\\
&\quad={}_4\phi_3\left({a^2,q^2b^2,ab,qab \atop -qab,-q^2ab,a^2b^2};q^2,t^2\right)\\
&\qquad\qquad+\frac{bt(1-a^2)}{(1+ab)(1+qab)}\,{}_4\phi_3\left({q^2a^2,q^2b^2,qab,q^2ab \atop -q^2ab,-q^3ab,q^2a^2b^2};q^2,t^2\right).
\end{aligned}
$$

q-quadratic product transformation from A3

- q-quadratic product transformation from A3:

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a,-qa \\ qa^2 \end{matrix};q,t\right)\,{}_2\phi_1\left(\begin{matrix} b,-qb \\ qb^2 \end{matrix};q,-t\right)\\
&\quad = {}_4\phi_3\left(\begin{matrix} ab,\{\pm\}qab,-q^2ab \\ qa^2,qb^2,q^2a^2b^2 \end{matrix};q^2,t^2\right)\\
&\qquad\qquad + \frac{(b-a)(1+qab)t}{(1-qa^2)(1-qb^2)}\,{}_4\phi_3\left(\begin{matrix} qab,\{\pm\}q^2ab,-q^3ab \\ q^3a^2,q^3b^2,q^2a^2b^2 \end{matrix};q^2,t^2\right).
\end{aligned}
$$

- The $q \to 1^-$ limit provides:

$$
\begin{aligned}
{}_1F_1\left(\begin{matrix} a \\ 2a+1 \end{matrix};z\right)\,{}_1F_1\left(\begin{matrix} b \\ 2b+1 \end{matrix};-z\right) &= {}_2F_3\left(\begin{matrix} \frac{a+b}{2},\frac{a+b+1}{2} \\ a+\frac12,b+\frac12,a+b+1 \end{matrix};\frac{z^2}{4}\right)\\
&\quad + \frac{(a-b)z}{(2a+1)(2b+1)}\,{}_2F_3\left(\begin{matrix} \frac{a+b+1}{2},\frac{a+b+2}{2} \\ a+\frac32,b+\frac32,a+b+1 \end{matrix};\frac{z^2}{4}\right)
\end{aligned}
$$

q-quadratic product transformation from A5

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix}-c, q^2c\\ q^2c^2\end{matrix}; q, t\right)\,{}_2\phi_1\left(\begin{matrix}\{\pm\}a\\ a^2\end{matrix}; q, -t\right)\\
&= \frac{ct(1+q)}{(1-q^2c^2)}\,{}_4\phi_3\left(\begin{matrix}\{\pm\}qac, \{\pm\}q^2ac\\ qa^2, q^3c^2, q^2a^2c^2\end{matrix}; q^2, t^2\right)\\
&\quad+\frac{(1-qc^2)(1-qa^2c^2)}{(1-q^2c^2)(1-a^2c^2)}\,{}_5\phi_4\left(\begin{matrix}q^3a^2c^2, \{\pm\}ac, \{\pm\}qac\\ qa^2, qc^2, qa^2c^2, q^2a^2c^2\end{matrix}; q^2, t^2\right)\\
&\quad+\frac{qc^2(1-q)(1-\frac{a^2}{q})}{(1-q^2c^2)(1-a^2c^2)}\,{}_5\phi_4\left(\begin{matrix}q^3, \{\pm\}ac, \{\pm\}qac\\ q, \frac{a^2}{q}, q^3c^2, q^2a^2c^2\end{matrix}; q^2, t^2\right)
\end{aligned}
$$

q-quadratic product transformation from B1

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix}\{\pm\}a\\ \frac{a^2}{q}\end{matrix};q,z\right){}_2\phi_1\left(\begin{matrix}\{\pm\}b\\ qb^2\end{matrix};q,-qz\right)\\
&\qquad={}_4\phi_3\left(\begin{matrix}\{\pm\}ab,\{\pm\}qab\\ \frac{a^2}{q},qb^2,a^2b^2\end{matrix};q^2,z^2\right)\\
&\qquad\qquad+\frac{z(1-a^2b^2)}{(1-\frac{a^2}{q})(1-qb^2)}\,{}_4\phi_3\left(\begin{matrix}\{\pm\}qab,\{\pm\}q^2ab\\ qa^2,q^3b^2,a^2b^2\end{matrix};q^2,z^2\right).
\end{aligned}
$$

q-quadratic product transformation from B2

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix}\{\pm\}qa\\ qa^2\end{matrix};q,z\right)\,{}_2\phi_1\left(\begin{matrix}b,-qb\\ q^2b^2\end{matrix};q,-qz\right)\\
&\quad={}_5\phi_4\left(\begin{matrix}q^3b,\{\pm\}qab,\{\pm\}q^2ab\\ qa^2,qb,q^3b^2,q^2a^2b^2\end{matrix};q^2,z^2\right)\\
&\qquad+\frac{z(1-q^2a^2b)}{(1-qa^2)(1-qb)}\,{}_5\phi_4\left(\begin{matrix}\{\pm\}q^2ab,\{\pm\}q^3ab,q^4a^2b\\ q^3a^2,q^3b^2,q^2a^2b,q^4a^2b^2\end{matrix};q^2,z^2\right)
\end{aligned}
$$

q-quadratic product transformation from B3

$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} a, -qa \\ a^2 \end{matrix}; q, z\right) {}_2\phi_1\left(\begin{matrix} b, -qb \\ q^2b^2 \end{matrix}; q, -qz\right) \\
&\quad = \frac{(1+\frac{q}{a})(1-qab)}{q(1+\frac{1}{a})(1-ab)} {}_6\phi_5\left(\begin{matrix} q, q^3b, ab, -qab, -q^2ab, q^3ab \\ q, qb, qa^2, q^3b^2, q^2a^2b^2 \end{matrix}; q^2, z^2\right) \\
&\qquad - \frac{(1-q)(1+qb)}{q(1+\frac{1}{a})(1-ab)} {}_6\phi_5\left(\begin{matrix} q^3, q^4b^2, ab, \{\pm\}qab, -q^2ab \\ q, qa^2, q^2b^2, q^3b^2, q^2a^2b^2 \end{matrix}; q^2, z^2\right) \\
&\qquad + \frac{(1+qab)z}{(1+a)(1-qb)} {}_4\phi_3\left(\begin{matrix} qab, \{\pm\}q^2ab, -q^3ab \\ qa^2, q^3b^2, q^2a^2b^2 \end{matrix}; q^2, z^2\right)
\end{aligned}
$$

q-quadratic product transformation from D1

$$
{}_2\phi_1\left(\begin{matrix} a, q^{\frac12}a \\ a^2 \end{matrix};q,t\right)\,{}_2\phi_1\left(\begin{matrix} b, q^{\frac12}b \\ qb^2 \end{matrix};q^{\frac12}t\right)
$$

$$
= {}_8\phi_7\left(\begin{matrix} \{\pm\}\sqrt{ab}, \{\pm\}q^{\frac14}\sqrt{ab}, \{\pm\}q^{\frac12}\sqrt{ab}, \{\pm\}q^{\frac34}\sqrt{ab} \\ q^{\frac12}, -a, -q^{\frac12}a, -q^{\frac12}b, -qb, ab, q^{\frac12}ab \end{matrix};q,t^2\right)
$$

$$
+\frac{(1-q^{\frac12}ab)t}{(1-q^{\frac12})(1+a)(1+q^{\frac12}b)}\,{}_7\phi_6\left(\begin{matrix} q^{\frac32}ab, q^2ab, q^{\frac52}ab, 0, 0, 0, 0 \\ q^{\frac32}, -q^{\frac12}a, -qa, -qb, -q^{\frac32}b, q^{\frac12}ab \end{matrix};q,t^2\right)
$$

q-quadratic product transformation from C1, C2

$$\frac{(at;q)_\infty}{(t;q)_\infty} = {}_2\phi_1\left(\begin{matrix} a, qa \\ q \end{matrix}; q^2, t^2\right) + \frac{(1-a)t}{(1-q)}\,{}_2\phi_1\left(\begin{matrix} qa, q^2a \\ q^3 \end{matrix}; q^2, t^2\right),$$

$$\begin{aligned}{}_2\phi_1\left(\begin{matrix} qa^2, qab \\ ab \end{matrix}; q, t\right) &= \frac{(q^2a^2t;q)_\infty}{(t;q)_\infty}\left(1 + \frac{(1-\frac{b}{qa})qa^2t}{(1-ab)}\right) \\ &= \frac{(q^2a^2t;q^2)_\infty}{(qt;q^2)_\infty}\,{}_3\phi_2\left(\begin{matrix} qa^2, q^3a^2, q^4ab \\ q^2, ab \end{matrix}; q^4, t^2\right) \\ &\quad + \frac{(q^2a^2t;q^2)_\infty\,(1-qa^2)(1-q^2ab)t}{(qt;q^2)_\infty\,(1-q^2)(1-ab)}\,{}_3\phi_2\left(\begin{matrix} q^3a^2, q^5a^2, q^6ab \\ q^6, q^2ab \end{matrix}; q^4, t^2\right).\end{aligned}$$

Cayley-Orr type expansion formulas

- By using Cayley–Orr type expansion formulas stated in Gasper and Rahman's textbook.
- These were obtained by Singh (1959) and by Nassrallah (1982).
- These are Exercises 3.17–3.19 in G+R.

G+R Exercise 3.18.

Lemma (G+R Exercise 3.18.)

Let $0<|q|<1$, $a,b,c,z\in\mathbb{C}^*$, $|z|<1$, $|q^2cz|<|ab|$ *and*

$$\frac{(\frac{q^3cz}{ab};q^2)_\infty}{(z;q^2)_\infty}\,{}_2\phi_1\left(\begin{matrix}\frac{a}{q},\frac{b}{q}\\ c\end{matrix};q,\frac{q^2cz}{ab}\right)=\sum_{n=0}^\infty a_nz^n.$$

Then

$${}_2\phi_1\left(\begin{matrix}\frac{q^2c}{a},\frac{q^2c}{b}\\ q^2c\end{matrix};q^2,z\right)\,{}_2\phi_1\left(\begin{matrix}\frac{a}{q},\frac{b}{q}\\ c\end{matrix};q^2,\frac{q^2cz}{ab}\right)=\sum_{n=0}^\infty\frac{(qc;q^2)_n}{(q^2c;q^2)_n}a_nz^n.$$

- A1 transformation is verified with $c=ab/q$.

G+R Exercise 3.19.

Lemma (G+R Exercise 3.19.)

Let $0<|q|<1$, $a,b,c,z\in\mathbb{C}^*$, $|z|<1$, $|q^2cz|<|ab|$ *and*

$$\frac{(\frac{qcz}{ab};q^2)_\infty}{(z;q^2)_\infty}\,{}_2\phi_1\left(\begin{matrix}\frac{a}{q},b\\ \frac{c}{q}\end{matrix};q,\frac{cz}{ab}\right)=\sum_{n=0}^{\infty}a_nz^n.$$

Then

$${}_2\phi_1\left(\begin{matrix}\frac{qc}{a},\frac{c}{qb}\\ c\end{matrix};q^2,z\right)\,{}_2\phi_1\left(\begin{matrix}a,b\\ c\end{matrix};q^2,\frac{cz}{ab}\right)=\sum_{n=0}^{\infty}\frac{(\frac{c}{q};q^2)_n}{(c;q^2)_n}a_nz^n.$$

- Using $c=qab$, and Exercise 3.19, Nassrallah produced

$${}_2\phi_1\left(\begin{matrix}qa^2,qb^2\\ qa^2b^2\end{matrix};q^2,z\right)\,{}_2\phi_1\left(\begin{matrix}\frac{a^2}{q},qb^2\\ qa^2b^2\end{matrix};q^2,qz\right)={}_4\phi_3\left(\begin{matrix}a^2,qb^2,\{\pm\}ab\\ a^2b^2,\{\pm\}q^{\frac12}ab\end{matrix};q,z\right)$$

G+R Exercise 3.17.

Lemma (G+R Exercise 3.17.)

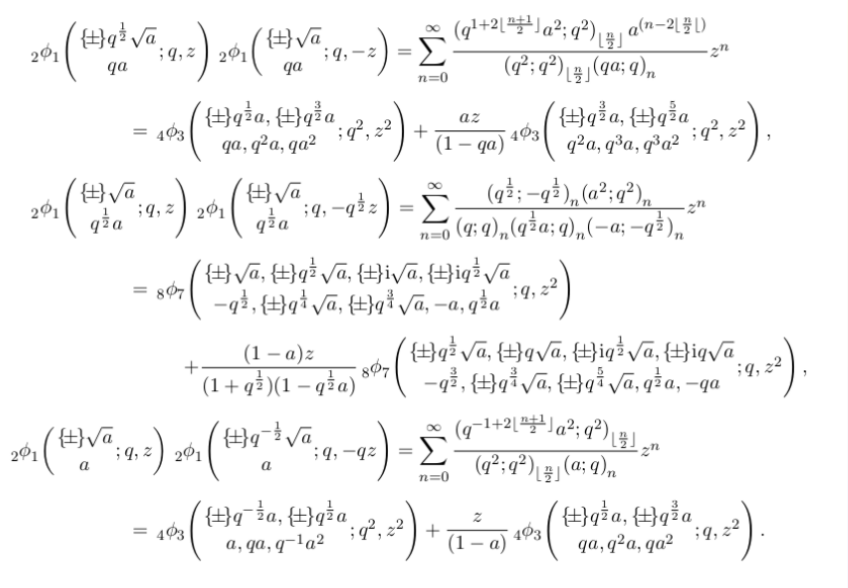
Let $0<|q|<1$, $a,b,c,z\in\mathbb{C}^*$, $|z|<1$, $|cz|<|qab|$ *and*

$$\frac{(\frac{cz}{ab};q^2)_\infty}{(z;q^2)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,\frac{cz}{qab}\right)=\sum_{n=0}^\infty a_n z^n.$$

Then

$${}_2\phi_1\left(\begin{matrix}\frac{c}{a},\frac{c}{b}\\qc\end{matrix};q^2,z\right)\,{}_2\phi_1\left(\begin{matrix}a,b\\qc\end{matrix};q^2,\frac{cz}{qab}\right)=\sum_{n=0}^\infty\frac{(c;q^2)_n}{(qc;q^2)_n}a_n z^n.$$

G+R Exercise 3.17



$$
{}_2\phi_1\left(\begin{matrix}\{\pm\}q^{\frac12}\sqrt{a}\\ qa\end{matrix};q,z\right){}_2\phi_1\left(\begin{matrix}\{\pm\}\sqrt{a}\\ qa\end{matrix};q,-z\right)=\sum_{n=0}^{\infty}\frac{(q^{1+2\lfloor\frac{n+1}{2}\rfloor}a^2;q^2)_{\lfloor\frac{n}{2}\rfloor}\,a^{(n-2\lfloor\frac{n}{2}\lfloor)}}{(q^2;q^2)_{\lfloor\frac{n}{2}\rfloor}(qa;q)_n}z^n
$$

$$
={}_4\phi_3\left(\begin{matrix}\{\pm\}q^{\frac12}a,\{\pm\}q^{\frac32}a\\ qa,q^2a,qa^2\end{matrix};q^2,z^2\right)+\frac{az}{(1-qa)}\,{}_4\phi_3\left(\begin{matrix}\{\pm\}q^{\frac32}a,\{\pm\}q^{\frac52}a\\ q^2a,q^3a,q^3a^2\end{matrix};q^2,z^2\right),
$$

$$
{}_2\phi_1\left(\begin{matrix}\{\pm\}\sqrt{a}\\ q^{\frac12}a\end{matrix};q,z\right){}_2\phi_1\left(\begin{matrix}\{\pm\}\sqrt{a}\\ q^{\frac12}a\end{matrix};q,-q^{\frac12}z\right)=\sum_{n=0}^{\infty}\frac{(q^{\frac12};-q^{\frac12})_n(a^2;q^2)_n}{(q;q)_n(q^{\frac12}a;q)_n(-a;-q^{\frac12})_n}z^n
$$

$$
={}_8\phi_7\left(\begin{matrix}\{\pm\}\sqrt{a},\{\pm\}q^{\frac12}\sqrt{a},\{\pm\}\mathrm{i}\sqrt{a},\{\pm\}\mathrm{i}q^{\frac12}\sqrt{a}\\ -q^{\frac12},\{\pm\}q^{\frac14}\sqrt{a},\{\pm\}q^{\frac34}\sqrt{a},-a,q^{\frac12}a\end{matrix};q,z^2\right)
$$

$$
+\frac{(1-a)z}{(1+q^{\frac12})(1-q^{\frac12}a)}\,{}_8\phi_7\left(\begin{matrix}\{\pm\}q^{\frac12}\sqrt{a},\{\pm\}q\sqrt{a},\{\pm\}\mathrm{i}q^{\frac12}\sqrt{a},\{\pm\}\mathrm{i}q\sqrt{a}\\ -q^{\frac32},\{\pm\}q^{\frac34}\sqrt{a},\{\pm\}q^{\frac54}\sqrt{a},q^{\frac12}a,-qa\end{matrix};q,z^2\right),
$$

$$
{}_2\phi_1\left(\begin{matrix}\{\pm\}\sqrt{a}\\ a\end{matrix};q,z\right){}_2\phi_1\left(\begin{matrix}\{\pm\}q^{-\frac12}\sqrt{a}\\ a\end{matrix};q,-qz\right)=\sum_{n=0}^{\infty}\frac{(q^{-1+2\lfloor\frac{n+1}{2}\rfloor}a^2;q^2)_{\lfloor\frac{n}{2}\rfloor}}{(q^2;q^2)_{\lfloor\frac{n}{2}\rfloor}(a;q)_n}z^n
$$

$$
={}_4\phi_3\left(\begin{matrix}\{\pm\}q^{-\frac12}a,\{\pm\}q^{\frac12}a\\ a,qa,q^{-1}a^2\end{matrix};q^2,z^2\right)+\frac{z}{(1-a)}\,{}_4\phi_3\left(\begin{matrix}\{\pm\}q^{\frac12}a,\{\pm\}q^{\frac32}a\\ qa,q^2a,qa^2\end{matrix};q,z^2\right).
$$

Another application of G+R Exercise 3.17

In Exercise 3.17, which is specific terminating 2-balanced ${}_3\phi_2$ series which sums with two terms. The latter can be evaluated with $c = q^2ab$. Then one can obtain:

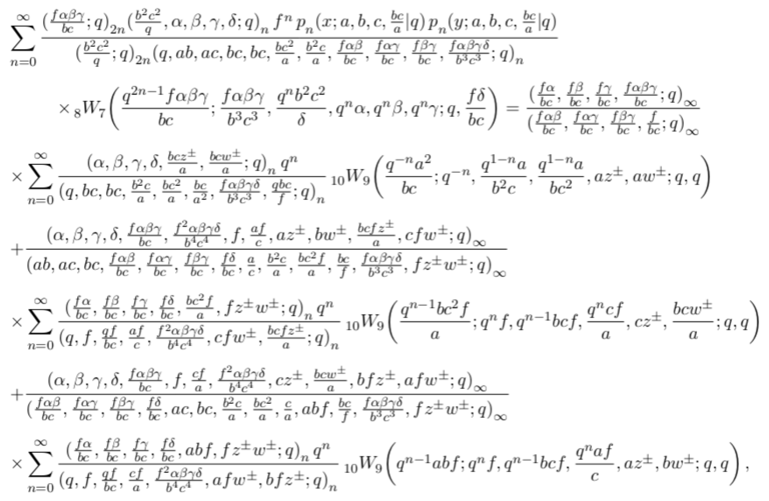
$$
\begin{aligned}
&{}_2\phi_1\left(\begin{matrix} q^2a, q^2b \\ q^3ab \end{matrix}; q^2, z\right) {}_2\phi_1\left(\begin{matrix} a, b \\ q^3ab \end{matrix}; q^2, qz\right) \\
&\qquad = \frac{(1-q)(1-qab)}{(1-qa)(1-qb)}\, {}_4\phi_3\left(\begin{matrix} qa, qb, \{\pm\} q\sqrt{ab} \\ qab, \{\pm\}\sqrt{q^3ab} \end{matrix}; q, z\right) \\
&\qquad\qquad + \frac{q\,(1-a)(1-b)}{(1-qa)(1-qb)}\, {}_4\phi_3\left(\begin{matrix} qa, qb, \{\pm\} q\sqrt{ab} \\ q^2ab, \{\pm\}\sqrt{q^3ab} \end{matrix}; q, z\right)
\end{aligned}
$$

Bilinear expansions of Askey–Wilson polynomials

- There are at least 11 known alternative linear and bilinear expansion formulas for Askey–Wilson polynomials, including the Askey–Rahman–Suslov nonsymmetric Poisson kernel for Askey–Wilson polynomials.
- Aside: also, through this exercise we've noticed, corrected and reported several errors in these expansion formulas.
- One may attempt to utilize the above mentioned q-quadratic special values for Askey–Wilson polynomials to obtain new q-quadratic transformation and summation formulas.
- Also produced new bilinear sums for Askey–Wilson polynomials

Rahman's (1985) bilinear ${}_8W_7$ sum (G+R Exercise 9.18)

- Let $x=\frac{1}{2}(z+z^{-1})$, $y=\frac{1}{2}(w+w^{-1})$. Then



$$\sum_{n=0}^{\infty}\frac{(\frac{f\alpha\beta\gamma}{bc};q)_{2n}(\frac{b^2c^2}{q},\alpha,\beta,\gamma,\delta;q)_n\,f^n\,p_n(x;a,b,c,\frac{bc}{a}|q)\,p_n(y;a,b,c,\frac{bc}{a}|q)}{(\frac{b^2c^2}{q};q)_{2n}(q,ab,ac,bc,bc,\frac{bc^2}{a},\frac{b^2c}{a},\frac{f\alpha\beta}{bc},\frac{f\alpha\gamma}{bc},\frac{f\beta\gamma}{bc},\frac{f\alpha\beta\gamma\delta}{b^3c^3};q)_n}$$

$$\times\,{}_8W_7\left(\frac{q^{2n-1}f\alpha\beta\gamma}{bc};\frac{f\alpha\beta\gamma}{b^3c^3},\frac{q^nb^2c^2}{\delta},q^n\alpha,q^n\beta,q^n\gamma;q,\frac{f\delta}{bc}\right)=\frac{(\frac{f\alpha}{bc},\frac{f\beta}{bc},\frac{f\gamma}{bc},\frac{f\alpha\beta\gamma}{bc};q)_\infty}{(\frac{f\alpha\beta}{bc},\frac{f\alpha\gamma}{bc},\frac{f\beta\gamma}{bc},\frac{f}{bc};q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(\alpha,\beta,\gamma,\delta,\frac{bcz^\pm}{a},\frac{bcw^\pm}{a};q)_n\,q^n}{(q,bc,bc,\frac{b^2c}{a},\frac{bc^2}{a},\frac{bc}{a^2},\frac{f\alpha\beta\gamma\delta}{b^3c^3},\frac{qbc}{f};q)_n}\,{}_{10}W_9\left(\frac{q^{-n}a^2}{bc};q^{-n},\frac{q^{1-n}a}{b^2c},\frac{q^{1-n}a}{bc^2},az^\pm,aw^\pm;q,q\right)$$

$$+\frac{(\alpha,\beta,\gamma,\delta,\frac{f\alpha\beta\gamma}{bc},\frac{f^2\alpha\beta\gamma\delta}{b^4c^4},f,\frac{af}{c},az^\pm,bw^\pm,\frac{bcfz^\pm}{a},cfw^\pm;q)_\infty}{(ab,ac,bc,\frac{f\alpha\beta}{bc},\frac{f\alpha\gamma}{bc},\frac{f\beta\gamma}{bc},\frac{f\delta}{bc},\frac{a}{c},\frac{b^2c}{a},\frac{bc^2f}{a},\frac{bc}{f},\frac{f\alpha\beta\gamma\delta}{b^3c^3},fz^\pm w^\pm;q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(\frac{f\alpha}{bc},\frac{f\beta}{bc},\frac{f\gamma}{bc},\frac{f\delta}{bc},\frac{bc^2f}{a},fz^\pm w^\pm;q)_n\,q^n}{(q,f,\frac{qf}{bc},\frac{af}{c},\frac{f^2\alpha\beta\gamma\delta}{b^4c^4},cfw^\pm,\frac{bcfz^\pm}{a};q)_n}\,{}_{10}W_9\left(\frac{q^{n-1}bc^2f}{a};q^nf,q^{n-1}bcf,\frac{q^ncf}{a},cz^\pm,\frac{bcw^\pm}{a};q,q\right)$$

$$+\frac{(\alpha,\beta,\gamma,\delta,\frac{f\alpha\beta\gamma}{bc},f,\frac{cf}{a},\frac{f^2\alpha\beta\gamma\delta}{b^4c^4},cz^\pm,\frac{bcw^\pm}{a},bfz^\pm,afw^\pm;q)_\infty}{(\frac{f\alpha\beta}{bc},\frac{f\alpha\gamma}{bc},\frac{f\beta\gamma}{bc},\frac{f\delta}{bc},ac,bc,\frac{b^2c}{a},\frac{bc^2}{a},\frac{c}{a},abf,\frac{bc}{f},\frac{f\alpha\beta\gamma\delta}{b^3c^3},fz^\pm w^\pm;q)_\infty}$$

$$\times\sum_{n=0}^{\infty}\frac{(\frac{f\alpha}{bc},\frac{f\beta}{bc},\frac{f\gamma}{bc},\frac{f\delta}{bc},abf,fz^\pm w^\pm;q)_n\,q^n}{(q,f,\frac{qf}{bc},\frac{cf}{a},\frac{f^2\alpha\beta\gamma\delta}{b^4c^4},afw^\pm,bfz^\pm;q)_n}\,{}_{10}W_9\left(q^{n-1}abf;q^nf,q^{n-1}bcf,\frac{q^naf}{c},az^\pm,bw^\pm;q,q\right),$$

Example 1: three-term q-quadratic transformation

- Setting $\{a,b,c,w,z\}\mapsto\{\{\pm\}ia,ib,i,i\}$. Then one obtains $|ab|<|c|$,

$$
\begin{aligned}
&{}_8W_7\left(\frac{a^2b^2}{q};q,a^2,b^2,c,qc;q^2,\frac{a^2b^2}{c^2}\right)\\
&\quad=\frac{(a^2b^2;q)_\infty}{(a^2b^2,\frac{a^2b^2}{c^2};q^2)_\infty(-ab,-\frac{ab}{c},\frac{a^2b^2}{c};q)_\infty}\\
&\qquad\qquad\times\overset{a;b}{\Bigg[\!\!\Bigg]}\frac{(a^2,a^2,\frac{a^2b^4}{c^2},\frac{a^2b^4}{c^2};q^2)_\infty(\frac{a^2}{c};q)_\infty}{(a^2,\frac{a}{b},\frac{ab^3}{c};q)_\infty}\\
&\qquad\qquad\qquad\times{}_{10}W_9\left(\frac{ab^3}{qc};\{\pm\}b,\{\pm\}b,\frac{b^2}{c},\frac{ab}{c},\frac{a^2b^2}{qc};q,q\right).
\end{aligned}
$$

Example 2: four-term q-quadratic transformation

- Setting $\{a,b,c,w,z\}\mapsto\{ib,-iq/a,-ia,i,i\}$. Then if $|q|<|c|$,

$$
\begin{aligned}
&{}_{10}W_9\left(q;q^2,ab,\frac{q^2}{ab},\frac{qa}{b},\frac{qb}{a},c,qc;q^2,\frac{q^2}{c^2}\right)\\
&\qquad+\frac{q^2(1-q^3)(1-\frac{a}{b})(1-\frac{b}{a})(1-\frac{q}{ab})(1-\frac{ab}{q})(1-c)}{c(1-q)(1-ab)(1-\frac{qa}{b})(1-\frac{qb}{a})(1-\frac{q^2}{ab})(1-\frac{q^2}{c})}\\
&\qquad\qquad\times{}_{10}W_9\left(q^3;q^2,qab,\frac{q^3}{ab},\frac{q^2a}{b},\frac{q^2b}{a},qc,q^2c;q^2,\frac{q^2}{c^2}\right)\\
&=\frac{(q^2;q)_\infty}{(\frac{q^2}{c^2};q^2)_\infty(-q,ab,\frac{q^2}{ab},\frac{q}{c},\frac{q^2}{c};q)_\infty}\overset{a;b}{\mathbb{I}}\frac{(b^2,\frac{q^2}{a^2},\frac{q^2a^2}{c^2},\frac{q^4}{b^2c^2};q^2)_\infty(\frac{qb}{ac};q)_\infty}{(\frac{qb}{a},-\frac{b}{a},-\frac{q^2a}{bc};q)_\infty}\\
&\qquad\qquad\times{}_{10}W_9\left(-\frac{qa}{bc};\{\pm\}a,\{\pm\}\frac{q}{b},\{\pm\}\frac{q}{c},\frac{qa}{bc};q,q\right).
\end{aligned}
$$

Another bilinear sum over Askey–Wilson polynomials

- Let $x=\frac{1}{2}(z+z^{-1})$, $y=\frac{1}{2}(w+w^{-1})$. Setting $\alpha=bc/f$ causes the first term to vanish and for the second term only the $n=0$ term contributes. Then replace $f\to t$ and utilize

$$ {}_8W_7\left(q^{2n-1}\beta\gamma;\frac{\beta\gamma}{b^2c^2},q^n\beta,q^n\gamma,\frac{q^nb^2c^2}{\delta},\frac{q^nbc}{t};q,\frac{\delta t}{bc}\right)=\frac{(\delta,\beta\gamma,bct,\frac{\beta\gamma\delta t}{b^3c^3};q)_\infty(\frac{\beta\gamma\delta}{b^2c^2},\frac{\beta\gamma t}{bc};q)_n(b^2c^2;q)_{2n}}{(b^2c^2,\frac{\beta\gamma\delta}{b^2c^2},\frac{\delta t}{bc},\frac{\beta\gamma t}{bc};q)_\infty(\delta,bct;q)_n(\beta\gamma;q)_{2n}}, $$

which produces the bilinear sum over Askey–Wilson polynomials

$$ \sum_{n=0}^{\infty}\frac{(b^2c^2;q)_{2n}(\frac{b^2c^2}{q},\frac{bc}{t};q)_n\,t^n\,p_n(x;a,b,c,\frac{bc}{a}|q)p_n(y;a,b,c,\frac{bc}{a}|q)}{(\frac{b^2c^2}{q};q)_{2n}(q,ab,ac,bc,bc,\frac{bc^2}{a},\frac{b^2c}{a},bct;q)_n} $$

$$ =\frac{(b^2c^2,t,\frac{at}{c},az^\pm,bw^\pm,\frac{bctz^\pm}{a},ctw^\pm;q)_\infty}{(ab,ac,bc,\frac{a}{c},\frac{b^2c}{a},bct,\frac{bc^2t}{a},tz^\pm w^\pm;q)_\infty}\,{}_{10}W_9\left(\frac{bc^2t}{qa};t,\frac{ct}{a},\frac{bct}{q},cz^\pm,\frac{bcw^\pm}{a};q,q\right) $$

$$ +\frac{(b^2c^2,t,\frac{ct}{a},cz^\pm,\frac{bcw^\pm}{a},btz^\pm,atw^\pm;q)_\infty}{(ac,bc,\frac{c}{a},\frac{b^2c}{a},\frac{bc^2}{a},abt,bct,tz^\pm w^\pm;q)_\infty}\,{}_{10}W_9\left(\frac{abt}{q};t,\frac{at}{c},\frac{bct}{q},az^\pm,bw^\pm;q,q\right), $$

Examples 3: q-quadratic nonterminating summation

- Setting $\{a,b,c,d\}\mapsto\{-iq/a,-ia,ib,iq/b\}$ produces $|t|<|1|$,

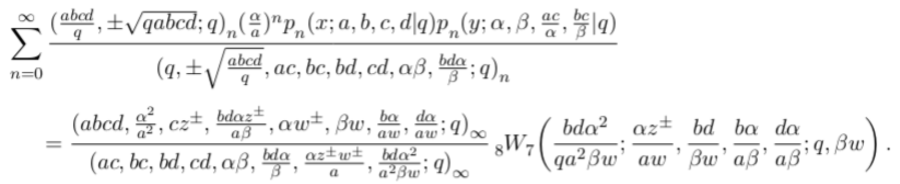
$$
{}_8W_7\left(q;q^2,\frac{qa}{b},\frac{q^2}{ab},\frac{b}{t},\frac{qb}{t};q^2,t^2\right)
$$

$$
-\frac{(1-q^3)(1-\frac{a}{b})(1-\frac{ab}{q})(1-\frac{t}{b})}{(1-q)(1-ab)(1-\frac{a}{qb})(1-\frac{q^2t}{b})}\,{}_8W_7\left(q^3;q^2,\frac{q^2a}{b},\frac{q^3}{ab},\frac{qb}{t},\frac{q^2b}{t};q^2,t^2\right)
$$

$$
=\frac{(b^2;q^2)_\infty(q^2,at,-\frac{qt}{b},\frac{qt}{a};q)_\infty}{(t^2;q^2)_\infty(-q,ab,\frac{qb}{a},\frac{q^2t}{b};q)_\infty}
$$

Askey–Rahman–Suslov Poisson kernel for AW polynomials

- Let $x=\frac{1}{2}(z+z^{-1})$, $y=\frac{1}{2}(w+w^{-1})$. ARS found the following bilinear generating function for Askey–Wilson polynomials at $t=1$, namely



$$\sum_{n=0}^{\infty}\frac{(\frac{abcd}{q},\pm\sqrt{qabcd};q)_n(\frac{\alpha}{a})^n p_n(x;a,b,c,d|q)p_n(y;\alpha,\beta,\frac{ac}{\alpha},\frac{bc}{\beta}|q)}{(q,\pm\sqrt{\frac{abcd}{q}},ac,bc,bd,cd,\alpha\beta,\frac{bd\alpha}{\beta};q)_n}$$

$$=\frac{(abcd,\frac{\alpha^2}{a^2},cz^{\pm},\frac{bd\alpha z^{\pm}}{a\beta},\alpha w^{\pm},\beta w,\frac{b\alpha}{aw},\frac{d\alpha}{aw};q)_\infty}{(ac,bc,bd,cd,\alpha\beta,\frac{bd\alpha}{\beta},\frac{\alpha z^{\pm}w^{\pm}}{a},\frac{bd\alpha^2}{a^2\beta w};q)_\infty}\,{}_8W_7\left(\frac{bd\alpha^2}{qa^2\beta w};\frac{\alpha z^{\pm}}{aw},\frac{bd}{\beta w},\frac{b\alpha}{a\beta},\frac{d\alpha}{a\beta};q,\beta w\right).$$

Example 4: q-quadratic nonterminating summation

- Replacing $\{a,b,c,d,\alpha,\beta,z,w\}\mapsto\{\mathrm{i}b,\mathrm{i}q/b,-\mathrm{i}a,-\frac{\mathrm{i}q}{a},-\mathrm{i}a,\frac{\mathrm{i}q}{b},\mathrm{i},\mathrm{i}\}$. Then for $|a|<|b|$,

$$
\begin{aligned}
&{}_8W_7\left(q;q^2,ab,\frac{qb}{a},\frac{qb}{a},\frac{q^2}{ab};q^2,\frac{a^2}{b^2}\right)\\
&\quad+\frac{q^2}{ab}\frac{(1-q^3)(1-\frac{ab}{q})(1-\frac{ab}{q})(1-\frac{a}{b})(1-\frac{a}{b})}{(1-q)(1-\frac{qa}{b})(1-\frac{qa}{b})(1-\frac{q^2}{ab})(1-ab)}\\
&\qquad\times{}_8W_7\left(q^3;q^2,qab,\frac{q^2b}{a},\frac{q^2b}{a},\frac{q^3}{ab};q^2,\frac{a^2}{b^2}\right)\\
&\qquad=\frac{(q^2,\frac{a^2}{b^2};q)_\infty(a^2,a^2,\frac{q^2}{b^2},\frac{q^2}{b^2};q^2)_\infty}{(-q,-q,ab,\frac{qa}{b},\frac{qa}{b},\frac{q^2}{ab};q)_\infty(\frac{a^2}{b^2},\frac{a^2}{b^2};q^2)_\infty}.
\end{aligned}
$$