Bilateral discrete and continuous orthogonality relations in the q^{-1}-symmetric Askey scheme

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Finite and infinite q-shifted factorials–building blocks

- The finite q-shifted factorial,

$$(a;q)_n := (1-a)(1-qa)(1-q^2a)\cdots(1-q^{n-1}a) = \prod_{k=0}^{n-1}(1-aq^k)$$

- The infinite q–shifted factorial, $q \in \mathbb{C}^\dagger$,

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

- Other adopted product notations:

$$(a_1,\ldots,a_r)_n := (a_1)_n\cdots(a_r)_n,$$

$$(a_1,\ldots,a_r;q)_n := (a_1;q)_n\cdots(a_r;q)_n,$$

$$(a_1,\ldots,a_r;q)_\infty := (a_1;q)_\infty\cdots(a_r;q)_\infty,$$

Basic hypergeometric series

- Note that for $a=0$, $(a;q)_k=(a;q)_\infty=1$.
- For $a=q^{-n}$, $n\in\mathbb{N}_0$, $(a;q)_\infty=0$.
- For $a=q^{-n}$, $n\in\mathbb{N}_0$, $(a;q)_k=0$ for $k>n$.
- Infinite series representation for nonterminating basic hypergeometric series

$$ {}_{r+1}\phi_s\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_s \end{matrix};z\right)=\sum_{k=0}^\infty \frac{(a_1,\ldots,a_{r+1};q)_k}{(q,b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k, $$

divergent for $r<s$, convergent for $|z|<1$ if $r=s$ and entire if $s>r$.

- If one of the numerator parameters is of the form q^{-n}, $n\in\mathbb{N}_0$, then the infinite series terminates. This is fundamental to the study of q-orthogonal polynomials.

Nonterminating basic bilateral hypergeometric series

- Infinite series representation for nonterminating basic bilateral hypergeometric series

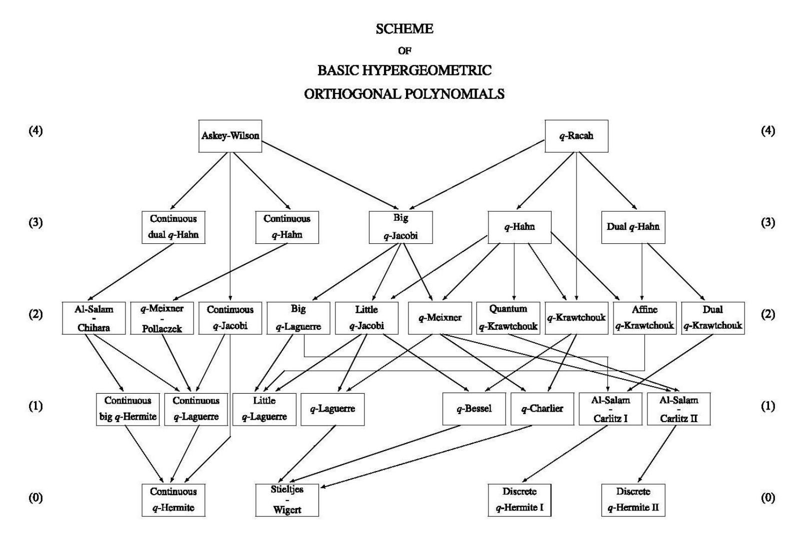
$$ {}_r\psi_s\left(\begin{matrix} a_1,\ldots,a_r \\ b_1,\ldots,b_s \end{matrix};z\right) = \sum_{k=-\infty}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{s-r} z^k, $$

convergent when $s \geq r$ provided that

$$ |b_1 \cdots b_s| < |a_1 \cdots a_r z|, $$

and also, in the case $s = r$, $|z| < 1$.

The q-Askey scheme of basic hypergeometric OPs



Symmetric basic and generalized hypergeometric OPs

Askey–Wilson polynomials
(4 parameters: a, b, c, d)

↓ $d \to 0$

continuous dual q-Hahn polynomials
(3 parameters: a, b, c)

↓ $c \to 0$

Al-Salam-Chihara polynomials
(2 parameters: a, b)

↓ $b \to 0$

continuous big q-Hermite polynomials
(1 parameter: a)

↓ $a \to 0$

continuous q-Hermite polynomials
(0 parameters)

Wilson polynomials
(4 parameters: a, b, c, d)

↓

continuous dual Hahn
(3 parameters: a, b, c)

↓

Hermite polynomials
(0 parameters)

Orthogonal polynomials in the q-symmetric Askey scheme

- Askey–Wilson polynomials

$$p_n(x;a,b,c,d|q) := a^{-n}(ab,ac,ad;q)_n\,{}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az^{\pm} \\ ab, ac, ad \end{matrix}; q, q\right)$$

- continuous dual q-Hahn polynomials

$$p_n(x;a,b,c|q) := a^{-n}(ab,ac;q)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, az^{\pm} \\ ab, ac \end{matrix}; q, q\right)$$

- Al-Salam–Chihara polynomials

$$Q_n(x;a,b|q) := a^{-n}(ab;q)_n\,{}_3\phi_2\left(\begin{matrix} q^{-n}, az^{\pm} \\ ab, 0 \end{matrix}; q, q\right)$$

- continuous big q-Hermite polynomials

$$H_n(x;a|q) := a^{-n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, az^{\pm} \\ 0, 0 \end{matrix}; q, q\right)$$

Orthogonal polynomials in the q-symmetric Askey scheme

- continuous q-Hermite polynomials

$$H_n(x|q) := z^n \, {}_2\phi_0\left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix}; q, \frac{q^n}{z^2}\right)$$

- Symmetric limits in the q-symmetric Askey-scheme

$$p_n(x;a,b,c|q) = \lim_{d\to 0} p_n(x;a,b,c,d|q),$$

$$Q_n(x;a,b|q) = \lim_{c\to 0} p_n(x;a,b,c|q),$$

$$H_n(x;a|q) = \lim_{b\to 0} Q_n(x;a,b|q),$$

$$H_n(x|q) = \lim_{a\to 0} H_n(x;a|q).$$

The q^{-1}-symmetric Askey-scheme

- Relation for finite q^{-1}-shifted factorial

$$(a;q^{-1})_n = q^{-\binom{n}{2}}(-a)^n(a^{-1};q)_n$$

- Relation between q^{-1}-AW polynomials and Askey–Wilson polynomials

$$p_n(x;a,b,c,d|q^{-1}) = q^{-3\binom{n}{2}}(-abcd)^n p_n(x;\tfrac{1}{a},\tfrac{1}{b},\tfrac{1}{c},\tfrac{1}{d}|q).$$

- Symmetric limits of q^{-1}-Askey-scheme

$$Q_n(x;a,b|q^{-1}) = \lim_{c\to 0} p_n(x;a,b,c|q^{-1}),$$

$$H_n(x;a|q^{-1}) = \lim_{b\to 0} Q_n(x;a,b|q^{-1}),$$

$$H_n(x|q^{-1}) = \lim_{a\to 0} H_n(x;a|q^{-1}).$$

Square bracket notation for q and q^{-1}-symmetric basic hypergeometric OPs

- Consider a q or q^{-1}-symmetric orthogonal polynomial p_n in x with a set of free parameters $\mathbf{a}$.
- Then if $x = \frac{1}{2}(z + z^{-1})$, we write

$$p_n[z; \mathbf{a}|q] := p_n(x; \mathbf{a}|q) = p_n(\tfrac{1}{2}(z + z^{-1}); \mathbf{a}|q).$$

- Similarly, if $x = \frac{1}{2}(z - z^{-1})$, we write

$$p_n[z; \mathbf{a}|q] := p_n(x; \mathbf{a}|q) = p_n(\tfrac{1}{2}(z - z^{-1}); \mathbf{a}|q).$$

Orthogonal polynomials in the q^{-1}-symmetric Askey scheme

These polynomials have continuous orthogonality relations on the imaginary axis. We start with $x = \frac{1}{2}(z + z^{-1})$. Now let $z \mapsto iz$.

$$x \mapsto \frac{1}{2}(iz + \frac{1}{iz}) = \frac{i}{2}(z - z^{-1}).$$

Define the following orthogonal polynomials in the q^{-1}-symmetric family of orthogonal polynomials in the q^{-1}-Askey-scheme, namely

$$\mathbf{p}_n(x; a, b, c, d|q) := \mathbf{p}_n[z; a, b, c, d|q] := i^{-n} p_n[iz; ia, ib, ic, id|q^{-1}],$$

$$\mathbf{p}_n(x; a, b, c|q) := \mathbf{p}_n[z; a, b, c|q] := i^{-n} p_n[iz; ia, ib, ic|q^{-1}],$$

$$\mathbf{Q}_n(x; a, b|q) := \mathbf{Q}_n[z; a, b|q] := i^{-n} Q_n[iz; ia, ib|q^{-1}],$$

$$\mathbf{H}_n(x; a|q) := \mathbf{H}_n[z; a|q] := i^{-n} H_n[iz; ia|q^{-1}],$$

$$\mathbf{H}_n(x|q) := \mathbf{H}_n[z|q] := i^{-n} H_n[iz|q^{-1}],$$

where $x = \frac{1}{2}(z - z^{-1})$.

Symmetric limits of q^{-1}-symmetric q-Askey scheme

- The orthogonal polynomials in the q^{-1}-symmetric Askey scheme can also be obtained as the $d \to c \to b \to a \to 0$ limit cases, namely,

$$\mathbf{p}_n(x;a,b,c,d|q) = q^{-3\binom{n}{2}}(iabcd)^n p_n(x;-\tfrac{i}{a},-\tfrac{i}{b},-\tfrac{i}{c},-\tfrac{i}{d}|q)$$

$$\mathbf{p}_n(x;a,b,c|q) = \lim_{d\to 0}\mathbf{p}_n(x;a,b,c,d|q),$$

$$\mathbf{Q}_n(x;a,b|q) = \lim_{c\to 0}\mathbf{p}_n(x;a,b,c|q),$$

$$\mathbf{H}_n(x;a|q) = \lim_{b\to 0}\mathbf{Q}_n(x;a,b|q),$$

$$\mathbf{H}_n(x|q) = \lim_{a\to 0}\mathbf{H}_n(x;a|q).$$

- Note that since the q^{-1}-Askey–Wilson polynomials are just renormalized Askey–Wilson polynomials, they are not really new polynomials, whereas the other q^{-1}-subfamilies are distinct.

Orthogonal polynomials in the q^{-1}-symmetric Askey scheme

- q^{-1}-Askey–Wilson polynomials, $\mathbf{a} := \{a, b, c, d\}$

$$\mathbf{p}_n(x;\mathbf{a}|q)$$

$$= q^{-3\binom{n}{2}}(-a^2bcd)^n\left(\tfrac{-1}{ab}, \tfrac{-1}{ac}, \tfrac{-1}{ad}; q\right)_n {}_4\phi_3\left(\begin{matrix} q^{-n}, \frac{q^{n-1}}{abcd}, \frac{z}{a}, -\frac{1}{az} \\ \frac{-1}{ab}, \frac{-1}{ac}, \frac{-1}{ad} \end{matrix}; q, q\right)$$

$$= q^{-3\binom{n}{2}}(-abcdz)^n\left(\tfrac{-1}{ab}, \tfrac{-1}{cz}, \tfrac{-1}{dz}; q\right)_n {}_4\phi_3\left(\begin{matrix} q^{-n}, -q^{1-n}cd, \frac{z}{a}, \frac{z}{b} \\ -\frac{1}{ab}, -q^{1-n}cz, -q^{1-n}dz \end{matrix}; q, q\right)$$

- continuous dual q^{-1}-Hahn polynomials, $\mathbf{a} := \{a, b, c\}$

$$\mathbf{p}_n(x;\mathbf{a}|q) = q^{-\binom{n}{2}}(-a)^n(\tfrac{z}{a}, -\tfrac{1}{az}; q)_n \, {}_3\phi_2\left(\begin{matrix} q^{-n}, -q^{1-n}ab, -q^{1-n}ac \\ -q^{1-n}az, \frac{q^{1-n}a}{z} \end{matrix}; q, q\right)$$

$$= q^{-2\binom{n}{2}}(-abc)^n(-\tfrac{1}{ab}, -\tfrac{1}{ac}; q)_n \, {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{z}{a}, \frac{-1}{az} \\ \frac{-1}{ab}, \frac{-1}{ac} \end{matrix}; q, -\frac{q^n}{bc}\right)$$

Orthogonal polynomials in the q^{-1}-symmetric Askey scheme

- q^{-1}-Al-Salam–Chihara polynomials

$$\mathbf{Q}_n(x;a,b|q) = q^{-\binom{n}{2}}(-b)^n(-\frac{1}{ab};q)_n\, {}_3\phi_1\left(\begin{matrix} q^{-n}, \frac{z}{a}, \frac{-1}{az} \\ \frac{-1}{ab} \end{matrix};q,\frac{q^n a}{b}\right)$$

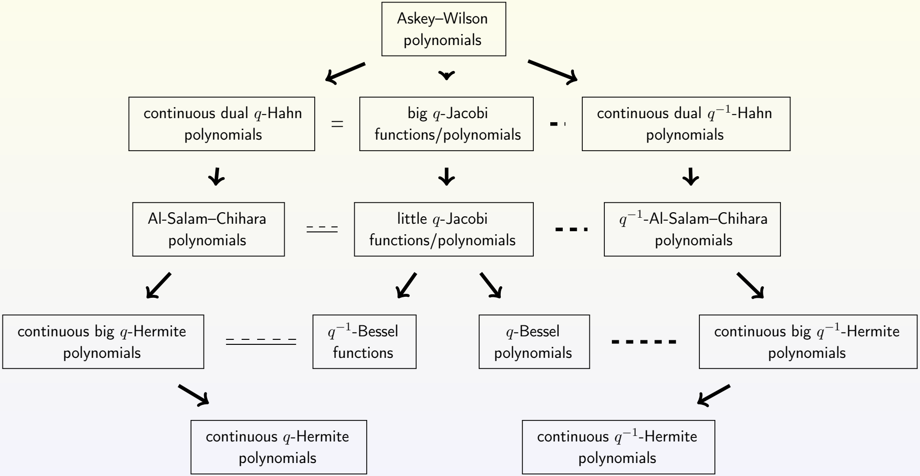
- continuous big q^{-1}-Hermite polynomials

$$\mathbf{H}_n(x;a|q) = (-\tfrac{1}{a})^n\, {}_3\phi_0\left(\begin{matrix} q^{-n}, \frac{z}{a}, \frac{-1}{az} \\ - \end{matrix};q,-q^n a^2\right)$$

- continuous q^{-1}-Hermite polynomials

$$\mathbf{H}_n(x|q) = z^n\, {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix};q,-\frac{q}{z^2}\right)$$

Duality for the q and q^{-1}-symmetric subfamilies



Example of duality: The little q-Jacobi polynomials vs. the q and q^{-1}-Al-Salam–Chihara polynomials

$$p_n(x;a,b;q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1}ab \\ qa \end{matrix}; q, qx\right).$$

$$Q_n\left[\frac{q^{-m}}{a};a,b|q\right] = Q_n\left[q^m a;a,b|q\right] = Q_n\left(\tfrac{1}{2}\left(q^m a + \frac{q^{-m}}{a}\right);a,b|q\right)$$

$$= q^{\binom{m}{2}} \frac{(-ab)^m}{a^n} \frac{(ab;q)_n \left(\frac{qa}{b};q\right)_m}{(ab;q)_m} p_m\left(\frac{q^{-n}}{ab};\frac{a}{b},\frac{ab}{q};q\right),$$

$$Q_n\left[\frac{q^m}{a};a,b|q^{-1}\right] = Q_n[q^{-m}a;a,b|q^{-1}] = Q_n\left(\tfrac{1}{2}\left(\frac{q^m}{a} + q^{-m}a\right);a,b|q^{-1}\right)$$

$$= q^{-\binom{n}{2}-\binom{m}{2}}(-b)^n\left(-\frac{a}{qb}\right)^m \frac{\left(\frac{1}{ab};q\right)_n \left(\frac{qb}{a};q\right)_m}{\left(\frac{1}{ab};q\right)_m} p_m\left(q^n;\frac{b}{a},\frac{1}{qab};q\right).$$

Recently performed survey of known orthogonality relations for q-symmetric, q^{-1}-symmetric and their dual families

- Submitted in 2024 a joint paper with Roberto Costas-Santos and Xiang-Sheng Wang:

  Orthogonality relations for the q and q^{-1}-symmetric and dual polynomials in the q-Askey scheme

Infinite continuous bilateral orthogonality of q^{-1}-AW

- The first orthogonality relation found for this family is that which comes from the weight function which Askey found for the continuous q^{-1}-Hermite polynomials in 1989 *Continuous q-Hermite polynomials when $q>1$.* According to Ismail & Masson (1994), Askey proved the following orthogonality relation such that $m,n\le N$, $|q^3abcd|<|q|^{2N}$:

$$\int_{-\infty}^{\infty}\mathbf{p}_m[q^x;\mathbf{a}|q]\mathbf{p}_n[q^x;\mathbf{a}|q]\frac{(-q^{1+x}\mathbf{a},q^{1-x}\mathbf{a};q)_\infty}{(-q^{2x+1},-q^{1-2x};q)_\infty}\,\mathrm{d}x$$

$$=\frac{(q,-qab,-qac,-qad,-qbc,-qbd,-qcd;q)_\infty}{(qabcd;q)_\infty}$$

$$\times q^{-6\binom{n}{2}}(-a^2b^2c^2d^2)^n\frac{(q,-\frac{1}{ab},-\frac{1}{ac},-\frac{1}{ad},-\frac{1}{bc},-\frac{1}{bd},-\frac{1}{cd};q)_n(\frac{1}{qabcd};q)_{2n}}{(\frac{1}{qabcd};q)_n(\frac{1}{abcd};q)_{2n}}\delta_{m,n}$$

This orthogonality relation is explicitly given in Ismail–Zhang–Zhou (2022). The total mass of this orthogonality relation corresponding to the $m=n=0$ case is the famous Askey q-beta integral (1989).

Infinite discrete bilateral orthogonality of q^{-1}-AW

Theorem (Ismail-Zhang–Zhou (2022))

Let $m, n, N \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a, b, c, d \in \mathbb{C}^*$, $n, m \le N$ *such that* $|abcd| < |q|^{2N-1}$. *Then the* q^{-1}*-Askey–Wilson polynomials satisfy the following infinite discrete bilateral orthogonality relation:*

$$\sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathbf{p}_m[q^k\alpha; \mathbf{a}|q] \mathbf{p}_n[q^k\alpha; \mathbf{a}|q] \frac{(\frac{\alpha}{\mathbf{a}}; q)_k}{(-q\alpha\mathbf{a}; q)_k} (qabcd)^k$$

$$= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}, -qab, -qac, -qad, -qbc, -qbd, -qcd; q)_\infty}{(-q\alpha\mathbf{a}, \frac{q\mathbf{a}}{\alpha}, qabcd; q)_\infty}$$

$$\times q^{-6\binom{n}{2}} (a^2b^2c^2d^2)^n \frac{(q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{ad}, -\frac{1}{bc}, -\frac{1}{bd}, -\frac{1}{cd}; q)_n (\frac{1}{qabcd}; q)_{2n}}{(\frac{1}{qabcd}; q)_n (\frac{1}{abcd}; q)_{2n}} \delta_{m,n}$$

These polynomials are orthogonal over a finite family.

Total Mass is equivalent to Bailey's ${}_6\psi_6$ summation

- If $|abcd| < |q|^{-1}$, then one has

$$\sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\frac{(\frac{\alpha}{\mathbf{a}};q)_k}{(-q\alpha\mathbf{a};q)_k}(qabcd)^k$$

$$=\frac{(q,-\alpha^2,-\frac{q}{\alpha^2},-qab,-qac,-qad,-qbc,-qbd,-qcd;q)_\infty}{(-q\alpha\mathbf{a},\frac{q\mathbf{a}}{\alpha},qabcd;q)_\infty},$$

which is equivalent to Bailey's ${}_6\psi_6$ summation

$${}_6\psi_6\left(\begin{matrix}\pm q\sqrt{\alpha},b,c,d,e\\ \pm\sqrt{\alpha},\frac{q\alpha}{b},\frac{q\alpha}{c},\frac{q\alpha}{d},\frac{q\alpha}{e}\end{matrix};q,\frac{q\alpha^2}{bcde}\right)$$

$$=\frac{(q,q\alpha,\frac{q}{\alpha},\frac{q\alpha}{bc},\frac{q\alpha}{bd},\frac{q\alpha}{be},\frac{q\alpha}{cd},\frac{q\alpha}{ce},\frac{q\alpha}{de};q)_\infty}{(\frac{q}{b},\frac{q}{c},\frac{q}{d},\frac{q}{e},\frac{q\alpha}{b},\frac{q\alpha}{c},\frac{q\alpha}{d},\frac{q\alpha}{e},\frac{q\alpha^2}{bcde};q)_\infty},$$

Infinite discrete bilateral orthogonality relations for the infinite families (cdqiH and qiASC)

$$\sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\mathbf{p}_m[q^k\alpha;\mathbf{a}|q]\mathbf{p}_n[q^k\alpha;\mathbf{a}|q]\frac{(\frac{\alpha}{\mathbf{a}};q)_k}{(-q\alpha\mathbf{a};q)_k}q^{\binom{k}{2}}(-q\alpha abc)^k$$

$$=\frac{(q,-\alpha^2,-\frac{q}{\alpha^2},-qab,-qac,-qbc;q)_\infty}{(-q\alpha\mathbf{a},\frac{q\mathbf{a}}{\alpha};q)_\infty}$$

$$\times q^{-4\binom{n}{2}}\left(\frac{a^2b^2c^2}{q}\right)^n\left(q,-\frac{1}{ab},-\frac{1}{ac},-\frac{1}{bc};q\right)_n\delta_{m,n}.$$

$$\sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\mathbf{Q}_m[q^k\alpha;\mathbf{a}|q]\mathbf{Q}_n[q^k\alpha;\mathbf{a}|q]\frac{(\frac{\alpha}{\mathbf{a}};q)_k}{(-q\alpha\mathbf{a};q)_k}q^{2\binom{k}{2}}(q\alpha^2ab)^k$$

$$=\frac{(q,-\alpha^2,-\frac{q}{\alpha^2},-qab;q)_\infty}{(-q\alpha\mathbf{a},\frac{q\mathbf{a}}{\alpha};q)_\infty}q^{-2\binom{n}{2}}\left(\frac{ab}{q}\right)^n\left(q,-\frac{1}{ab};q\right)_n\delta_{m,n}.$$

Infinite discrete bilateral orthogonality relations for the infinite families (cbqiH and cqiH)

$$\sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\mathsf{H}_m[q^k\alpha;a|q]\mathsf{H}_n[q^k\alpha;a|q]\frac{(\frac{\alpha}{a};q)_k}{(-q\alpha a;q)_k}q^{3\binom{k}{2}}(-q\alpha^3a)^k$$

$$=\frac{(q,-\alpha^2,-\frac{q}{\alpha^2};q)_\infty}{(-q\alpha a,\frac{qa}{\alpha};q)_\infty}\frac{q^{-\binom{n}{2}}(q;q)_n}{q^n}\delta_{m,n}.$$

$$\sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\mathsf{H}_m[q^k\alpha|q]\mathsf{H}_n[q^k\alpha|q]q^{4\binom{k}{2}}(q\alpha^4)^k$$

$$=(q,-\alpha^2,-\frac{q}{\alpha^2};q)_\infty\frac{q^{-\binom{n}{2}}(q;q)_n}{q^n}\delta_{m,n}.$$

Ismail's survey for continuous q^{-1}-Hermite polynomials

In Ismail (2009), Theorem 21.6.4 for $w_2(x)$ which was studied by Atakishiyev–Frank–Wolf (1994) *A simple difference realization of the Heisenberg q-algebra*, the orthogonality measure for continuous q^{-1}-Hermite polynomials, namely

$$w_2(x) = \exp\left(-\frac{2}{\log q^{-1}}\left[\log(x+\sqrt{x^2+1})\right]^2\right).$$

If you choose $x = \frac{1}{2}(z - z^{-1})$, then the above definition reduces to

$$w_2(x) = \exp\left(-\frac{2\,(\log z)^2}{\log q^{-1}}\right),$$

Ismail's survey for continuous q^{-1}-Hermite polynomials

In terms of the polynomials $\mathbf{H}_n[z|q]$, the equivalent orthogonality relation is

$$\int_{-\infty}^{\infty}(q^x+q^{-x})\mathbf{H}_m[q^x|q]\mathbf{H}_n[q^x|q]\exp\left(-2x^2\log q^{-1}\right)\,\mathrm{d}x$$

$$=q^{-\binom{n}{2}}\frac{(q;q)_n}{q^{n+\frac{1}{8}}}\sqrt{\frac{2\pi}{\log q^{-1}}}\delta_{m,n},$$

which has a positive definite measure of orthogonality for $q\in(0,1)$. Define

$$\omega_2(x):=\exp\left(-2x^2\log q^{-1}\right)=q^{2x^2},$$

which follows using the laws of logarithms, or

$$\int_{-\infty}^{\infty}(1+q^{2x})\mathbf{H}_m[q^x|q]\mathbf{H}_n[q^x|q]q^{2x^2-x}\,\mathrm{d}x=q^{-\binom{n}{2}}\frac{(q;q)_n}{q^{n+\frac{1}{8}}}\sqrt{\frac{2\pi}{\log q^{-1}}}\delta_{m,n}.$$

Ismail's survey for continuous q^{-1}-Hermite polynomials

Cursory observation clearly identifies the corresponding infinite discrete bilateral orthogonality relation given by

$$\sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\mathbf{H}_m[q^k\alpha|q]\mathbf{H}_n[q^k\alpha|q]q^{2k^2-k}\alpha^4$$

$$=(q,-\alpha^2,-\frac{q}{\alpha^2};q)_\infty\frac{q^{-\binom{n}{2}}(q;q)_n}{q^n}\delta_{m,n},$$

which is clearly the infinite discrete bilateral analogue for $\alpha=1$ and is corresponds to a positive definite orthogonality measure for $q\in(0,1)$.

The Ismail–Rahman discrete/continuous bilateral correspondence

- In Ismail & Rahman (1995) *Some basic bilateral sums and integrals*, Pacific Journal of Mathematics Vol. **170**, No. 2, they considered given a function defined via an infinite bilateral sum, namely

$$g(\alpha;q) := \sum_{k=-\infty}^{\infty} f(k;\alpha;q),$$

the corresponding continuous integral given by

$$G := \int_{-\infty}^{\infty} f(x;q^{2x};q)\,\omega(x;q)\,\mathrm{d}x.$$

where $\omega(x;q)$ is a bounded continuous unit-periodic function on $x \in \mathbb{R}$, namely $\omega(x \pm 1;q) = \omega(x;q)$.

The Ismail–Rahman discrete/continuous bilateral correspondence

- They applied this method to several important bilateral sums, e.g., ${}_6\psi_6$ sum and, for instance, obtained the Ismail–Masson q-beta integral

$$\int_{-\infty}^{\infty} \frac{(q^x+q^{-x})(-iq^{-x}\mathbf{a}, iq^x\mathbf{a};q)_\infty}{(q^{-x}f, q^{-x}g, -\frac{q^{1-x}}{f}, -\frac{q^{1-x}}{g}, -q^xf, -q^xg, \frac{q^{x+1}}{f}, \frac{q^{x+1}}{g};q)_\infty}\,\mathrm{d}x$$

$$= \frac{2\pi i(\frac{ab}{q}, \frac{ac}{q}, \frac{ad}{q}, \frac{bc}{q}, \frac{bd}{q}, \frac{cd}{q};q)_\infty}{f\log q^{-1}(q, \frac{abcd}{q^3}, \frac{g}{f}, \frac{qf}{g}, -fg, -\frac{q}{fg};q)_\infty}.$$

Continuous integral correspondence for continuous q^{-1}-Hermite polynomials

Let $m, n \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha \in \mathbb{C}^*$ and define

$$\Psi_{m,n}(\alpha|q) := \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2)\mathbf{H}_n[q^k\alpha|q]\mathbf{H}_m[q^k\alpha|q]q^{2k^2-k}\alpha^{4k}$$

$$= (q, -\alpha^2, -\frac{q}{\alpha^2}; q)_\infty \frac{q^{-\binom{n}{2}}(q;q)_n}{q^n}\delta_{m,n}.$$

Continuous integral correspondence for continuous q^{-1}-Hermite polynomials

Then

$$\mathsf{K}_{m,n}(\alpha|q) := \int_{-\infty}^{\infty}(1+q^{2x}\alpha^2)\mathbf{H}_n[q^x\alpha|q]\mathbf{H}_m[q^x\alpha|q]q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x$$

$$= \int_0^1 \Psi_{m,n}(q^x\alpha|q)q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x$$

$$= (q;q)_\infty \frac{q^{-\binom{n}{2}}(q;q)_n\delta_{m,n}}{q^n}\mathsf{J}(\alpha|q)$$

$$:= (q;q)_\infty \frac{q^{-\binom{n}{2}}(q;q)_n\,\delta_{m,n}}{q^n}\int_0^1(-q^{2x}\alpha^2,-\frac{q^{1-2x}}{\alpha^2};q)_\infty q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x,$$

Necessary integral

- In order to derive the correspondence, it was clear that one needed to compute the following integral.

Lemma (integral of a unit periodic function)

Let $0 < q < 1$, $\Re\alpha > 0$. *Then*

$$\mathsf{J}(\alpha|q) = \int_0^1 (-q^{2x}\alpha^2, -\frac{q^{1-2x}}{\alpha^2}; q)_\infty q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x$$

$$= \frac{1}{(q;q)_\infty}\int_{-\infty}^{\infty}(1+q^{2x}\alpha^2)\,q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x = \frac{\sqrt{2\pi}\,\alpha\exp\left(\frac{2(\log\alpha)^2}{\log q^{-1}}\right)}{q^{\frac{1}{8}}\sqrt{\log q^{-1}}(q;q)_\infty}.$$

Necessary integral

Proof.

The correspondence is natural when considering the $n = m = 0$ component. Then we can convert the integral into a sum of two Gaussian integrals by making the substitution $q^x \mapsto \mathrm{e}^{-x}$. After completing the square, both integrands are shifted with respect to each other, and

$$J(\alpha|q) = \frac{2\alpha \exp\left(\frac{2(\log \alpha)^2}{\log q^{-1}}\right)}{q^{\frac{1}{8}} \log q^{-1} (q;q)_\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{2x^2}{\log q^{-1}}} \, \mathrm{d}x,$$

which completes the proof. □

Infinite continuous bilateral orthogonality relations

- Define the following continuous function which represents the infinite discrete bilateral orthogonality relation for the q^{-1}-Askey–Wilson polynomials, namely

$$\Psi_{m,n}(\alpha;\mathbf{a}|q)$$

$$:=\sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\mathbf{p}_n[q^k\alpha;\mathbf{a}|q]\mathbf{p}_m[q^k\alpha;\mathbf{a}|q](-q^{k+1}\alpha\mathbf{a},\frac{q^{1-k}}{\alpha}\mathbf{a};q)_\infty q^{2k^2-k}\alpha^{4k}$$

$$=\frac{(q,-\alpha^2,-\frac{q}{\alpha^2},-qab,-qac,-qad,-qbc,-qbd,-qcd;q)_\infty}{(qabcd;q)_\infty}$$

$$\times\frac{q^{-6\binom{n}{2}}(-a^2b^2c^2d^2)^n(q,-\frac{1}{ab},-\frac{1}{ac},-\frac{1}{ad},-\frac{1}{bc},-\frac{1}{bd},-\frac{1}{cd};q)_n(\frac{1}{qabcd};q)_{2n}}{(\frac{1}{qabcd};q)_n(\frac{1}{abcd};q)_{2n}}\delta_{m,n}$$

Continuous integral correspondence for q^{-1}-AW polynomials

Theorem

Let $n, m, N \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a, b, c, d \in \mathbb{C}^*$, $\mathbf{a}$ *be the multiset given by* $\{a, b, c, d\}$, $n, m \le N$ *such that* $|abcd| < |q|^{2N-1}$. *Then*

$$\mathsf{K}_{m,n}(\alpha;\mathbf{a}|q) = \int_{-\infty}^{\infty} (1+q^{2x}\alpha^2)\mathbf{p}_n[q^x\alpha;\mathbf{a}|q]\mathbf{p}_m[q^x\alpha;\mathbf{a}|q] \times\left(-q^{x+1}\alpha\mathbf{a}, \frac{q^{1-x}}{\alpha}\mathbf{a};q\right)_\infty q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x$$

$$= \int_0^1 \Psi_{m,n}(q^x\alpha;\mathbf{a}|q)q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x$$

$$= \frac{\sqrt{2\pi}\,\alpha\exp\left(\frac{2(\log\alpha)^2}{\log q^{-1}}\right)(-qab,-qac,-qad,-qbc,-qbd,-qcd;q)_\infty}{q^{\frac18}\sqrt{\log q^{-1}}(qabcd;q)_\infty}$$

$$\times\frac{q^{-6\binom{n}{2}}(-a^2b^2c^2d^2)^n(q,-\frac{1}{ab},-\frac{1}{ac},-\frac{1}{ad},-\frac{1}{bc},-\frac{1}{bd},-\frac{1}{cd};q)_n(\frac{1}{qabcd};q)_{2n}}{(\frac{1}{qabcd};q)_n(\frac{1}{abcd};q)_{2n}}\delta_{m,n}$$

Continuous integral correspondence for continuous dual q^{-1}-Hahn polynomials

Let $n, m \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a, b, c \in \mathbb{C}^*$, $\mathbf{a}$ be the multiset given by $\{a, b, c\}$ and define

$$\Psi_{m,n}(\alpha; \mathbf{a}|q)$$

$$:= \sum_{k=-\infty}^{\infty} (1 + q^{2k}\alpha^2) \mathsf{p}_n[q^k\alpha; \mathbf{a}|q] \mathsf{p}_m[q^k\alpha; \mathbf{a}|q] (-q^{k+1}\alpha\mathbf{a}, \frac{q^{1-k}}{\alpha}\mathbf{a}; q)_\infty q^{2k^2-k}\alpha^{4k}$$

$$= (q, -\alpha^2, -\frac{q}{\alpha^2}, -qab, -qac, -qbc; q)_\infty$$

$$\times q^{-4\binom{n}{2}} \left(\frac{a^2b^2c^2}{q}\right)^n \left(q, -\frac{1}{ab}, -\frac{1}{ac}, -\frac{1}{bc}; q\right)_n \delta_{m,n},$$

which provides the infinite discrete bilateral orthogonality relation for the continuous dual q^{-1}-Hahn polynomials.

Continuous integral correspondence for continuous dual q^{-1}-Hahn polynomials

Then one has the following continuous integral correspondence

$$\mathsf{K}_{m,n}(\alpha;\mathbf{a}|q)$$

$$:= \int_{-\infty}^{\infty}(1+q^{2x}\alpha^2)\mathsf{p}_n[q^x\alpha;\mathbf{a}|q]\mathsf{p}_m[q^x\alpha;\mathbf{a}|q](-q^{x+1}\alpha\mathbf{a},\frac{q^{1-x}}{\alpha}\mathbf{a};q)_\infty q^{2x^2-x}\alpha^{4x}$$

$$= \int_0^1 \Psi_{m,n}(q^x\alpha;\mathbf{a}|q)q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x$$

$$= \frac{\sqrt{2\pi}\,\alpha\exp\left(\frac{2(\log\alpha)^2}{\log q^{-1}}\right)(-qab,-qac,-qbc;q)_\infty}{q^{\frac{1}{8}}\sqrt{\log q^{-1}}}$$

$$\times q^{-4\binom{n}{2}}\left(\frac{a^2b^2c^2}{q}\right)^n\left(q,-\frac{1}{ab},-\frac{1}{ac},-\frac{1}{bc};q\right)_n\delta_{m,n}.$$

Continuous integral correspondence for q^{-1}-Al-Salam–Chihara polynomials

Let $n, m \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a, b \in \mathbb{C}^*$, $\mathbf{a}$ be the multiset given by $\{a, b\}$ and define

$$\begin{aligned}
&\Psi_{m,n}(\alpha;\mathbf{a}|q)\\
&\quad := \sum_{k=-\infty}^{\infty}(1+q^{2k}\alpha^2)\mathbf{Q}_n[q^k\alpha;\mathbf{a}|q]\mathbf{Q}_m[q^k\alpha;\mathbf{a}|q]\\
&\qquad\qquad\qquad \times\left(-q^{k+1}\alpha\mathbf{a},\frac{q^{1-k}}{\alpha}\mathbf{a};q\right)_\infty q^{2k^2-k}\alpha^{4k}\\
&\quad = \left(q,-\alpha^2,-\frac{q}{\alpha^2},-qab;q\right)_\infty q^{-2\binom{n}{2}}\left(\frac{ab}{q}\right)^n\left(q,-\frac{1}{ab};q\right)_n\delta_{m,n}.
\end{aligned}$$

Continuous integral correspondence for q^{-1}-Al-Salam–Chihara polynomials

Then

$$\begin{aligned}
&\mathsf{K}_{m,n}(\alpha;\mathbf{a}|q) \\
&\quad := \int_{-\infty}^{\infty} (1+q^{2x}\alpha^2)\mathbf{Q}_n[q^x\alpha;\mathbf{a}|q]\mathbf{Q}_m[q^x\alpha;\mathbf{a}|q] \\
&\qquad\qquad \times(-q^{x+1}\alpha\mathbf{a}, \frac{q^{1-x}}{\alpha}\mathbf{a};q)_\infty q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x \\
&\quad = \int_0^1 \Psi_{m,n}(q^x\alpha;\mathbf{a}|q)q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x \\
&\quad = \frac{\sqrt{2\pi}\,\alpha\exp\left(\frac{2(\log\alpha)^2}{\log q^{-1}}\right)(-qab;q)_\infty}{q^{\frac{1}{8}}\sqrt{\log q^{-1}}} q^{-\binom{n}{2}}\left(\frac{ab}{q}\right)^n\left(q,-\frac{1}{ab};q\right)_n\delta_{m,n}.
\end{aligned}$$

Continuous integral correspondence for continuous big q^{-1}-Hermite polynomials

Let $n, m \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, a \in \mathbb{C}^*$ and define

$$\Psi_{m,n}(\alpha; a|q) := \sum_{k=-\infty}^{\infty} (1+q^{2k}\alpha^2)\mathbf{H}_n[q^k\alpha; a|q]\mathbf{H}_m[q^k\alpha; a|q] \times\left(-q^{k+1}\alpha a, \frac{q^{1-k}a}{\alpha}; q\right)_\infty q^{2k^2-k}\alpha^{4k}$$

$$= \frac{(q, -\alpha^2, -\frac{q}{\alpha^2}; q)_\infty}{(-q\alpha a, \frac{qa}{\alpha}; q)_\infty} \frac{q^{-\binom{n}{2}}(q;q)_n}{q^n}\delta_{m,n}.$$

Continuous integral correspondence for continuous big q^{-1}-Hermite polynomials

Then

$$\begin{aligned}
&\mathsf{K}_{m,n}(\alpha;a|q)\\
&\quad := \int_{-\infty}^{\infty}(1+q^{2x}\alpha^2)\mathbf{H}_n[q^x\alpha;a|q]\mathbf{H}_m[q^x\alpha;a|q]\\
&\qquad\qquad \times\left(-q^{x+1}\alpha a, \frac{q^{1-x}a}{\alpha};q\right)_\infty q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x\\
&\quad = \int_0^1 \Psi_{m,n}(q^x\alpha;a|q)q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x\\
&\quad = (q;q)_\infty \frac{q^{-\binom{n}{2}}(q;q)_n\,\delta_{m,n}}{q^n}\int_0^1\left(-q^{2x}\alpha^2,-\frac{q^{1-2x}}{\alpha^2};q\right)_\infty q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x\\
&\quad = \frac{\sqrt{2\pi}\,\alpha\exp\left(\frac{2(\log\alpha)^2}{\log q^{-1}}\right)}{q^{\frac18}\sqrt{\log q^{-1}}}\,\frac{q^{-\binom{n}{2}}(q;q)_n\delta_{m,n}}{q^n}.
\end{aligned}$$

Specialize to Askey–Wilson polynomials

- One can specialize from the q^{-1}-Askey–Wilson polynomials to the Askey–Wilson polynomials using

$$\mathbf{p}_n(x;a,b,c,d|q) = q^{-3\binom{n}{2}}(iabcd)^n p_n(x;-\tfrac{i}{a},-\tfrac{i}{b},-\tfrac{i}{c},-\tfrac{i}{d}|q)$$

- Then one can obtain orthogonality relations for other orthogonal polynomials in the q-Askey scheme. For instance for continuous q-Jacobi and continuous q-ultraspherical polynomials through specialization.
- Although limits are usually problematic.

Orthogonality relation for cts q-Jacobi polynomials

Theorem

Let $n, m \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, \beta \in \mathbb{C}^*$. *Then*

$$\int_{-\infty}^{\infty} (1+q^{2x}) P_n^{(\alpha,\beta)}(i(q^x - q^{-x})|q) P_m^{(\alpha,\beta)}(i(q^x - q^{-x})|q)$$

$$\times w^{(\alpha,\beta)}(x|q) q^{2x^2-x}\,\mathrm{d}x = h_n^{(\alpha,\beta)}(q)\delta_{m,n},$$

$$w^{(\alpha,\beta)}(x|q) := (iq^{\frac14-\frac12\alpha+x}, -iq^{\frac14-\frac12\beta+x}, iq^{\frac34-\frac12\alpha+x}, -iq^{\frac34-\frac12\beta+x},$$

$$iq^{\frac14-\frac12\beta-x}, -iq^{\frac14-\frac12\alpha-x}, iq^{\frac34-\frac12\beta-x}, -iq^{\frac34-\frac12\alpha-x}; q)_\infty,$$

$$h_n^{(\alpha,\beta)}(q) := \frac{\sqrt{2\pi}}{q^{\frac18}\sqrt{\log q^{-1}}} \frac{(q^{-\alpha}, q^{-\beta}, -q^{-\frac12(\alpha+\beta)}, -q^{-\frac12(\alpha+\beta-1)}; q)_\infty}{(q^{-\frac12(\alpha+\beta)}, q^{-\frac12(\alpha+\beta+1)}; q)_\infty}$$

$$\times q^{(\alpha+\frac12)n} \frac{(q^{\alpha+1}, q^{\beta+1}, -q^{\frac12(\alpha+\beta+3)}; q)_n (q^{\alpha+\beta+1}; q)_{2n}}{(q, q^{\alpha+\beta+1}, -q^{\frac12(\alpha+\beta+1)}; q)_n (q^{\alpha+\beta+2}; q)_{2n}}.$$

Orthogonality relation for cts q-ultraspherical polynomials

By setting $\beta = \alpha$ in cts q-Jacobi polynomial orthogonality relation, one obtains the an orthogonality relation for cts q-ultraspherical polynomials.

Theorem

Let $n, m \in \mathbb{N}_0$, $q \in \mathbb{C}^\dagger$, $\alpha, \beta \in \mathbb{C}^*$. *Then*

$$\int_{-\infty}^{\infty} (1+q^{2x}) C_n(i(q^x - q^{-x}), \beta|q) C_m(i(q^x - q^{-x}); \beta|q) \times(-\frac{q^{1+2x}}{\beta}, -\frac{q^{1-2x}}{\beta}; q)_\infty q^{2x^2-x}\,\mathrm{d}x = h_n(\beta|q)\delta_{m,n},$$

where

$$h_n(\beta|q) := \frac{\sqrt{2\pi}}{q^{\frac{1}{8}}\sqrt{\log q^{-1}}} \frac{(\frac{q}{\beta^2}; q)_\infty}{(\frac{1}{\beta}, \frac{q}{\beta}; q)_\infty} \frac{(-q\beta, \beta^2; q)_n (\beta^2; q)_{2n}}{(q, -\beta; q)_n (q\beta^2; q)_{2n}}.$$

Total mass of q^{-1}-AW OR provides q-beta integral

Let $q \in \mathbb{C}^\dagger$, $\alpha, a, b, c, d \in \mathbb{C}^*$, $\mathbf{a}$ be the multiset given by $\{a, b, c, d\}$. Then if one examines $\mathsf{K}_{0,0}(\alpha; \mathbf{a}|q)$ in the above continuous integral correspondence for q^{-1}-Askey–Wilson polynomials, one has the following q-beta integral,

$$\int_{-\infty}^{\infty} (1 + q^{2x}\alpha^2)(-q^{x+1}\alpha\mathbf{a}, \frac{q^{1-x}}{\alpha}\mathbf{a}; q)_\infty q^{2x^2-x}\alpha^{4x}\,\mathrm{d}x$$

$$= \frac{\sqrt{2\pi}\,\alpha \exp\left(\frac{2(\log\alpha)^2}{\log q^{-1}}\right)(-qab, -qac, -qad, -qbc, -qbd, -qcd; q)_\infty}{q^{\frac{1}{8}}\sqrt{\log q^{-1}}(qabcd; q)_\infty},$$

where $|abcd| < |q|^{-1}$ is required for convergence of the integral.

$q \to 1^-$ limit gives new beta integral

The $q \to 1^-$ limit gives the following new symmetric beta integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\Gamma(2x, -2x, 1+a+x, 1+a-x, 1+b+x, 1+b-x, 1+c+x, 1+c-x, 1+d+x, 1+d-x)}$$
$$= -\frac{1}{2\pi^2} \frac{\Gamma(a+b+c+d+1)}{\Gamma(a+b+1, a+c+1, a+d+1, b+c+1, b+d+1, c+d+1)},$$

where $\Re(a+b+c+d) > -1$ and we are using the convention that a comma delineated list of argument to the gamma function represents multiplication by separate gamma functions with their corresponding arguments.

Ongoing related work

- Volkmer and Cohl are working on a preprint where we consider integrals of Ramanujan-type in terms of bilateral hypergeometric series, limiting cases and q-analogues.
- Costas-Santos and Cohl are working on finding limiting orthogonality relations in the Askey scheme.
- Define the Wilson polynomials

$$W_n(x^2;a,b,c,d) := (a+b)_n(a+c)_n(a+d)_n \times {}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a\pm ix \\ a+b, a+c, a+d\end{matrix};1\right)$$

- Define the continuous dual Hahn polynomials

$$S_n(x^2;a,b,c) := (a+b)_n(a+c)_n\, {}_3F_2\left(\begin{matrix}-n, a\pm ix \\ a+b, a+c\end{matrix};1\right).$$

Orthogonality relation for Wilson polynomials

Let $\mathbf{a} := \{a, b, c, d\}$,

$$\int_{-\infty}^{\infty} W_n(-x^2; \mathbf{a}) W_m(-x^2; \mathbf{a}) \mathrm{w}(x; \mathbf{a}) \, \mathrm{d}x$$

$$= \frac{(-1)^{n+1} n!}{2\pi^2}$$

$$\times \frac{\Gamma(-a-b-c-d+1-2n)\Gamma(-a-b-c-d+2-n)/\Gamma(2-a-b-c-d-2n)\delta_{m,n}}{\Gamma(1-a-b-n)\Gamma(1-a-c-n)\Gamma(1-a-d-n)\Gamma(1-b-c-n)\Gamma(1-b-d-n)\Gamma(1-c-d-n)}$$

where

$$\mathrm{w}(x; \mathbf{a}) := \frac{1}{\Gamma(\pm 2x)\Gamma(1-a\pm x)\Gamma(1-b\pm x)\Gamma(1-c\pm x)\Gamma(1-d\pm x)}.$$

Orthogonality relation for continuous dual Hahn polynomials

Let $\mathbf{a} := \{a, b, c\}$,

$$\int_{-\infty}^{\infty} S_n(-x^2;\mathbf{a}) S_m(-x^2;\mathbf{a}) \mathsf{w}(x;\mathbf{a})\,\mathrm{d}x$$

$$= \frac{(-1)^{n+1} n!}{2\pi^2} \frac{\delta_{m,n}}{\Gamma(1-a-b-n)\Gamma(1-a-c-n)\Gamma(1-b-c-n)}$$

where

$$\mathsf{w}(x;\mathbf{a}) := \frac{1}{\Gamma(\pm 2x)\Gamma(1-a\pm x)\Gamma(1-b\pm x)\Gamma(1-c\pm x)}.$$

Future work

- Using the new beta integral to investigate orthogonality relations for Jacobi and Hermite polynomials by examining limits of orthogonality relations
- Investigate generalizations of other known orthogonality relations for the continuous q^{-1}-Hermite polynomials such as those which correspond to the Ismail–Masson q-beta integral

$$\int_{-\infty}^{\infty} \frac{(q^x+q^{-x})H_m[iq^x|q^{-1}]H_n[iq^x|q^{-1}]}{(q^x\beta, q^x\bar{\beta}, -\frac{q^{x+1}}{\beta}, -\frac{q^{x+1}}{\bar{\beta}}, -q^{-x}\beta, -q^{-x}\bar{\beta}, \frac{q^{1-x}}{\beta}, \frac{q^{1-x}}{\bar{\beta}}; q)_\infty}\,\mathrm{d}x$$
$$= \frac{q^{-\binom{n}{2}}2\pi i(q;q)_n}{\beta(-q)^n \log q^{-1}(q, -\beta\bar{\beta}, -q/(\beta\bar{\beta}), \bar{\beta}/\beta, q\beta/\bar{\beta}; q)_\infty}\delta_{m,n}.$$

- Others?