An Overpartition Companion of Andrews and Keith's 2-colored q-series Identity

Hunter Waldron
Michigan Technological University

Partition Notation

Throughout this talk, we use the convention that a **partition**

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$$

is a weakly decreasing sequence of integers

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

with $\lambda_i = 0$ if $i > k$. The $\lambda_i \neq 0$ are the **parts** of λ, the **length** is the number of parts, and the **size** is $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \cdots$.

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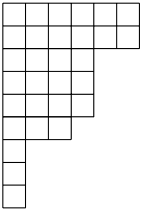
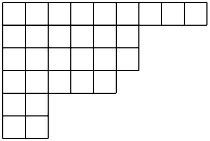


The Young diagram of (6,6,4,4,4,3,1,1,1).



The Young diagram of the **conjugate** (9,6,6,5,2,2).

Schmidt's Theorem

Theorem (Schmidt, 1999)

We have the equality

$$\sum_{\lambda \in \mathcal{D}} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} = \frac{1}{(q;q)_\infty}.$$

Above, $\mathcal{P}$ and $\mathcal{D}$ are the sets of all partitions and partitions with distinct parts, respectively.

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Example: The Schmidt type partitions that contribute to q^5 are

$$5, \quad 5+4, \quad 5+3, \quad 5+2, \quad 5+1,$$

$$4+3+1, \quad 4+2+1$$

Uncu's Similar Result

Theorem (Uncu, 2018)

We have the equality

$$\sum_{\lambda \in \mathcal{P}} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{1}{(q;q)_\infty^2}.$$

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The product on the right can be interpreted as the generating function for 2-colored partitions.

Example: The Schmidt type partitions that contribute to q^3 are

$$3,\ \ 3+1,\ \ 3+2,\ \ 3+3,\ \ 2+2+1,$$

$$2+1+1,\ \ 2+2+1+1,\ \ 2+1+1+1,\ \ 2+1+1+1,\ \ 1+1+1+1+1$$

t-Colored Partitions

For fixed $t \geq 1$, A t-colored partition allows each appearance of a part size to appear in t distinct ways, which are usually marked with integer subscripts.

Example: 2-colored partitions of 3 are

$$3_2, \;\; 3_1, \;\; 2_2 + 1_2, \;\; 2_2 + 1_1, \;\; 2_1 + 1_1, \;\; 2_1 + 1_2,$$

$$1_2 + 1_2 + 1_2, \;\; 1_2 + 1_2 + 1_1, \;\; 1_2 + 1_1 + 1_1, \;\; 1_1 + 1_1 + 1_1$$

Elementary Proof of Schmidt and Uncu's Theorems

Every partition $\lambda \in \mathcal{P}$ is uniquely determined by the columns in λ's Young diagram. The generating function for partitions with only the column height of n present is $(1 - q^{\lceil n/2 \rceil})^{-1}$, counted by the Schmidt weight $\lambda_1 + \lambda_3 + \lambda_5 + \cdots$.

Elementary Proof of Schmidt and Uncu's Theorems

Every partition $\lambda \in \mathcal{P}$ is uniquely determined by the columns in λ's Young diagram. The generating function for partitions with only the column height of n present is $(1 - q^{\lceil n/2 \rceil})^{-1}$, counted by the Schmidt weight $\lambda_1 + \lambda_3 + \lambda_5 + \cdots$.

Now take the product over all n to get

$$\frac{1}{1-q}\frac{1}{1-q}\frac{1}{1-q^2}\frac{1}{1-q^2}\frac{1}{1-q^3}\frac{1}{1-q^3}\cdots = \frac{1}{(q;q)_\infty^2}$$

which gives Uncu's Theorem

$$\sum_{\lambda \in \mathcal{P}} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{1}{(q;q)_\infty^2}.$$

Now for Schmidt's Theorem

Schmidt's Theorem follows from Uncu's Theorem once we establish the relationship

$$\sum_{\lambda \in \mathcal{P}} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{1}{(q;q)_\infty} \sum_{\lambda \in \mathcal{D}} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots}.$$

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This is easily seen from the fact that any partition in $\lambda \in \mathcal{P}$ is uniquely determined by a pair (ν, μ) where $\nu \in \mathcal{D}$ and μ has all parts repeating in multiples of 2. To construct λ, combine the parts of ν and μ.

This Schmidt weight of λ is equal to that of ν plus μ in this construction.

Moving on to q-Series

Recall that $(z;q)_0 = 1$, and for $n \geq 1$ or $n = \infty$,

$$(z;q)_n = (1-z)(1-zq)\cdots(1-zq^{n-1}),$$

and that the q-multinomial coefficient is given by

$$\begin{bmatrix} n \\ k_1, \ldots, k_t \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{k_1}\cdots(q;q)_{k_t}}$$

where $k_1 + \cdots + k_t = n$.

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where $k_1 + \cdots + k_t = n$.

When $t = 2$, this becomes the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

A q-Series identity of Andrews and Keith

Theorem (Andrews-Keith, 2023)

We have the equality

$$\sum_{n\geq 0}\sum_{\substack{j+k\geq n\\ j,k\leq n}} \frac{(-1)^{j+k+n} t_1^j t_2^k q^{\binom{n}{2}+\binom{j+1}{2}+\binom{k+1}{2}} \begin{bmatrix} n \\ n-j, n-k, j+k-n \end{bmatrix}_q}{(t_1 q; q)_n (t_2 q; q)_n (q; q)_n}$$

$$= \frac{1}{(t_1 q; q)_\infty (t_2 q; q)_\infty}.$$

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$$= \frac{1}{(t_1 q; q)_\infty (t_2 q; q)_\infty}.$$

Observe that the product side is the generating function

$$\sum_{\lambda\in\mathcal{P}_2} t_1^{\ell_1(\lambda)} t_2^{\ell_2(\lambda)} q^{|\lambda|} = \frac{1}{(t_1 q; q)_\infty (t_2 q; q)_\infty}$$

where $\ell_1(\lambda)$ and $\ell_2(\lambda)$ count the number of parts by color.

The Underlying Combinatorics

Theorem (Andrews-Keith, 2023)

Fix $m \geq 2$ and $S = \{s_1, \ldots, s_i\} \subseteq \{1, 2, \ldots, m\}$ with $1 \in S$, and S ordered $s_1 < s_2 < \cdots < s_i$. For all $n \geq 1$, The partitions λ with all part sizes appearing fewer than m times such that

$$n = \sum_{\substack{k \equiv S \\ \pmod{m}}} \lambda_k = \lambda_{s_1} + \cdots + \lambda_{s_i} + \lambda_{s_1+m} + \cdots + \lambda_{s_i+m} + \cdots$$

$$\rho_j = \sum_{k \geq 0} (\lambda_{mk+j} - \lambda_{mk+j+1}) = \lambda_j - \lambda_{j+1} + \lambda_{j+m} - \lambda_{j+m+1} + \cdots$$

for $1 \leq j < m$ are equinumerous with the partitions of size n in $\mathcal{P}$ where any parts congruent to k modulo i appears in the $s_{k+1} - s_k$ colors $\{s_k, \ldots, s_{k+1} - 1\}$ where we take $s_{i+1} = m$, and parts of color j appear ρ_j times.

Stockhofe's Partition Bijection

Before proceeding we need to introduce the following.

Fix $m \geq 2$. A partition λ is *m*-flat if $\lambda_i - \lambda_{i+1} < m$ for all $i \geq 1$, and is *m*-distinct if any part size appears fewer than m times.

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A partition is m-regular if no part is divisble by m.

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A partition is m-regular if no part is divisble by m. The classic Glaisher's bijection maps m-distinct to m-regular partitions. Stockhofe's (restricted) bijection maps m-flat to m-regular partitions.

The Definition

Stockhofe's bijection ϕ is defined on an m-flat partition λ through the following algorithm:

1. Let $\lambda = \hat{\lambda}$. Work from $\hat{\lambda}$'s last part to the first, removing any part that is a multiple of m and leaves $\hat{\lambda}$ m-flat. Form μ from these parts.
2. Now, again work from $\hat{\lambda}$'s last part to the first. For each part of the form $\hat{\lambda}_i = km$, remove the part then subtract m from all larger parts. Add the part $\hat{\lambda}_i + (i-1)m$ to μ.
3. Now $\hat{\lambda}$ is both m-regular and m-flat, and all parts of μ are multiples of m. We can write $\mu = m\nu$, treating partitions as vectors.
4. Finally, $\phi(\lambda) = \hat{\lambda} + m\nu'$.

The Andrews-Keith Theorem Example

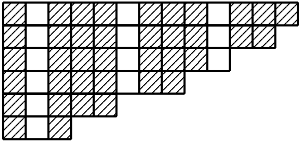
Let $m = 4$ and consider the Schmidt weight

$$\sum_{i \not\equiv 2 \pmod 4} \lambda_i = \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_7 + \cdots .$$

Take for example the 4-distinct partition

$$\lambda = (6, 6, 6, 5, 5, 4, 4, 4, 3, 3, 2, 2, 1).$$

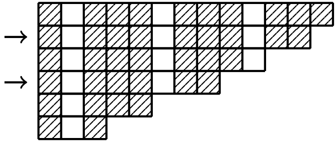
Conjugating λ gives a 4-flat partition $\lambda' = (13, 12, 10, 8, 5, 3)$.



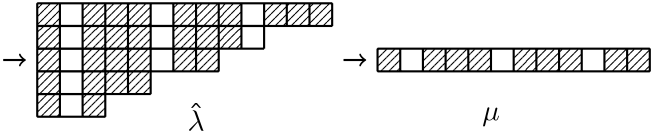
We proceed now to compute $\phi(\lambda')$.

Calculating $\phi(\lambda')$

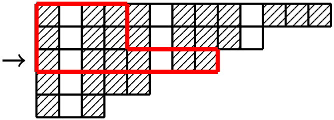
Set $\hat{\lambda} = \lambda'$.



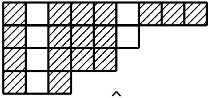
There are two parts in $\hat{\lambda}$ that are multiples of m. Ignore $\hat{\lambda}_4$ since removing this part leaves $\hat{\lambda}$ no longer 4-flat. We form μ by first removing $\hat{\lambda}_2$.



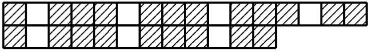
Calculating $\phi(\lambda')$



To deal with the remaining part that is a multiple of 4, we remove the part and take 4 from every part above. The amount removed gets added to μ as a single part.



$\hat{\lambda}$



μ

Calculating $\phi(\lambda')$

We are almost finished. Observe all parts of $\mu = (16, 12)$ are multiples of 4, so we can write $\mu = 4\nu$ where $\nu = (4, 3)$. Then

$$\phi(\lambda') = \hat{\lambda} + 4\nu' = (9, 6, 5, 3) + 4(2, 2, 2, 1) = (17, 14, 13, 7)$$

where addition is done coordiniatewise.

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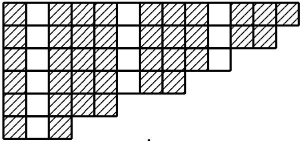
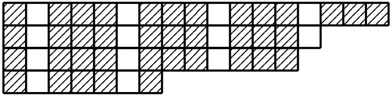


λ'

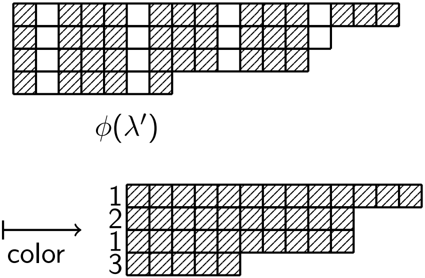


$\phi(\lambda')$

Since all operations were done on rows, in multiples of *m*, the Schmidt weight is preserved.

Assigning Colors

We now delete any cell not shaded in, and assign the ith part the color c if $\phi(\lambda')_i \equiv c \pmod{m}$.



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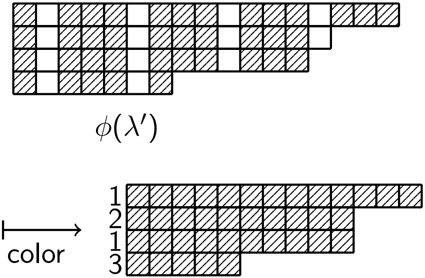

We finish with the partition appearing in 3 colors, $(13_1, 10_2, 10_1, 5_3)$.

Finishing The Example

We started by considering 4-distinct partitions λ with the Schmidt weight

$$\sum_{i \not\equiv 2 \pmod 4} \lambda_i = \lambda_1 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_7 + \cdots .$$

The image of this bijection is the set of partitions appearing in 3 colors $\{1, 2, 3\}$ such that parts $\equiv 1 \pmod 3$ appear in the colors $\{1, 2\}$ and parts $\equiv 2 \pmod 3$ appear in the color $\{3\}$.

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Remark. The generating function for these Schmidt type partitions is thus
$$\frac{1}{(q; q^3)_\infty^2 (q^2; q^3)_\infty}.$$

Extending The Bijection

Consider now the same Schmidt weight and $m = 4$, but now suppose λ is 8-distinct.

Form μ by taking 4 copies of any part that appears 4 or more times, and added them to μ as a single part. Let $\hat{\lambda}$ be what remains.

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Form μ by taking 4 copies of any part that appears 4 or more times, and added them to μ as a single part. Let $\hat{\lambda}$ be what remains. Then:

- All parts of μ are multiples of 4 and appear at most once, and $\hat{\lambda}'$ is 4-flat.
- The Schmidt weight of λ is that of μ plus $\hat{\lambda}$.

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Now combine (**not** vector addition) the parts of σ' and $\phi(\hat{\lambda}')$, and do the same process as before to get a colored partition.

What Is The Image Now?

Plainly any 4-distinct partition is also 8-distinct, and for these partitions μ is the empty partition, so the same colored partitions from before are in the image.

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Now, parts $\equiv 0 \pmod 3$ are allowed and get assigned the color 4. The generating function becomes

$$\frac{(-q^3;q^3)_\infty}{(q;q^3)_\infty^2(q^2;q^3)_\infty}.$$

Using These Maps To Get q-series Identities

A special case of the Andrews-Keith partition identity says that the following two families of partitions are equinumerous:

1. The 2-colored partitions of n with m_1 and m_2 being the number of times parts in each respective colors appear.
2. The partitions λ with parts repeating < 3 times, such that

$$n = \lambda_1 + \lambda_4 + \lambda_7 + \cdots$$
$$m_1 = \lambda_1 - \lambda_2 + \lambda_4 - \lambda_5 + \cdots$$
$$m_2 = \lambda_2 - \lambda_3 + \lambda_5 - \lambda_6 + \cdots .$$

A Recurrence for the Schmidt Type Partitions

Define $Q_i = q^{\lfloor i/3 \rfloor} t(i)$ where

$$t(i) = \begin{cases} t_1 & i \equiv 1 \pmod 3 \\ t_2 & i \equiv 2 \pmod 3 \\ 1 & i \equiv 0 \pmod 3 \end{cases}$$

Let $G_0 = 1$, $G_1 = (1 - Q_1)^{-1}$, and for $n \geq 2$ let

$$G_n = \frac{Q_{n-2} G_{n-2}}{1 - Q_n} + \frac{Q_{n-1} G_{n-1}}{1 - Q_n}.$$

Then

$$\sum_{n \geq 0} G_{3n} = \frac{1}{(t_1 q; q)_\infty (t_2 q; q)_\infty}.$$

Jumping Now to Overpartitions

The overpartitions $\overline{\mathcal{P}}$ are partitions with the first appearance of any part optionally overlined.

Example: The overpartitions of 3 are

$$3, \ \ \overline{3}, \ \ 2+1, \ \ \overline{2}+1, \ \ 2+\overline{1}, \ \ \overline{2}+\overline{1}, \ 1+1+1 \ \ \overline{1}+1+1$$

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If $\ell_0(\lambda)$ and $\ell_n(\lambda)$ count the overlined and non-overlined parts of λ,

$$\sum_{\lambda \in \overline{\mathcal{P}}} t_1^{\ell_o(\lambda)} t_2^{\ell_n(\lambda)} q^{|\lambda|} = \frac{(-t_1 q; q)_\infty}{(t_2 q; q)_\infty}.$$

A Companion to the Andrews-Keith Identity

Theorem (2024)

We have the equality

$$\sum_{n\geq 0}\sum_{\substack{j+k\geq n\\ j,k\leq n}} \frac{(-1)^{j+k+n} t_1^j t_2^k q^{\binom{n}{2}+\binom{k+1}{2}+j^2-nj+j} \begin{bmatrix} n \\ n-j, n-k, j+k-n \end{bmatrix}_q}{(t_2 q; q)_n (q;q)_n}$$

$$= \frac{(-t_1 q; q)_\infty}{(t_2 q; q)_\infty}.$$

The Schmidt Type Partitions

The Schmidt type partitions that describe this identity do not come from the Andrews-Keith partition identity, we have to extend their map.

The desired generating function equality turns out to be

$$\sum_{\lambda \in \mathcal{D}_4} t_1^{e(\lambda)} t_2^{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{(-qt_1; q)_\infty}{(qt_2; q)_\infty}$$

where $\mathcal{D}_4$ is the set of partitions with parts appearing < 4 times, and $e(\lambda)$ counts the number of part sizes appearing 2 or 3 times.

An Example

The coefficient of $t_1 t_2^2 q^6$ on both sides of

$$\sum_{\lambda \in \mathcal{D}_4} t_1^{e(\lambda)} t_2^{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots}$$

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is 6. The Schmidt type partitions are

$$5 + 3 + 1 + 1, \quad 4 + 3 + 1 + 1 + 1, \quad 3 + 3 + 3 + 1$$

$$4 + 4 + 2, \quad 3 + 2 + 2 + 2 + 1, \quad 4 + 2 + 2 + 2$$

and the overpartitions are

$$\overline{4} + 1 + 1,\ 4 + \overline{1} + 1,\ \overline{2} + 2 + 2,\ \overline{3} + 2 + 1,\ 3 + \overline{2} + 1,\ 3 + 2 + \overline{1}.$$

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The coefficient of $t_1 t_2^2 q^6$ on both sides of

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Remark. This identity also generalizes Schmidt's theorem.

The Recurrence

The q-series identity was discovered from the following recurrence relation (but proved differently!).

Let $Q_{2k} = q^k$ and $Q_{2k+1} = t_2 q^{k+1}$, and

$$L_n = \frac{Q_n}{1 - Q_n} \left(L_{n-1} + t_1 L_{n-2} + t_1 L_{n-3} \right)$$

where the base cases are $L_0 = 1, L_1 = Q_1/(1 - Q_1)$, and

$$L_2 = \frac{Q_2 t_1}{1 - Q_2} + \frac{Q_1 Q_2}{(1 - Q_1)(1 - Q_2)}.$$

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Here L_n counts the number of relevant Schmidt type partitions with exactly n parts.

A Proof Using q-Series Techniques

Recall we are trying to show

$$\sum_{n\geq 0}\sum_{\substack{j+k\geq n\\ j,k\leq n}} \frac{(-1)^{j+k+n} t_1^j t_2^k q^{\binom{n}{2}+\binom{k+1}{2}+j^2-nj+j} \begin{bmatrix} n \\ n-j, n-k, j+k-n \end{bmatrix}_q}{(t_2 q;q)_n (q;q)_n}$$

$$= \frac{(-t_1 q;q)_\infty}{(t_2 q;q)_\infty}.$$

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$$= \frac{(-t_1q;q)_\infty}{(t_2q;q)_\infty}.$$

Write

$$\frac{(-t_1q;q)_\infty}{(t_2q;q)_\infty} = \sum_{n\geq 0} \frac{t_1^n q^{\binom{n+1}{2}}}{(q;q)_n} \times \sum_{m\geq 0} \frac{t_2^m q^{m^2}}{(q;q)_m (t_2q;q)_m}$$

so for any fixed $J \geq 0$, the coefficient of t_1^J on the product side is

$$\frac{q^{\binom{J+1}{2}}}{(q;q)_J} \sum_{m\geq 0} \frac{t_2^m q^{m^2}}{(q;q)_m (qt_2;q)_m}.$$

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The coefficient of t_1^J on the sum side is

$$(-1)^J q^{J^2+J} \sum_{n \geq J} \sum_{\substack{J+k \geq n \\ k \leq n}} \frac{(-1)^{k+n} t_2^k q^{\binom{n}{2}+\binom{k+1}{2}-nJ} \begin{bmatrix} n \\ n-J, n-k, J+k-n \end{bmatrix}_q}{(t_2 q; q)_n (q;q)_n}$$

$$= \frac{q^{\binom{J+1}{2}}}{(q;q)_J} \sum_{n \geq 0} \frac{t_2^n q^{n^2}}{(t_2 q; q)_{n+J} (q;q)_n} \sum_{k=0}^{J} (-1)^k t_2^k q^{\binom{k}{2}+(n+1)k} \begin{bmatrix} J \\ k \end{bmatrix}_q$$

after many omitted reindexings and calculations.

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$$= \frac{q^{\binom{J+1}{2}}}{(q;q)_J} \sum_{n \geq 0} \frac{t_2^n q^{n^2}}{(t_2 q; q)_{n+J} (q;q)_n} \sum_{k=0}^{J} (-1)^k t_2^k q^{\binom{k}{2}+(n+1)k} \begin{bmatrix} J \\ k \end{bmatrix}_q$$

after many omitted reindexings and calculations. Using Cauchy's identity

$$(z;q)_N = \sum_{k=0}^{N} (-1)^k z^k q^{\binom{k}{2}} \begin{bmatrix} N \\ k \end{bmatrix}_q$$

with $z = t_2 q^{n+1}$ gives the result.

Questions and Conjectures

The generality of the Andrews-Keith partition identity gives strong evidence there are many other related identities.

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The generality of the Andrews-Keith partition identity gives strong evidence there are many other related identities. Their q-series identity seems to be the $k = 2$ case of an infinite family.

Conjecture.

For each $k \geq 2$, there exists polynomials $f(n, i_1, \cdots, i_k, q)$ in q with non-negative coefficients and a constant term 1 such that

$$\sum_{n \geq 0} \sum_{\substack{i_1 + \cdots + i_k \geq n \\ i_1, \ldots, i_k \leq n}} \frac{(-1)^{n+i_1+\cdots+i_k} t_1^{i_1} \cdots t_k^{i_k} q^{\binom{n}{2}+\binom{i_1+1}{2}+\cdots+\binom{i_k+1}{2}} f(n, i_1, \cdots, i_k, q)}{(q;q)_n (t_1 q; q)_n \cdots (t_k q; q)_n}$$

$$= \frac{1}{(t_1 q, t_2 q, \ldots, t_k q; q)_\infty}.$$

A Systematic Method to Solve Recurrences?

The recurrence from the Andrews-Keith q-series generalizes to

$$G_n = \frac{1}{1 - Q_n} \left(Q_{n-1} G_{n-1} + \cdots Q_{n-k} G_{n-k} \right)$$

where now $Q_i = q^{\lceil i/(k+1) \rceil} t(i)$ and $t(i) = t_i$ if $i \not\equiv 0 \pmod{k+1}$, with $t(i) = 1$ otherwise.

The initial conditions are $G_0 = 1$, and for $1 \leq i < k$,

$$G_i = \frac{1}{(1 - Q_1) \cdots (1 - Q_i)}.$$

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Question 1. Is there a general method to solve this recurrence?

Question 2. How can this be established from looking at colored partitions alone? Do we need the Schmidt type perspective?

Some Data Supporting This Conjecture

Let's look at the $k = 3$ case. The Mathematica code here will compute the recurrence for us:

$$\mathtt{T[n\_]} := \mathtt{t}_1^{\mathtt{If[Mod[n,4]==1,\ 1,0]}} * \mathtt{t}_2^{\mathtt{If[Mod[n,4]==2,\ 1,0]}} * \mathtt{t}_3^{\mathtt{If[Mod[n,4]==3,\ 1,0]}}$$

$$\mathtt{Q[n\_]} := \mathtt{q}^{\mathtt{Ceiling[n/4]}} * \mathtt{T[n]}$$

$$\mathtt{G[0]} := 1;$$

$$\mathtt{G[1]} := \frac{1}{1 - \mathtt{Q[1]}};$$

$$\mathtt{G[2]} := \frac{1}{(1 - \mathtt{Q[1]}) * (1 - \mathtt{Q[2]})};$$

$$\mathtt{G[3]} := \frac{1}{(1 - \mathtt{Q[1]}) * (1 - \mathtt{Q[2]}) * (1 - \mathtt{Q[3]})};$$

$$\mathtt{G[n\_]} := \frac{\mathtt{Q[n-1]}}{1 - \mathtt{Q[n]}} * \mathtt{G[n-1]} + \frac{\mathtt{Q[n-2]}}{1 - \mathtt{Q[n]}} * \mathtt{G[n-2]} + \frac{\mathtt{Q[n-3]}}{1 - \mathtt{Q[n]}} * \mathtt{G[n-3]}$$

Some Data Supporting This Conjecture

If we expand out for instance G_4, which corresponds to $n = 1$, we get

$$\frac{q^3 t_1 t_2 t_3 - q^2 t_1 t_2 - q^2 t_1 t_3 - q^2 t_2 t_3 + q t_1 + q t_2 + q t_3}{(1-q)(1-qt_1)(1-qt_2)(1-qt_3)}.$$

The denominator is $(q;q)_1 (qt_1;q)_1 (qt_2;q)_1 (qt_3;q)_1$, and each term on top has the form

$$(-1)^{1+i_1+i_2+i_3} t_1^{i_1} t_2^{i_2} t_3^{i_3} q^{\binom{n}{2}+\binom{i_1+1}{2}+\binom{i_2+1}{2}+\binom{i_3+1}{2}} f(1, i_1, i_2, i_3, q).$$

where $0 \leq i_1, i_2, i_3 \leq 1$ and $i_1 + i_2 + i_3 \geq 1$.

A Closer Look at $f(n, i_1, i_2, i_3, q)$

Here are some particular examples:

$$f(4,3,3,2,q) = (q+1)\left(q^2+1\right)\left(q^2+q+1\right)\left(q^4+2q^3+2q^2+q+1\right)$$

$$f(4,2,2,2,q) = \left(q^2+1\right)\left(q^2+q+1\right)\left(q^5+5q^4+5q^3+5q^2+2q+1\right)$$

$$f(4,0,3,4,q) = (1+q)(1+q^2) = \begin{bmatrix} 4 \\ 4-3, 4-4, 3+4-4 \end{bmatrix}_q$$

Thank you!