Bijective Approaches for Schmidt Type Theorems

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Notation

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The q-Pochhammer symbol is defined:

$$(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k)$$

$$(z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k)$$

Background

Theorem (Frank Schmidt)

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Example

The partitions of 5 are

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), \text{ and } (1,1,1,1,1)$$

The corresponding partitions with distinct parts are

$$(5), (4,3,1), (4,2,1), (5,4), (5,3), (5,2), (5,1)$$

We call sums of the form $\lambda_r + \lambda_{t+r} + \lambda_{2t+r} + \cdots$ *Schmidt weights* of λ.

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Let $\mathcal{D}$ and $\mathcal{P}$ be the sets of partitions with distinct parts and unrestricted parts, respectively. Schmidt's result implies:

$$\sum_{\lambda \in \mathcal{D}} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} = \frac{1}{(q;q)_\infty}$$

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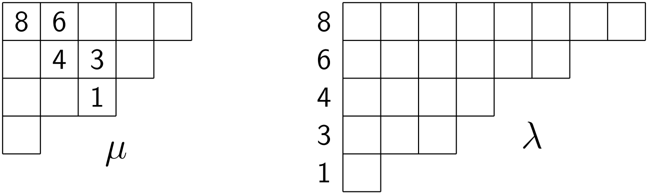
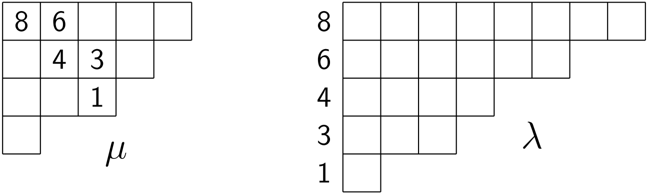
Peter Mork's bijection proves this:



Mork's bijection maps $\mu = (5, 4, 3, 1)$ to $\lambda = (8, 6, 4, 3, 1)$.

The diagonal hooks h_i^{μ} become the odd indexed parts λ_{2i-1}, the hooks to their right become the even indexed parts.

Mork's bijection



$$\lambda_1 + \lambda_3 + \lambda_5 + \cdots = |\mu|$$

$$\lambda_2 + \lambda_4 + \lambda_6 + \cdots = \ ?$$

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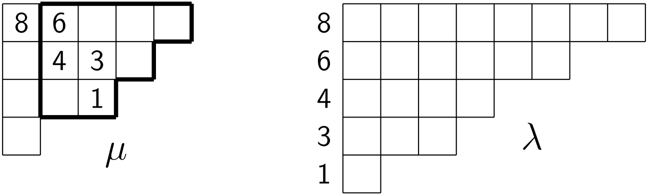

$$\lambda_1 + \lambda_3 + \lambda_5 + \cdots = |\mu|$$

$$\lambda_2 + \lambda_4 + \lambda_6 + \cdots = |\mu| - \ell(\mu)$$

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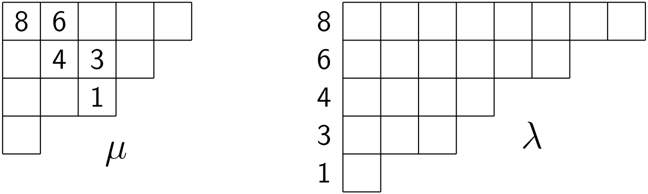

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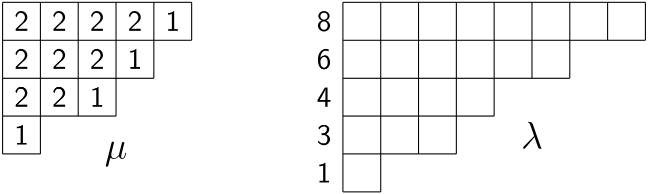

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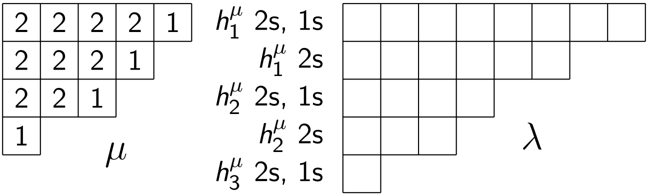
If we write μ as a 2-modular diagram with remainder 1 on each part, this becomes a size preserving bijection from partitions with odd parts to partitions with distinct parts.

This is exactly Bessenrodt's bijection!



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The relationship between these maps

The following diagram commutes:

$$
\begin{tikzcd}
 & \mathcal{D} \arrow[dl, leftrightarrow, "\text{Bessenrodt}"'] \arrow[d, leftrightarrow, "\text{Mork}"] \\
\mathcal{O} \arrow[r, leftrightarrow, "\substack{\text{Delete numbers}\\ \text{from 2-modular diagram}}"'] & \mathcal{P}
\end{tikzcd}
$$

$\mathcal{P}$ - partitions
$\mathcal{O}$ - partitions with odd parts
$\mathcal{D}$ - partitions with distinct parts

Identities implied by Mork's bijection

$$\sum_{\lambda \in \mathcal{D}} z^{|\lambda|} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{1}{(qz; qz^2)_\infty}$$

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Proof. We have by Mork's bijection:

$$\sum_{\lambda \in \mathcal{D}} z^{|\lambda|} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \sum_{\lambda \in \mathcal{P}} z^{2|\lambda| - \ell(\lambda)} q^{|\lambda|}$$

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Replace z with qz and q with q^{-1} to get the second identity.

A closer look at the first identity

$$\frac{1}{(qz;qz^2)_\infty} = 1 + zq + (z^2+z^3)q^2 + (z^3+z^4+z^5)q^3 + (z^4+z^5+2z^6+z^7)q^4 + \cdots$$

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Setting $z=1$ gives the values of the partition function:

$$\frac{1}{(q;q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

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These polynomials have two interpretations:

$$\sum_{\substack{\lambda\in\mathcal{D},\\ \lambda_1+\lambda_3+\lambda_5+\cdots=n}} z^{|\lambda|} = \sum_{\substack{\lambda\in\mathcal{P},\\ |\lambda|=n}} z^{2|\lambda|-\ell(\lambda)}$$

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For example, $(3,2,1),(4,2)$ and $(3,1),(2,2)$, respectively, contribute to the $2z^6$ factored out of q^4 above.

A closer look at the first identity

These polynomials seem to be unimodal. For example, the following is the coefficient of q^{20}:

$$z^{20} + z^{21} + 2z^{22} + 3z^{23} + 5z^{24} + 7z^{25} + 11z^{26} +$$
$$15z^{27} + 22z^{28} + 30z^{29} + 42z^{30} + 54z^{31} + 70z^{32} +$$
$$82z^{33} + 90z^{34} + 84z^{35} + 64z^{36} + 33z^{37} + 10z^{38} + z^{39}$$

Background

Theorem (Ali Uncu)

$$\sum_{\lambda \in \mathcal{P}} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{1}{(q;q)_\infty^2}$$

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Later, George Andrews and Peter Paule independently found the same result, and gave the following interpretation.

Theorem

The number of 2-colored partitions of n is equal to the number of partitions λ such that $\lambda_1 + \lambda_3 + \lambda_5 + \cdots = n$.

Background

Example

There are 10 2-colored partitions of 3.

$$(3^1), (3^2), (2^1, 1^1), (2^2, 1^1), (2^1, 1^2), (2^2, 1^2),$$

$$(1^1, 1^1, 1^1), (1^2, 1^1, 1^1), (1^2, 2^1, 1^1), (1^2, 1^2, 1^2)$$

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$$(1^1, 1^1, 1^1), (1^2, 1^1, 1^1), (1^2, 2^1, 1^1), (1^2, 1^2, 1^2)$$

There are 10 partitions λ such that $\lambda_1 + \lambda_3 + \lambda_5 + \cdots = 3$.

$$({\color{red}3}), ({\color{red}3}, 1), ({\color{red}3}, 2), ({\color{red}3}, 3), ({\color{red}2}, 2, {\color{red}1}), ({\color{red}2}, 2, {\color{red}1}, 1),$$

$$({\color{red}2}, 1, {\color{red}1}), ({\color{red}2}, 1, {\color{red}1}, 1), ({\color{red}1}, 1, {\color{red}1}, 1, {\color{red}1}), ({\color{red}1}, 1, {\color{red}1}, 1, {\color{red}1}, 1)$$

Background

Theorem (Walter Bridges and Ali Uncu)

$$\sum_{\lambda \in \mathcal{P}} z^{\lambda_1} q^{\lambda_1 + \lambda_3 + \lambda_5 + \cdots} = \frac{1}{(qz;q)_\infty^2}$$

$$\sum_{\lambda \in \mathcal{P}} z^{\lambda_1} q^{\lambda_2 + \lambda_4 + \lambda_6 + \cdots} = \frac{1}{(1-z)(qz;q)_\infty^2}$$

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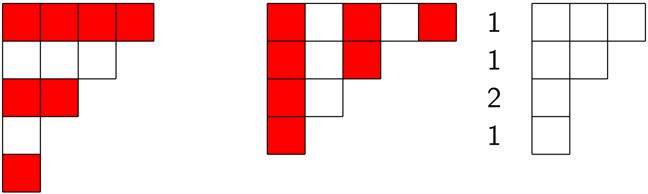
William Keith and George Andrews remarked on a bijection that explains the first of these identities. Conjugate a partition λ, and use the uncounted places $\lambda_2, \lambda_4, \lambda_6, \ldots$ to get the colors:



Extending the Bijection

Theorem

The number of partitions λ such that $\lambda_1 = k$, $\lambda_r = m$, and $\lambda_r + \lambda_{t+r} + \lambda_{2t+r} + \cdots = n$ is equal to the number of pairs (μ, ν) such that μ is a partition with at most $r-1$ parts where $\mu_1 = k - m$, and ν is a t-colored partition of n with m parts.

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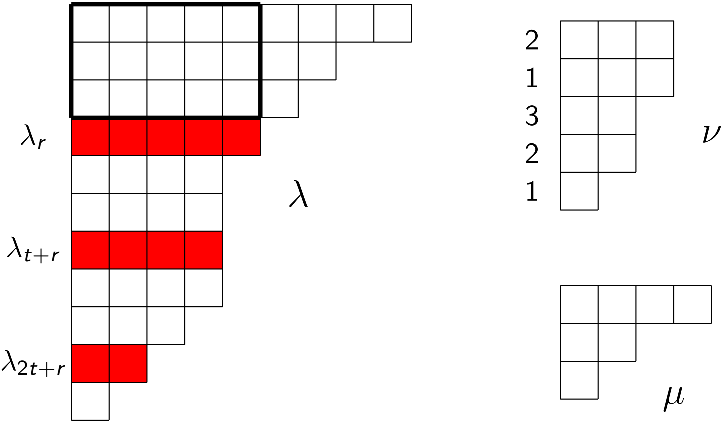
This implies:

$$\sum_{\lambda \in \mathcal{P}} z^{\lambda_1} q^{\lambda_r + \lambda_{t+r} + \lambda_{2t+r} + \cdots} = \sum_{\substack{\lambda \in \mathcal{P}, \\ \ell(\lambda) \leq r-1}} z^{\lambda_1} \times \sum_{\lambda \in \mathcal{P}^t} z^{\ell(\lambda)} q^{|\lambda|}$$

$$= \frac{1}{(1-z)^{r-1}(qz;q)_\infty^t}$$

where $\mathcal{P}^t$ is the set of t-colored partitions.

Extending the Bijection



Example of the bijection where $r = 4$ and $t = 3$:
$\lambda = (9, 7, 6, 5, 4, 4, 4, 4, 3, 2, 1)$ maps to $\mu = (4, 2, 1)$ and
$\nu = (3^2, 3^1, 2^3, 1^2, 1^1)$.

Restricting the bijection to distinct parts

Restricting this bijection to $r = 1$ and partitions with only distinct parts gives the following.

Corollary

The number of partitions λ *with distinct parts such that* $\lambda_1 = k$ *and* $\lambda_1 + \lambda_{t+1} + \lambda_{2t+1} + \cdots = n$ *is equal to the number of* t*-colored partitions* ν *of* n *with* k *parts in which every possible part size from 1 to* ν_1 *appears, and in all* t *colors, except the largest which only appears in the colors* $1, \ldots, s$ *for some* $s \leq t$*.*

A better identity

Using a much different approach, we can do better:

$$\sum_{\lambda \in \mathcal{P}} z^{\lambda_1} s^{|\lambda|} q^{\lambda_r + \lambda_{t+r} + \lambda_{2t+r} + \cdots}$$

$$= \frac{1}{(sz;s)_{r-1}} \prod_{n=0}^{\infty} \frac{1}{(s^{nt+r} q^{n+1} z; s)_t}$$

The proof when $r = 1$

Define

$$f_n(q,s) = \sum_{\substack{\lambda\in\mathcal{P},\\ \lambda_1=n}} s^{|\lambda|} q^{\lambda_1 + \lambda_{t+1} + \lambda_{2t+1} + \cdots}$$

So then

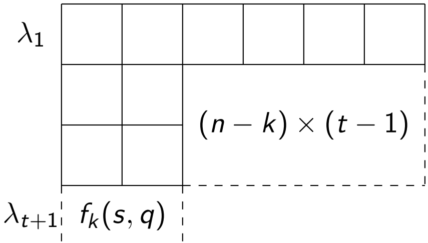
$$F(s,q,z) = \sum_{\lambda\in\mathcal{P}} z^{\lambda_1} s^{|\lambda|} q^{\lambda_1 + \lambda_{t+1} + \lambda_{2t+1} + \cdots} = \sum_{n=0}^{\infty} f_n(q,s) z^n$$

is the generating function of f_n.

The proof when $r = 1$

First we find a recurrence relation for

$$f_n(q,s) = \sum_{\substack{\lambda \in \mathcal{P}, \\ \lambda_1 = n}} s^{|\lambda|} q^{\lambda_1 + \lambda_{t+1} + \lambda_{2t+1} + \cdots}.$$



$$f_n(q,s) = (qs)^n \sum_{k=0}^{n} s^{k(t-1)} \begin{bmatrix} n-k+t-1 \\ t-1 \end{bmatrix}_s f_k(q,s)$$

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Now we replace f_n with the recurrence relation:

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This factors to $F(s,q,z) = \frac{1}{(sqz;s)_t} F(s,q,s^t qz)$.

The proof when $r = 1$

Finally, expand $F(s,q,z) = \frac{1}{(sqz;s)_t} F(s,q,s^t qz)$ N times:

$$F(s,q,z) = \prod_{n=0}^{N} \frac{1}{(s^{nt+1}q^{n+1}z;s)_t} F(s,q,s^{Nt}q^N z)$$

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The $r > 1$ case follows easily from this.

The proof when $r > 1$

Let $F_r(s,q,z) = \sum_{\lambda \in \mathcal{P}} z^{\lambda_1} s^{|\lambda|} q^{\lambda_r + \lambda_{t+r} + \lambda_{2t+r} + \cdots}$.

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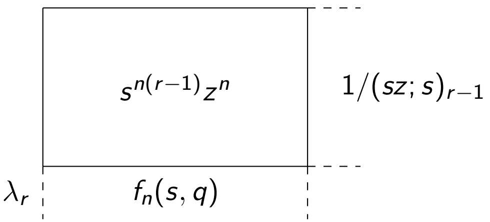

$F_r(s,q,z) =$

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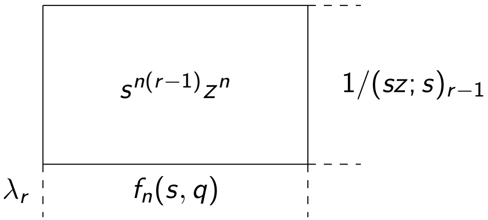
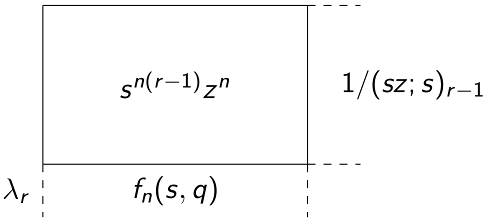


$$F_r(s,q,z) = \sum_{n=0}^{\infty} s^{n(r-1)} z^n \frac{1}{(sz;s)_{r-1}} f_n(s,q)$$

The proof when $r > 1$

Let $F_r(s,q,z) = \sum_{\lambda \in \mathcal{P}} z^{\lambda_1} s^{|\lambda|} q^{\lambda_r + \lambda_{t+r} + \lambda_{2t+r} + \cdots}$.



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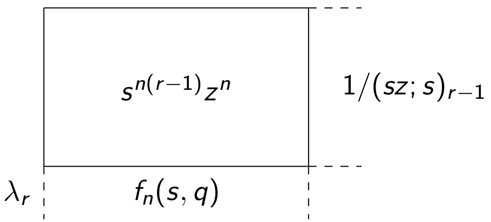

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$$= \frac{1}{(sz;s)_{r-1}} F(s,q,s^{r-1}z) = \frac{1}{(sz;s)_{r-1}} \prod_{n=0}^{\infty} \frac{1}{(s^{nt+r} q^{n+1} z;s)_t}.$$

Reversing the Schmidt weight

Setting s to q and q to q^{-1} in

$$\sum_{\lambda \in \mathcal{P}} z^{\lambda_1} s^{|\lambda|} q^{\lambda_r + \lambda_{t+r} + \lambda_{2t+r} + \cdots}$$ produces

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This has a colored partition interpretation if $r > 1$: *The number of partitions λ with $\lambda_1 = k$ and all parts except at the indices $r, t+r, 2t+r, \ldots$ summing to n is equal to the number of 2-colored partitions μ of n such that $\ell(\mu) = k$, and the second color appears only in part sizes $r-1, t-1+r-1, 2(t-1)+r-1, \ldots$.*

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- Generalize Mork's bijection, perhaps by first generalizing Bessenrodt's bijection.
- Adapt the new bijection or the proof of the main theorem to work with more general Schmidt weights, or to certain subsets of $\mathcal{P}$.

Thank you!

A preprint of this work can be found at
https://arxiv.org/abs/2207.14586