A bijective proof and generalization of the non-negative crank-odd mex identity

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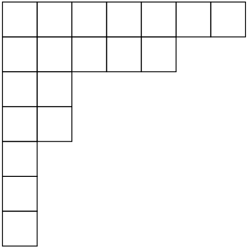
Online Partitions Seminar, April 28

Partition statistics

Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be an integer partition.

Example with $\lambda = (7, 5, 2, 2, 1, 1, 1)$:



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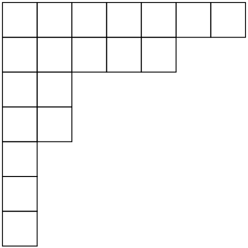

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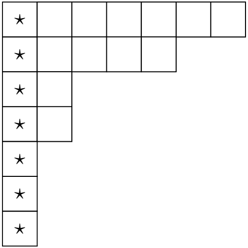

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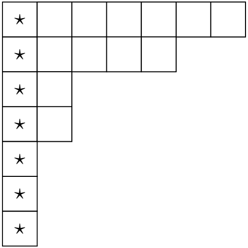

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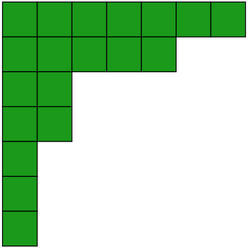

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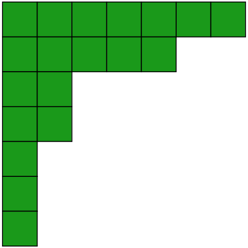

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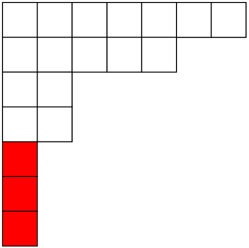

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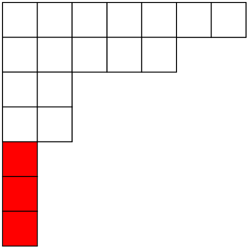

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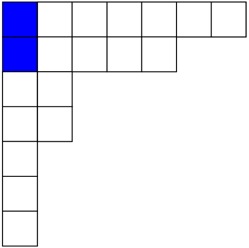

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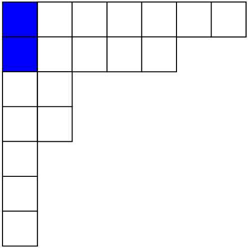

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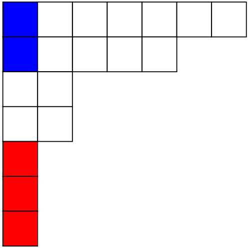

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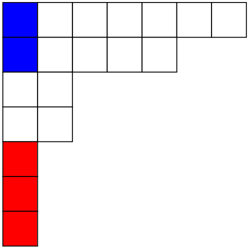

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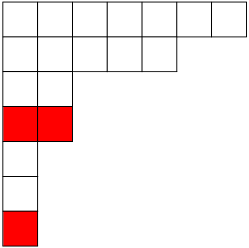

Non-negative crank–odd mex identity

Theorem 1: Andrews–Newman/Hopkins–Sellers

Let n be a non-negative integer. Then,

$$\sharp\{\lambda : |\lambda| = n, crank(\lambda) \geq 0\} = \sharp\{\lambda : |\lambda| = n, mex(\lambda) \equiv 1 \mod 2\}.$$

Analytic proof via the computation of the generating functions.

Refinement related to the parity of length and the congruences modulo 4 of the mex.

Combinatorial interpretations of Hopkins, Sellers and Yee related to the Durfee decomposition.

Extending the notion of mex

Let i be a non-negative integer and λ be an integer partition. The i-mex of λ, $mex_i(\lambda)$, is the smallest integer greater than i which is not a part of λ.

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For $\lambda = (7, 5, 2, 2, 1, 1, 1)$,

$$mex_0(\lambda) = mex_1(\lambda) = mex_2(\lambda) = 3,$$

$$mex_3(\lambda) = 4,$$

$$mex_4(\lambda) = mex_5(\lambda) = 6,$$

$$mex_6(\lambda) = 8,$$

$$mex_i(\lambda) = i + 1 \text{ for all } i \geq 7$$

Generalization of the non-negative crank–odd mex identity

Theorem 2: K.

Let n, i be two non-negative integers with $n \geq 2$. Then,

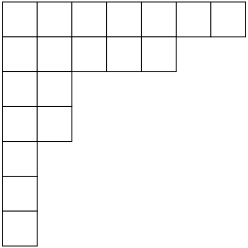
$$\sharp\{\lambda : |\lambda| = n,\ crank(\lambda) \geq i\} = \sharp\{\lambda : |\lambda| = n,\ i \in \lambda,\ mex_i(\lambda) - i \equiv 1 \mod 2\}$$

with the convention that there is a fictitious part 0 at the end of any integer partition.

When $i = 0$, $mex = mex_i$ and we recover the non-negative crank–odd mex identity for the weight greater than 1.

Durfee decomposition

Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be an integer partition. Set $\lambda_0 = \infty$ and $\lambda_{s+1} = 0$.



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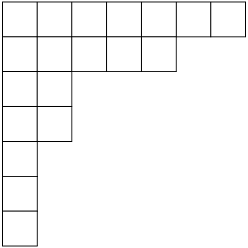
Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ be an integer partition. Set $\lambda_0 = \infty$ and $\lambda_{s+1} = 0$. For an non-negative integer i, define

$$d_i^\lambda = \max\{u \geq 0 : \lambda_u - u \geq i\}.$$



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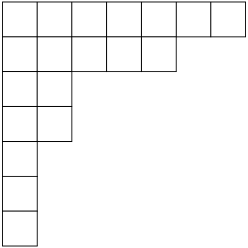

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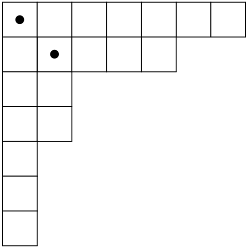
- $(d_i^\lambda)_{i \geq 0}$ is non-increasing and $(i + d_i^\lambda)_{i \geq 0}$ is non-decreasing.
- For $i = 0$, $d_0^\lambda = d$ is the length of the Durfee square, and the Durfee decomposition is $\lambda \equiv (d, \mu, \nu)$ where $\mu_u = \lambda_u - u$ and $\nu_u = \sharp\{v : \lambda_v \geq u\} - u$ for all $u \in \{1, \ldots, d\}$.

$$d_0^\lambda = 2.$$



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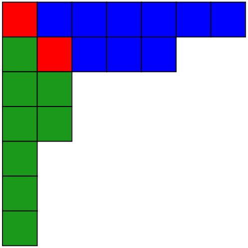
$$\lambda \equiv (2, (6, 3), (6, 2)).$$



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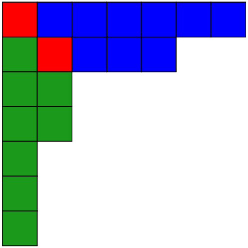

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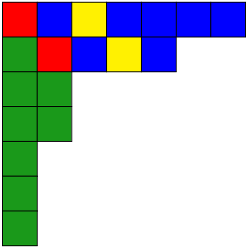

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Relating the *i*-mex to the Durfee decomposition

Theorem 3: K.

Let n, i be two non-negative integers. Then,

$$\sharp\{\lambda : |\lambda| = n,\ \lambda_{d_i^\lambda} > i + d_i^\lambda\} = \sharp\{\lambda : |\lambda| = n,\ mex_i(\lambda) - i \equiv 1 \mod 2\},$$

Bijection such that the length of the partitions and the parts $\leq i$ are conserved.

Relating the Durfee decomposition to the crank

Theorem 4: Hopkins–Sellers–Yee

Let n, i be two non-negative integers such that $n \geq 2$. Then,

$$\sharp\{\lambda : |\lambda| = n,\ i \in \lambda,\ \lambda_{d_i^\lambda} > i + d_i^\lambda\} = \sharp\{\lambda : |\lambda| = n,\ crank(\lambda) \leq -i\}.$$

Revised bijection from Hopkins, Sellers and Yee's paper.

Garvan's crank identity

Theorem 5: Garvan

We have

$$(x-1)q + \sum_{\lambda} x^{crank(\lambda)} q^{|\lambda|} = \frac{(q;q)_\infty}{(qx, qx^{-1}; q)_\infty}.$$

Hence, for n, i be two non-negative integers such that $n \geq 2$. Then,

$$\sharp\{\lambda : |\lambda| = n,\ crank(\lambda) = i\} = \sharp\{\lambda : |\lambda| = n,\ crank(\lambda) = -i\}.$$

Involution which transforms the crank into its opposite.

The setup

Let $\mathcal{P}$ be the set of partitions. For $i \geq 0$,

$$\mathcal{P}_i = \{\lambda : i \in \lambda\} \text{ and } \overline{\mathcal{P}}_i = \{\lambda : i \notin \lambda\}$$

$$\mathcal{F}_i = \{\lambda : \lambda_{d_i^\lambda} > i + d_i^\lambda\} \text{ and } \overline{\mathcal{F}}_i = \{\lambda : \lambda_{d_i^\lambda} = i + d_i^\lambda\}.$$

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For $i, k \geq 0$, set $\Delta_{i,k} = (i+k, \ldots, i+1)$ consisting of k consecutive integers ending by $i+1$. Hence,

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$$\bigsqcup_{k \geq 0} \{\Delta_{i,2k}\} \times \mathcal{P}_{i+2k+1}.$$

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- The set $\{\lambda : \lambda_{d_i^\lambda} > i + d_i^\lambda\}$ can be identified to
$$\{\Delta_{i,0}\} \times \mathcal{F}_i = \{\emptyset\} \times \mathcal{F}_i$$

Main idea

Build a transformation ϕ_i on

$$\left(\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\}\times\mathcal{P}\right)\setminus\left(\{\Delta_{i,0}\}\times\mathcal{F}_i\right)=\left(\bigsqcup_{k\geq 1}\{\Delta_{i,2k}\}\times\mathcal{F}_{i+2k}\right)\sqcup\left(\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\}\times\overline{\mathcal{F}}_{i+2k}\right)$$

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and iterate it on

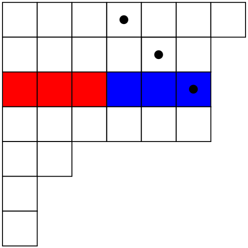
$$\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\}\times\mathcal{P}_{i+2k+1}.$$

Transformation ϕ_i

On $\{\Delta_{i,2k}\} \times \overline{\mathcal{F}}_{i+2k}$ for $k \geq 0$

Let $\lambda \in \overline{\mathcal{F}}_{i+2k}$. Then, $\lambda_{d^{\lambda}_{i+2k}} = i + 2k + d^{\lambda}_{i+2k}$.

Example with $i = 1$, $\lambda = (7, 6, 6, 6, 2, 1, 1)$ and $k = 1$. We have $d^{\lambda}_3 = 3$ and $\lambda_3 = 3 + 3 = 6$.



Transformation ϕ_i

On $\{\Delta_{i,2k}\} \times \overline{\mathcal{F}}_{i+2k}$ for $k \geq 0$

Let $\lambda \in \overline{\mathcal{F}}_{i+2k}$. Then, $\lambda_{d^{\lambda}_{i+2k}} = i + 2k + d^{\lambda}_{i+2k}$.

- Add one to λ_j for $1 \leq j < d^{\lambda}_{i+2k}$,

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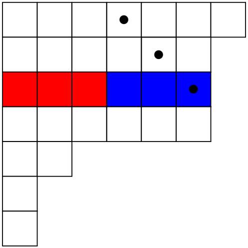

Transformation ϕ_i

On $\{\Delta_{i,2k}\} \times \overline{\mathcal{F}}_{i+2k}$ for $k \geq 0$

Let $\lambda \in \overline{\mathcal{F}}_{i+2k}$. Then, $\lambda_{d^\lambda_{i+2k}} = i + 2k + d^\lambda_{i+2k}$.

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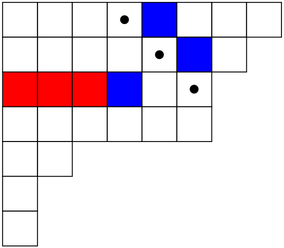

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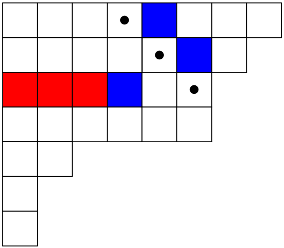

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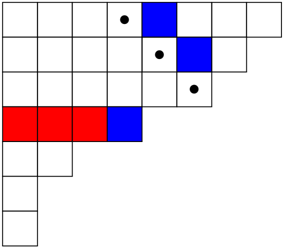

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- Add one to λ_j for $1 \leq j < d^{\lambda}_{i+2k}$,
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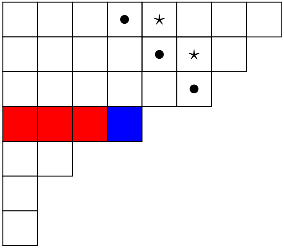
Let μ be the partition obtained. Then

$$\phi_i((\Delta_{i,2k}, \lambda)) = (\Delta_{i,2k}, \mu).$$

Example with $i = 1$, $\lambda = (7, 6, 6, 6, 2, 1, 1)$ and $k = 1$. We have $d^{\lambda}_3 = 3$ and $\lambda_3 = 3 + 3 = 6$. $\mu = (8, 7, 6, 4, 2, 1, 1)$.



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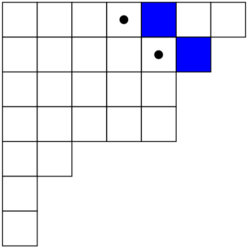
Example with $i = 1$, $\lambda = (7, 6, 5, 5, 2, 1, 1)$ and $k = 1$. We have $d^{\lambda}_3 = 2$ and $\lambda_2 = 6 > 3 + 2$.



Transformation ϕ_i

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Let $\lambda \in \mathcal{F}_{i+2k}$. Then, $\lambda_{d^\lambda_{i+2k}} > i + 2k + d^\lambda_{i+2k}$.

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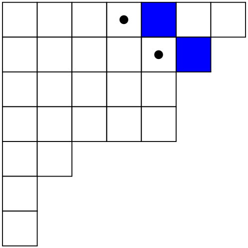

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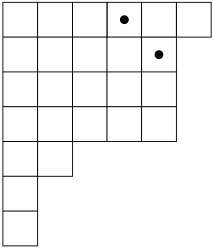

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- Subtract one from λ_j for $1 \leq j \leq d^{\lambda}_{i+2k}$,
- add the parts $i + 2k + d^{\lambda}_{i+2k}$ and $i + 2k - 1$.

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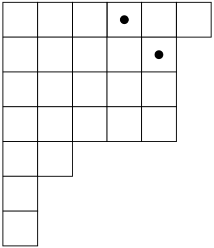

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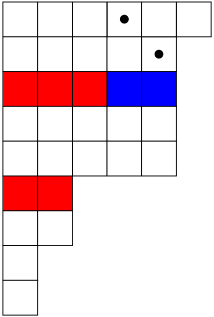

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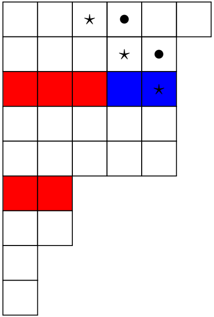
- Subtract one from λ_j for $1 \leq j \leq d^{\lambda}_{i+2k}$,
- add the parts $i + 2k + d^{\lambda}_{i+2k}$ and $i + 2k - 1$.

Let μ be the partition obtained. Then

$$\phi_i((\Delta_{i,2k}, \lambda)) = (\Delta_{i,2k-2}, \mu).$$

Example with $i = 1$, $\lambda = (7, 6, 5, 5, 2, 1, 1)$ and $k = 1$. We have $d^{\lambda}_3 = 2$ and $\lambda_2 = 6 > 3 + 2$. $\mu = (6, 5, 5, 5, 5, 2, 2, 1, 1)$.



The map Φ_i from $\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\} \times \mathcal{P}_{i+2k+1}$ to $\{\Delta_{i,0}\} \times \mathcal{F}_i$

For $(\Delta_{i,2k}, \lambda) \in \{\Delta_{i,2k}\} \times \mathcal{P}_{i+2k+1}$, iterate the transformation ϕ_i as long as it is possible.

Claim 1: Finite number of iterations

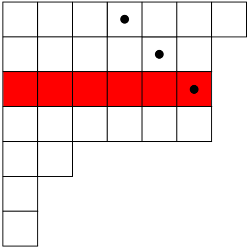
The number of iterations is finite, and the last pair belongs to $\{\Delta_{i,0}\} \times \mathcal{F}_i$.

We set $\Phi_i((\Delta_{i,2k}, \lambda))$ to be the last pair obtained after the iterations of ϕ_i on $(\Delta_{i,2k}, \lambda)$.

Example with $i = 1$ and the pair $(\Delta_{1,2}, (7,6,6,6,2,1,1))$

Step 0

$\Delta_{1,2}$



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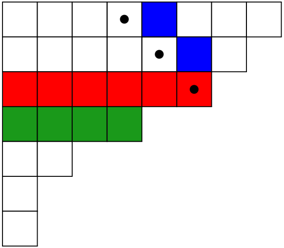
Step 1

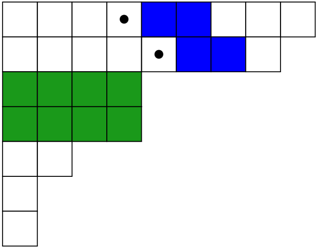
$\Delta_{1,2}$



Example with $i = 1$ and the pair $(\Delta_{1,2}, (7, 6, 6, 6, 2, 1, 1))$

Step 2

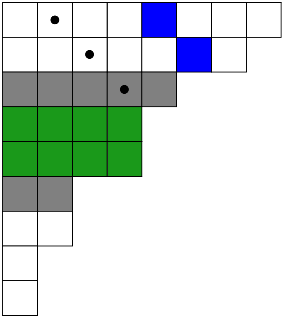
$\Delta_{1,2}$



Example with $i = 1$ and the pair $(\Delta_{1,2}, (7, 6, 6, 6, 2, 1, 1))$

Step 3

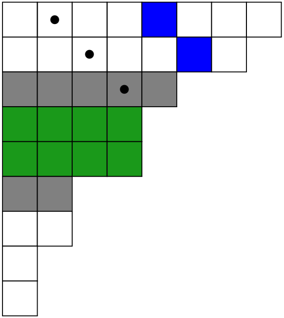
$\Delta_{1,0}$



Example with $i = 1$ and the pair $(\Delta_{1,2}, (7, 6, 6, 6, 2, 1, 1))$

$$\Phi_1((\Delta_{1,2}, (7, 6, 6, 6, 2, 1, 1))) = (\Delta_{1,0}, (8, 7, 5, 4, 4, 2, 2, 1, 1))$$

$\Delta_{1,0}$

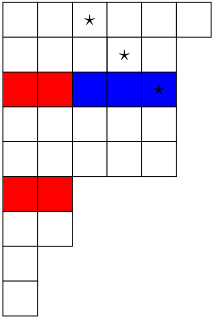


Transformation ψ_i on $\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\} \times \overline{\mathcal{P}}_{i+2k+1}$

On $\{\Delta_{i,2k}\} \times \left(\overline{\mathcal{P}}_{i+2k+1} \cap \overline{\mathcal{F}}_{i+2k+1}\right)$

Let $\lambda \in \overline{\mathcal{P}}_{i+2k+1} \cap \overline{\mathcal{F}}_{i+2k+1}$. Then, $\lambda_{d^{\lambda}_{i+2k+1}} = 1+i+2k+d^{\lambda}_{1+i+2k}$ and $1+i+2k \in \lambda$.

Example with $i=1$, $\lambda = (6,5,5,5,5,2,2,1,1)$ and $k=0$. We have $d^{\lambda}_2 = 3$ and $\lambda_3 = 5 = 2+3$.



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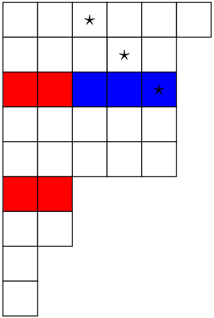

Transformation ψ_i on $\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\}\times\overline{\mathcal{P}}_{i+2k+1}$

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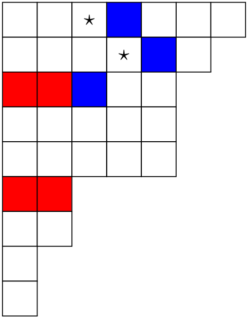

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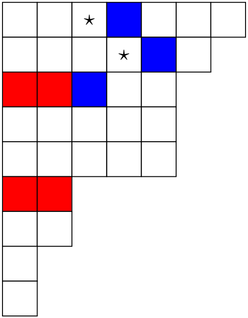
- Add one to λ_j for $1\leq j<d^\lambda_{i+2k+1}$,
- delete the parts $\lambda_{d^\lambda_{i+2k+1}}$ and $i+2k+1$.

Example with $i=1$, $\lambda=(6,5,5,5,5,2,2,1,1)$ and $k=0$. We have $d_2^\lambda=3$ and $\lambda_3=5=2+3$.



Transformation ψ_i on $\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\} \times \overline{\mathcal{P}}_{i+2k+1}$

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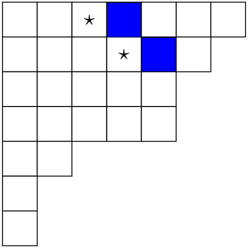

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Let μ be the partition obtained. Then

$$\psi_i((\Delta_{i,2k}, \lambda)) = (\Delta_{i,2k+2}, \mu).$$

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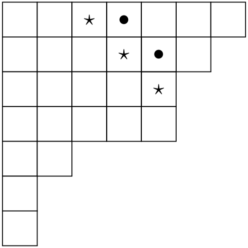

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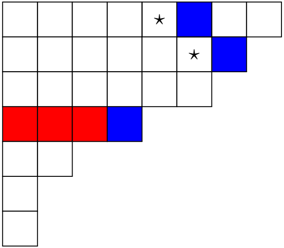
Example with $i = 1$, $\lambda = (8, 7, 6, 4, 2, 1, 1)$ and $k = 1$. We have $d^\lambda_4 = 2$ and $\lambda_2 = 7 > 4 + 2$.



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- Subtract from to λ_j for $1 \leq j \leq d^{\lambda}_{i+2k}$,

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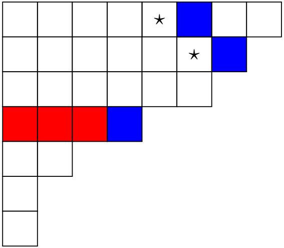

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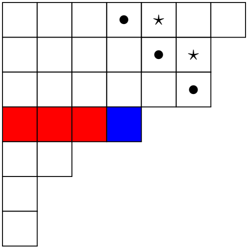

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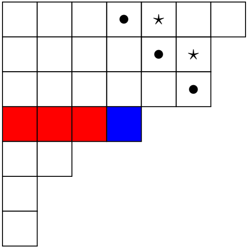
- Subtract from to λ_j for $1\leq j\leq d^\lambda_{i+2k}$,
- transform a part $i+2k+1$ into $i+2k+1+d^\lambda_{1+i+2k}$

Example with $i=1$, $\lambda=(8,7,6,4,2,1,1)$ and $k=1$. We have $d_4^\lambda=2$ and $\lambda_2=7>4+2$.



Transformation ψ_i on $\bigsqcup_{k\geq 0}\{\Delta_{i,2k}\} \times \overline{\mathcal{P}}_{i+2k+1}$

On $\{\Delta_{i,2k}\} \times \left(\overline{\mathcal{P}}_{i+2k+1} \cap \mathcal{F}_{i+2k+1}\right)$

Let $\lambda \in \overline{\mathcal{P}}_{i+2k+1} \cap \mathcal{F}_{i+2k+1}$. Then, $\lambda_{d^\lambda_{i+2k+1}} > 1 + i + 2k + d^\lambda_{i+2k+1}$ and $i + 2k + 1 \in \lambda$.

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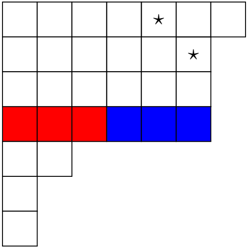

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- Subtract from to λ_j for $1\leq j\leq d^{\lambda}_{i+2k}$,
- transform a part $i+2k+1$ into $i+2k+1+d^{\lambda}_{1+i+2k}$

Let μ be the partition obtained. Then

$$\psi_i((\Delta_{i,2k},\lambda))=(\Delta_{i,2k},\mu).$$

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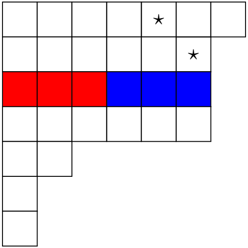

The map Ψ_i from $\{\Delta_{i,0}\} \times \mathcal{F}_i$ to $\bigsqcup_{k \geq 0}\{\Delta_{i,2k}\} \times \mathcal{P}_{i+2k+1}$

For $(\Delta_{i,0}, \lambda) \in \{\Delta_{i,0}\} \times \mathcal{F}_i$, iterate the transformation ψ_i as long as it is possible.

Claim 2: Finite number of iterations

The number of iterations is finite, and the last pair belongs to $\bigsqcup_{k \geq 0}\{\Delta_{i,2k}\} \times \mathcal{P}_{i+2k+1}$.

We set $\Psi_i((\Delta_{i,0}, \lambda))$ to be the last pair obtained after the iterations of ϕ_i.

Sketch of the proof of the well-definedness of the maps

- The maps ϕ_i and ψ_i describe inverse bijections between:
  - ⋆ $\{\Delta_{i,2k}\} \times \overline{\mathcal{F}}_{i+2k}$ and $\{\Delta_{i,2k}\} \times \overline{\mathcal{P}}_{i+2k+1} \cap \mathcal{F}_{i+2k+1}$ for $k \geq 0$,
  - ⋆ $\{\Delta_{i,2k}\} \times \mathcal{F}_{i+2k}$ and $\{\Delta_{i,2k-2}\} \times \overline{\mathcal{P}}_{i+2k-1} \cap \overline{\mathcal{F}}_{i+2k-1}$ for $k \geq 1$.

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- The pairs fixed by ϕ_i (and ψ_i) are those of the form $(\Delta_{i,2k}, \lambda)$ with $\lambda_1 = i + 2k + 1$, i.e. $\lambda \in \overline{\mathcal{F}}_{i+2k}$ with $d^{\lambda}_{i+2k} = 1$.

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- For all $(\Delta_{i,2k}, \lambda)$ not fixed by ϕ_i, the number of its iterations in $\{\Delta_{i,2k}\} \times \mathcal{P}$ is less than $\frac{|\Delta_{i,2k}|+|\lambda|}{1+i+2k}$.

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Sketch of the proof of the well-definedness of the maps

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- The maps Φ_i and Ψ_i describe inverse bijections.
- The parts $\leq i$ are conserved.

Relation between $crank(\lambda)$ and d_i^λ

Theorem 6: Hopkins–Sellers–Yee

Let i be a non-negative integer. Then,

$$crank(\lambda) \le -i \iff \omega(\lambda) \ge i + d_i^\lambda.$$

Proof.

Trivial when $|\lambda| = 0$. When $|\lambda| > 0$, $i + d_i^\lambda \ge d_0^\lambda > 0$.

If $\omega(\lambda) \ge i + d_i^\lambda > 0$, then $\eta(\lambda) \le d_i^\lambda$ and $crank(\lambda) \le -i$.

If If $0 < \omega(\lambda) < i + d_i^\lambda$, then $\eta(\lambda) \ge d_i^\lambda$ and $crank(\lambda) \le -i$. If $\omega(\lambda) = 0$, $crank(\lambda) = \lambda_1 > 0 \ge -i$.

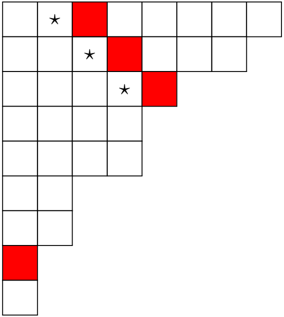
□

Set $\mathcal{C}_i = \{\lambda : crank(\lambda) \le -i\} = \{\lambda : \omega(\lambda) \ge i + d_i^\lambda\}$.

Bijective proof of Theorem 4

Let $\lambda \in \mathcal{F}_i \cap \overline{\mathcal{P}}_i$. Then, $\lambda_{d_i^\lambda} > i$ and $i \in \lambda$.

Example with $i = 1$, $\lambda = (8, 7, 5, 4, 4, 2, 2, 1, 1)$. We have $d_1^\lambda = 3$ and $\lambda_3 = 5 > 1 + 3$.



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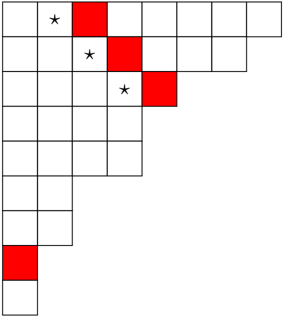

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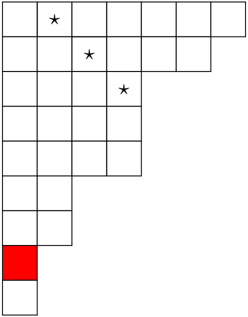

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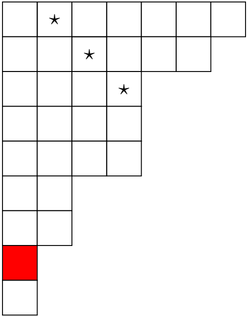
- Subtract one from λ_j for $1 \leq j \leq d_i^\lambda$,
- delete a part i and add $d_i^\lambda + i$ parts equal to 1



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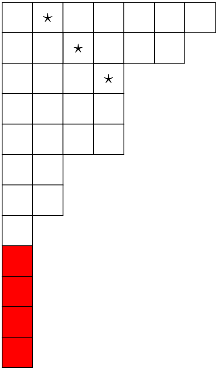

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The partition μ obtained satisfies $d_i^\mu = d_i^\lambda$ and $\omega(\mu) \geq i + d_i^\mu$, so that $\mu \in \mathcal{C}_i$.

Example with $i = 1$,
$\lambda = (8, 7, 5, 4, 4, 2, 2, 1, 1)$. We have
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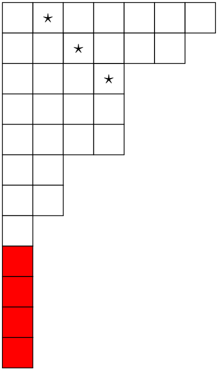
$\mu = (7, 6, 4, 4, 4, 2, 2, 1, 1, 1, 1, 1)$.



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For $\mu \in \mathcal{C}_i$ with $|\mu| > 1$, $\ell(\mu) \geq 2d_i^\mu + i$ and the transformation is invertible.

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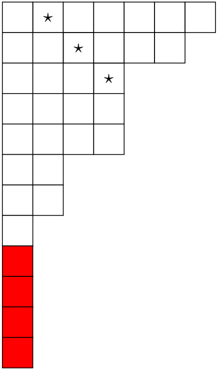


Involution transforming the crank into its opposite

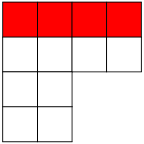
Let λ be a partition with $|\lambda| > 1$.

Involution transforming the crank into its opposite

Let λ be a partition with $|\lambda| > 1$.

- If $\omega(\lambda) = 0$, then transform the part λ_1 into λ_1 parts equals to 1 to obtain a partition μ with $\omega(\mu) = \lambda_1$ and $\eta(\mu) = 0$. Hence *crank*$(\mu) = -$*crank*(λ).

Example with $\lambda = (4, 4, 2, 2)$.



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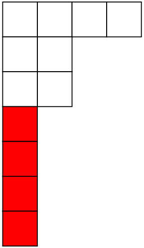

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- If $\omega(\lambda) > 0$ and $\eta(\lambda) = 0$, transform the $\omega(\lambda)$ parts equals to 1 into a part $\omega(\lambda)$ to obtain a partition μ with $\omega(\mu) = 0$ and $\mu_1 = \omega(\lambda)$. Hence, $\mathit{crank}(\mu) = -\mathit{crank}(\lambda)$.

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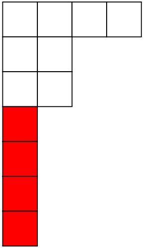

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These two cases are inverse each other.

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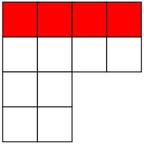
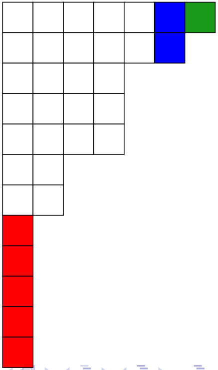


Involution transforming the crank into its opposite

Let λ be a partition with $|\lambda| > 1$.

- If $\omega(\lambda) > 0$ and $\eta(\lambda) > 0$, set $\rho(\lambda) = \max\{\omega(\lambda), \lambda_2 - 1\}$ and do the following.

Example with $\lambda = (7, 6, 4, 4, 4, 2, 2, 1, 1, 1, 1, 1)$. We have $\omega(\lambda) = 5$, $\eta(\lambda) = 2$ and $\rho(\lambda) = \max\{5, 5\} = 5$.



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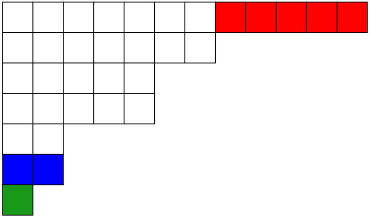

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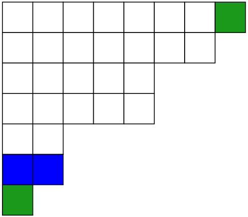

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Let λ be a partition with $|\lambda| > 1$.

- If $\omega(\lambda) > 0$ and $\eta(\lambda) > 0$, set $\rho(\lambda) = \max\{\omega(\lambda), \lambda_2 - 1\}$ and do the following.
  - ⋆ Transform λ into its conjugate λ^*.
  - ⋆ Transform λ_1^* into $\lambda_2^* + \lambda_1 - \rho(\lambda) - 1$.
  - ⋆ Add one to λ_i^* for $1 \leq i \leq \omega(\lambda)$.

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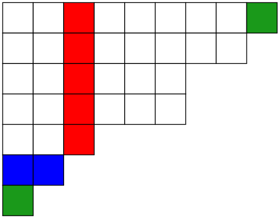
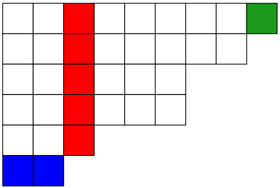


Involution transforming the crank into its opposite

Let λ be a partition with $|\lambda| > 1$.

- If $\omega(\lambda) > 0$ and $\eta(\lambda) > 0$, set $\rho(\lambda) = \max\{\omega(\lambda), \lambda_2 - 1\}$ and do the following.
  - ⋆ Transform λ into its conjugate λ^*.
  - ⋆ Transform λ_1^* into $\lambda_2^* + \lambda_1 - \rho(\lambda) - 1$.
  - ⋆ Add one to λ_i^* for $1 \leq i \leq \omega(\lambda)$.
  - ⋆ Delete $\lambda_1 - \rho(\lambda) - 1$ parts 1.

Example with $\lambda = (7, 6, 4, 4, 4, 2, 2, 1, 1, 1, 1, 1)$. We have $\omega(\lambda) = 5$, $\eta(\lambda) = 2$ and $\rho(\lambda) = \max\{5, 5\} = 5$.



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  - ⋆ Transform the part $\lambda^*_{\omega(\lambda)} = \eta(\lambda)$ into $\eta(\lambda)$ parts equal to 1.

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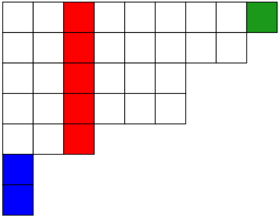

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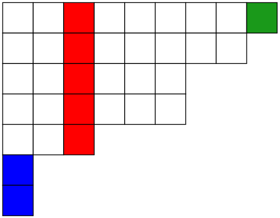
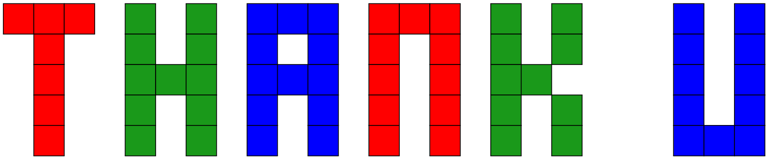
  The partition μ obtained satisfies $\omega(\mu) = \eta(\lambda)$ and $\eta(\mu) = \omega(\lambda)$ (and $\rho(\mu) = \lambda_2^*$).

Example with $\lambda = (7, 6, 4, 4, 4, 2, 2, 1, 1, 1, 1, 1)$. We have $\omega(\lambda) = 5$, $\eta(\lambda) = 2$ and $\rho(\lambda) = \max\{5, 5\} = 5$.
$\mu = (9, 8, 6, 6, 3, 1, 1)$




THANK U