Biases in Parts Among k-regular and k-indivisible Partitions

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July 2022

Partitions

Definition

- A *partition* of n, denoted $\lambda \vdash n$, is a non-increasing sequence of positive integers summing to n,

$$\lambda = (\lambda_1, \ldots, \lambda_k) \qquad \sum_{i=1}^{k} \lambda_i = n$$

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Example

We have $p(4) = 5$ since

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Hardy-Ramanujan's Asymptotic

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How does $p(n)$ grow as $n \to \infty$?

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Theorem (Hardy-Ramanujan)

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Parts in Congruence Classes

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We have that

$$T(r,t;n) = \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\pi t n^{1/2}\sqrt{2}}\left[\log(n) - 2\psi\left(\frac{r}{t}\right) + \alpha_t + O\left(n^{-\frac{1}{2}}\log(n)\right)\right],$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$.

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- The *second order term* implies a *BIAS*.
- If $r < s$, we eventually have $T(r,t;n) > T(s,t;n)$.

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- And for $2 \leq t \leq 10$, $r < s$, $D(r,t;n) \geq D(s,t;n)$ when $n \geq 9$.

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- *Better* explicit error terms than Craig when $k = 2$.

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$$(t) \text{ and } (t-1, 1).$$

Stable counterexamples for small k

Example

For $k = 3$, $(n, r, s, t) = (t + 2, t - 1, t, t)$ is also a counterexample:

$$\begin{array}{cc} (t, 2) & (t, 1, 1) \\ (t-1, 3) & (t-1, 2, 1) \end{array}$$

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Remark

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The ordering $\prec_{k,t}$

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Let $\mathscr{O}(t)$ be the *number* of such orderings on $\{1,\dots,t\}$ induced by $\prec_{k,t}$ for $k, t \geq 2$ coprime.

A Glimpse of k-indivisible Biases

$k = 2$	1	3	5	7	2	4	6

Figure: Biases among congruence classes mod t for k-indivisible partitions when $t = 7$, from most common to least common.

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$k = 3$	1	2	4	5	7	3	6
$k = 4$	1	2	3	5	6	7	4
$k = 5$	1	2	3	4	6	7	5

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$k = 5$	1	2	3	4	6	7	5
$k = 6, 10, 13, 20$	1	2	3	4	5	7	6
$k = 12$	1	2	3	4	6	5	7

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All other k	1	2	3	4	5	6	7

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- *Let $1 \leq r \leq t-k$, then $r \succ_{k,t} r+k$.*
- *Let $k \geq \frac{6(t^2-1)}{\pi^2}$, then for $1 \leq r < s \leq t$, we have $r \succ_{k,t} s$.*

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- *Let $t > 2$, then $\mathscr{O}(t) \geq \frac{\varphi(t)}{2}$, where $\varphi(t)$ is Euler's φ-function.*

Conjectures/Examples

Looking at $t = 7$ in a new light

$k = 2$	1	3	5	7	2	4	6
$k = 3$	1	2	4	5	7	3	6
$k = 4$	1	2	3	5	6	7	4
$k = 5$	1	2	3	4	6	7	5
$k = 6, 10, 13, 20$	1	2	3	4	5	7	6
$k = 12$	1	2	3	4	6	5	7
All other k	1	2	3	4	5	6	7

Figure: Biases among congruence classes mod t for k-indivisible partitions when $t = 7$, from most common to least common.

Plot versus the Digamma Function for $k = 3, t = 7$

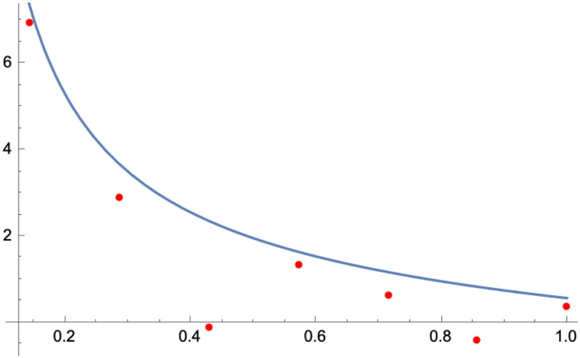

Figure: Second order term of each $1 \leq r \leq 7$ when $k = 3, t = 7$

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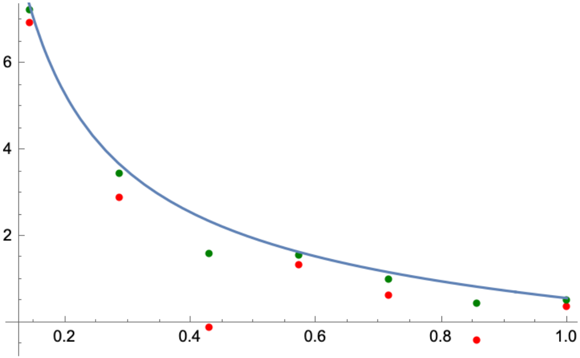
Plot versus the Digamma Function for $k = 10, t = 7$



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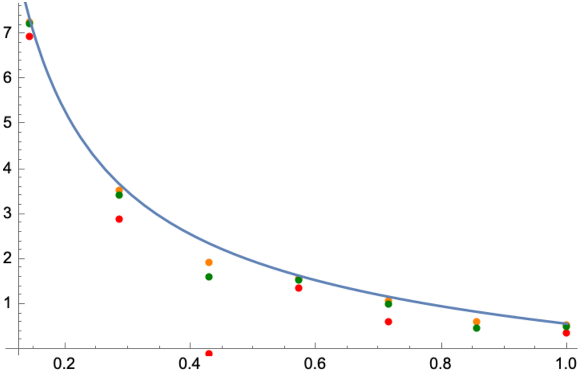

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A Conjecture concerning $\mathscr{O}(t)$

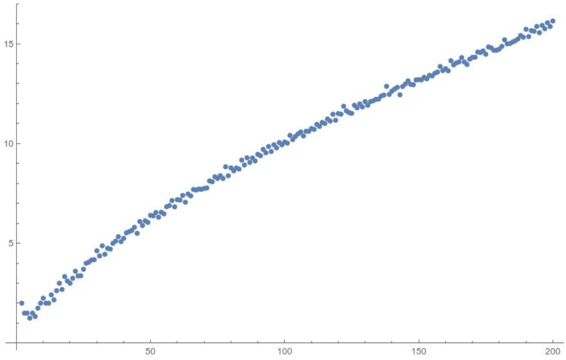

Figure: $\frac{\mathscr{O}(t)}{\varphi(t)}$ vs. t.

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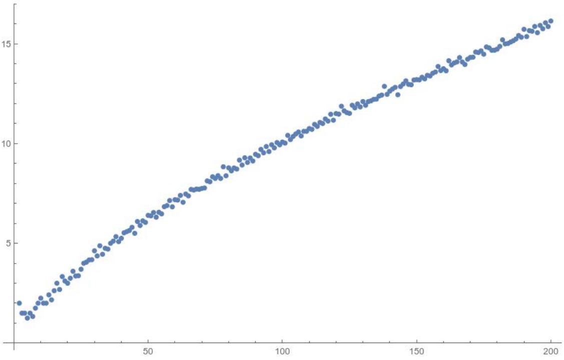

Figure: $\frac{\mathscr{O}(t)}{\varphi(t)}$ vs. t.

Conjecture (J.-O.)

$\frac{\mathscr{O}(t)}{\varphi(t)}$ *grows sublinearly as well as superlogarithmically*

There are no Ties

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- Shows there exists a t_0 such that if $\gcd(t, t_0) = 1$, and $\gcd(r,t) = \gcd(s,t) = 1$, then the conjecture holds.

Divisibility Relations Fail

Example

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$$k = 2, 3, 4, 5, 6, 7, 8, 9, 10, 15, \ldots, 42, 43, 54, 65.$$

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- But $14 \mid 42$, and 14 does not divide $10 + 11 \cdot j$ until 98 ($j = 8$). *Combinatorial heuristics aren't enough to see orderings*

Circle method

Idea

To find asymptotic for $a(n)$, study its generating function

$$F(q) = \sum_{n=0}^{\infty} a(n) q^n.$$

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Question

How do we estimate the integral?

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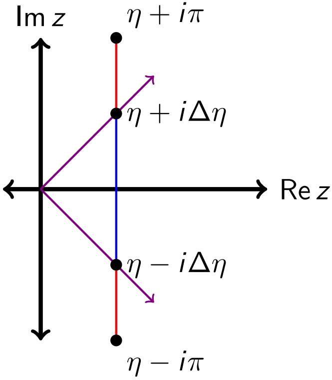
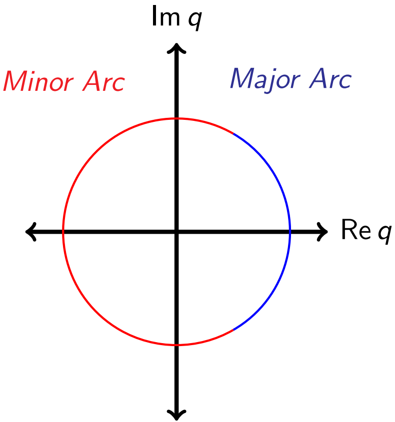
- The main contribution of the integral is the major arc, and the error term is the minor arc.

Path of Integration



$z = \eta + iy$ plane



$q = e^{-z}$ plane

The Generating Function for k-regular/k-indivisible Partitions

Proposition

The generating function for k-regular/k-indivisible partitions is

$$\xi_k(q) := \prod_{m=1}^{\infty} \frac{1-q^{mk}}{1-q^m} = \frac{\mathcal{P}(q)}{\mathcal{P}(q^k)}.$$

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Choice of term in the sum $\leftrightarrow$ How many parts have size m. □

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$$=\xi_k(q)\cdot\left(\frac{q^m}{1-q^m}-\frac{kq^{mk}}{1-q^{mk}}\right)$$

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counts the # of times m is a part in the k-indivisible ptns. when $k \nmid m$. We then sum over $m \equiv r \pmod{t}$ while excluding $k \mid m$.

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Proposition (Euler's Pentagonal Number Theorem)

$$\mathcal{P}(q)^{-1} = 1 + \sum_{m \geq 1} (-1)^m \left(q^{m(3m+1)/2} + q^{m(3m-1)/2} \right)$$

Estimating $L_k(r, t; q)$

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- Zagier gives an asymptotic version estimating $\sum_{m\geq 1} f(mz)$ under mild conditions on f.

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$$\sum_{m\geq 1} f(mx) \sim \frac{1}{x}\int_0^\infty f(u)du + \sum_{n=0}^{\infty} c_n \frac{B_{n+1}}{n+1}(-x)^n$$

as $x \to 0^+$.

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If $f(x) \sim \sum_{n\geq 0} c_n x^n$ and its derivatives have rapid decay at ∞, then

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as $x \to 0^+$.

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- Apply variant of Euler-Maclaurin, without a pole so no ψ. $\square$

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- Apply Euler-Maclaurin *twice*, with a pole, giving ψ. $\square$

Minor Arc Estimates for $L_k(r, t; q), L_k^{\times}(r, t; q)$

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- Apply two variants of Wright's Circle Method.

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Understanding $\psi_{k,t}$

$$\psi_{k,t}(r) = -\psi\left(\frac{r}{t}\right) + \frac{1}{k}\psi\left(\frac{\overline{r}}{t}\right)$$

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Estimating $\psi(a) - \psi(b)$ for $0 < b < a < 1$

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- Apply the previous two lemmas and rearrange. $\square$

Summary of Methods

Idea

- Want: Asymptotics for parts lying in residue classes.
- Use the circle method. How do we estimate?
  - Modular Transformation Laws.
  - Euler-Maclaurin Summation when you're not so lucky.
- $\psi_{k,t}$ is intricate. How do we understand it?
  - Approximate with a square sum.
  - Make the weighting factor $\frac{1}{k}$ so small it washes everything out.
  - Fix k mod t (i.e., $k = mt - 1$).

Summary of Results

Theorem (J.-O.)

Asymptotics for parts in k-regular/k-indivisible partitions of n which are r mod t (notationally, $D_k(r, t; n)$, $D_k^{\times}(r, t; n)$).

Corollary (J.-O.)

- *Bias towards lower congruence classes for $D_k(r, t; n)$.*
- *Intricate Bias for $D_k^{\times}(r, t; n)$.*

Theorem (J.-O.)

Basic properties of the biases in $D_k^{\times}(r, t; n)$

Acknowledgements

- This research was done as part of the 2022 UVA REU in Number Theory
- The authors would like to thank Ken Ono, the director, as well as their graduate student mentor William Craig.
- They are grateful for the support of grants from the NSF (DMS-2002265, DMS-2055118, DMS-2147273), the NSA (H98230-22-1-0020), and the Templeton World Charity Foundation.