Dissection of the General Quintuple Product, with Applications

Seminar in Partition Theory, q-Series and Related Topics

.

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Notation and Preliminary Results

q-products I

For $|q| < 1$, $(a; q)_\infty = (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3)\dots$,

$$(a_1, a_2, \dots a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \dots (a_n; q)_\infty,$$

$$f_1 := (q; q)_\infty := (1 - q)(1 - q^2)(1 - q^3)\cdots, f_j := (q^j; q^j)_\infty$$

$$(q^j; q^M)_\infty = (1 - q^j)(1 - q^{j+M})(1 - q^{j+2M})(1 - q^{j+3M})\dots,$$

$$\langle a; q \rangle_\infty := (a, q/a, q; q)_\infty$$

$$Q(z, q) := (z, q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty = \frac{\langle z^2; q \rangle_\infty}{(-z, -q/z; q)_\infty} \quad (1)$$

q-products II

From the previous slide:

$$Q(z,q) := (z, q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty$$

For later use, note that

$$Q(-1,q) := (-1, -q, q; q)_\infty (q, q; q^2)_\infty = 2(q;q)_\infty$$

$$Q(q,q^4) = (q, q^3, q^4; q^4)_\infty (q^2, q^6; q^8)_\infty = (q;q)_\infty$$

We use the notation (Hickerson?)

$$J_{a,m} := (q^a, q^{m-a}, q^m; q^m)_\infty,$$

$$\bar{J}_{a,m} := (-q^a, -q^{m-a}, q^m; q^m)_\infty. \tag{2}$$

Preliminary Results I

Two equivalent forms of the Jacobi triple product identity:

$$\sum_{n=-\infty}^{\infty} (-z)^n q^{n^2} = \langle zq; q^2 \rangle_\infty, \qquad \sum_{n=-\infty}^{\infty} (-z)^n q^{n(n-1)/2} = \langle z; q \rangle_\infty.$$

The Quintuple Product Identity (one formulation):

$$\langle z; q \rangle_\infty (qz^2, q/z^2; q^2)_\infty = \langle -qz^3; q^3 \rangle_\infty - z\langle -q^2 z^3; q^3 \rangle_\infty.$$

From Cooper's survey paper [3], it seems that the quintuple product identity was discovered independently by Fricke and Ramanujan around 1916.

Preliminary Results II

The m-dissection of a function $G(q) = \sum_{n=0}^{\infty} g_n q^n$:

$$G(q) = G_0(q^m) + qG_1(q^m) + \cdots + q^{m-1}G_{m-1}(q^m),$$
$$\text{where } G_i(q^m) = \sum_{n=0}^{\infty} g_{nm+i}q^{nm}. \tag{3}$$

m-dissections of the Jacobi triple product:

$$\langle z; q\rangle_\infty = \sum_{r=0}^{m-1} (-z)^r q^{(r^2-r)/2} \left\langle (-1)^{m+1} z^m q^{(m^2-m)/2+mr}; q^{m^2} \right\rangle_\infty,$$
$$\langle zq; q^2\rangle_\infty = \sum_{r=0}^{m-1} (-z)^r q^{r^2} \left\langle (-1)^{m+1} z^m q^{m^2+2mr}; q^{2m^2} \right\rangle_\infty.$$

Preliminary Results III

A *partition* of a non-negative integer n is a way of writing n as a sum of positive integers.
For example, there are 5 partitions of 4:

$$4, \; 3+1, \; 2+2, \; 2+1+1, \; 1+1+1+1.$$

We are also interested in various restricted partition functions.

Motivation and Previous Results

A Combinatorial Interpretation of a Trisection I

In a previous paper [7] we made the following observation. Starting with the 3-dissection of f_1,

$$f_1 = J_{12,27} - qJ_{6,27} - q^2 J_{3,27}, \tag{4}$$

and then dividing both sides by f_3 one gets

$$\sum_{n=0}^{\infty} D_S(n)q^n = (q, q^2; q^3)_\infty = \frac{f_1}{f_3} = \frac{J_{12,27}}{f_3} - q\frac{J_{6,27}}{f_3} - q^2\frac{J_{3,27}}{f_3} = \frac{1}{(q^3, q^6, q^9, q^{18}, q^{21}, q^{24}; q^{27})_\infty} - \frac{q}{(q^3, q^9, q^{12}, q^{15}, q^{18}, q^{24}; q^{27})_\infty} - \frac{q^2}{(q^6, q^9, q^{12}, q^{15}, q^{18}, q^{21}; q^{27})_\infty}. \tag{5}$$

Here S is the set of positive integers with no multiples of 3, $D_S(n)$ is the number of partitions of n into an even number of distinct parts from S minus the number of partitions of n into an odd number of distinct parts from S.

A Combinatorial Interpretation of a Trisection II

Next, trisecting the sum $\sum_{n=0}^{\infty} D_S(n)q^n$ (terms with $n \equiv 0, 1, 2 \pmod 3$), equating each with the corresponding product on the right, making the replacement $q \to q^{1/3}$ one gets some partition identities.
For $a \in \{1, 2, 4\}$ let $p_{a,9}(n)$ denote the number of partitions of n into parts $\not\equiv \pm a, 0 \pmod 9$. Then

$$\begin{aligned} D_S(3n) &= p_{4,9}(n), \\ D_S(3n+1) &= -p_{2,9}(n), \\ D_S(3n+2) &= -p_{1,9}(n). \end{aligned} \tag{6}$$

Since $f_1 = Q(q, q^4)$, possibly identities analogous to those at (6) could be produced via the m-dissection of quintuple products of the form $Q(q^j, q^M)$, which led us to consider the general m-dissection of the general quintuple product $Q(z, q)$.

An Earlier Result [9]

Theorem

Let $m \geq 5$, $\gcd(m,6)=1$ *and let* $t \in \{1,2,\ldots,(m-1)/2\}$, $\gcd(t,m)=1$. *Let* $\mathcal{Q}(t,m) = Q(q^t, q^m)$. *If* $m \equiv 1 \pmod 6$ *(similarly for* $m \equiv 1 \pmod 6$*), then*

$$\mathcal{Q}(t,m) = \sum_{r=0}^{\frac{m-1}{3}} (-1)^r q^{\frac{1}{2}r(m(3r-1)+6t)} \mathcal{Q}\left(\frac{1}{6}m\left(m^2+m(6r-1)+6t\right), m^3\right)$$
$$+ \sum_{r=0}^{\frac{m-4}{3}} (-1)^r q^{\frac{1}{6}(m-3r-2)\left(m^2-m(3r+1)-6t\right)} \mathcal{Q}\left(\frac{1}{2}m\left(m^2-m(2r+1)-2t\right), m^3\right)$$
$$+ \sum_{r=0}^{\frac{m-7}{6}} (-1)^r q^{\frac{1}{2}(3r+1)(mr+2t)} \mathcal{Q}\left(\frac{1}{6}m(m(m-6r-1)-6t), m^3\right)$$
$$+ \sum_{r=0}^{\frac{m-7}{6}} (-1)^{\frac{1}{6}(m+6r+11)} q^{\frac{1}{24}(m-6r-1)\left(m^2-6mr+m-12t\right)} \mathcal{Q}\left(m(mr+t), m^3\right). \quad (7)$$

The Ramanujan Function R(q) and its Reciprocal as Quintuple Products I

Ramanujan's function $R(q)$ and it reciprocal have expressions involving quintuple products:

$$\begin{aligned} R(q) &= \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty} = \frac{Q(-q, q^5)}{(q^5; q^5)_\infty}, \\ \frac{1}{R(q)} &= \frac{(q, q^4; q^5)_\infty}{(q^2, q^3; q^5)_\infty} = \frac{Q(-q^2, q^5)}{(q^5; q^5)_\infty} \end{aligned} \tag{8}$$

The 5-dissections of these, which were proven by Hirschhorn [5], follow from Theorem 1.

The Ramanujan Function R(q) and its Reciprocal as Quintuple Products II

Corollary

$$R(q) = \frac{(q^{125};q^{125})_\infty}{(q^5;q^5)_\infty}\left[\frac{(q^{40},q^{85};q^{125})_\infty}{(q^{20},q^{105};q^{125})_\infty} + q\frac{(q^{60},q^{65};q^{125})_\infty}{(q^{30},q^{95};q^{125})_\infty}\right.$$
$$\left. - q^7\frac{(q^{35},q^{90};q^{125})_\infty}{(q^{45},q^{80};q^{125})_\infty} - q^3\frac{(q^{10},q^{115};q^{125})_\infty}{(q^{5},q^{120};q^{125})_\infty} - q^{14}\frac{(q^{15},q^{110};q^{125})_\infty}{(q^{55},q^{70};q^{125})_\infty}\right],$$

$$R(q)^{-1} = \frac{(q^{125};q^{125})_\infty}{(q^5;q^5)_\infty}\left[\frac{(q^{30},q^{95};q^{125})_\infty}{(q^{15},q^{110};q^{125})_\infty} - q\frac{(q^{20},q^{105};q^{125})_\infty}{(q^{10},q^{115};q^{125})_\infty}\right.$$
$$\left. + q^2\frac{(q^{55},q^{70};q^{125})_\infty}{(q^{35},q^{90};q^{125})_\infty} - q^{18}\frac{(q^{5},q^{120};q^{125})_\infty}{(q^{60},q^{65};q^{125})_\infty} - q^{4}\frac{(q^{45},q^{80};q^{125})_\infty}{(q^{40},q^{85};q^{125})_\infty}\right].$$

Main Result - m dissection of the Quintuple Product $Q(z, q)$

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m dissection of the Quintuple Product $Q(z,q)$

Theorem

Let $|q|<1$ and $z\neq 0$ and m a positive integer such that $3\nmid m$.
(i) If $m\equiv 1 \pmod 3$, then

$$Q(z,q)=\sum_{r=0}^{m-1} q^{\frac{1}{2}r(3r-1)} z^{3r} Q\left(z^m q^{\frac{1}{6}m(m+6r-1)}, q^{m^2}\right). \tag{9}$$

(ii) If $m\equiv 2 \pmod 3$, then

$$Q(z,q)=\sum_{r=0}^{m-1} q^{\frac{1}{2}r(3r-1)} z^{3r} Q\left(z^{-m} q^{\frac{1}{6}m(m-6r+1)}, q^{m^2}\right). \tag{10}$$

Sketch of Proof I

Sketch of proof.

Start with the quintuple product identity:

$$Q(z,q) = \langle -qz^3; q^3\rangle_\infty - z\langle -q^2z^3; q^3\rangle_\infty,$$

then form the m-dissections of each of the triple products to get

$$Q(z,q) = \sum_{r=0}^{m-1} q^{r(3r-1)/2} z^{3r} \left\langle -q^{m(3m+6r-1)/2} z^{3m}; q^{3m^2} \right\rangle_\infty$$
$$- z\sum_{r=0}^{m-1} q^{r(3r+1)/2} z^{3r} \left\langle -q^{m(3m+6r+1)/2} z^{3m}; q^{3m^2} \right\rangle_\infty$$
$$=: \sum_{r=0}^{m-1} a_r - z\sum_{r=0}^{m-1} b_r.$$

□

Sketch of Proof II

Sketch of proof continued.

Each term in the first sum can be matched with a term in the second sum and combined, again using the Quintuple Product Identity, into quintuple products, leading to the claimed summations. For $m \equiv 1 \pmod 3$,

$$\begin{aligned} a_r - zb_{r+(m-1)/3} = q^{r(3r-1)/2} z^{3r} \\ \times \Bigg[\left\langle -\left(q^{m(m+6r-1)/6} z^m\right)^3 q^{m^2}; q^{3m^2} \right\rangle_\infty \\ - q^{m(m+6r-1)/6} z^m \left\langle -\left(q^{m(m+6r-1)/6} z^m\right)^3 q^{2m^2}; q^{3m^2} \right\rangle_\infty \Bigg] \\ = q^{r(3r-1)/2} z^{3r} Q(q^{m(m+6r-1)/6} z^m, q^{m^2}). \end{aligned}$$

Remark: The matching of terms in the first with terms in the second sum was discovered experimentally. □

Some Remark and A Special Case (for later use) I

- The m-dissections are only for m of the form $6t-1$ or $6t+1$, t a positive integer.
- For m of the form $6t+2$ or $6t+4$ we get $m/2$-dissections.
- For $m \equiv 0 \pmod 3$, the m-dissection of $Q(z,q)$ does not appear to have a straightforward description, at least not for all such m.

Upon making the replacements $q \to q^M$ and $z \to q^j$, where $M > 3$ is an integer and j is a positive integer satisfying $1 \leq j < M/2$, and using the product form of $Q(z,q)$, we get the following special case.

A Special Case (for later use) II

Corollary

Let $|q| < 1$ *and let* $M > 3$ *be an integer and let* j *is a positive integer satisfying* $1 \leq j < M/2$. *Let* m *be a positive integer such that* $3 \nmid m$.
(i) If $m \equiv 1 \pmod 3$, *then*

$$\left(q^j, q^{M-j}, q^M; q^M\right)_\infty \left(q^{M-2j}, q^{M+2j}; q^{2M}\right)_\infty = \sum_{r=0}^{m-1} q^{M(3r-1)r/2+3jr}$$
$$\times \left(q^{mM(m+6r-1)/6+jm}, q^{m^2M-mM(m+6r-1)/6-jm}, q^{m^2M}; q^{m^2M}\right)_\infty$$
$$\times \left(q^{m^2M+2jm+M(m+6r-1)m/3}, q^{m^2M-2jm-mM(m+6r-1)/3}; q^{2m^2M}\right)_\infty. \quad (11)$$

(ii) If $m \equiv 2 \pmod 3$, *then there is a similar formula.*

Application 1. Extension of a result of Evans and Ramanathan on the m dissection of $(q, q)_\infty$

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Extension of a result of Evans and Ramanathan I

Evans and Ramanathan independently stated the m dissection of $(q,q)_\infty$ in the case $\gcd(m,6)=1$ (with different formulae in the case $m \equiv 1 \pmod 6$ and $m \equiv 5 \pmod 6$).
Since

$$(q;q)_\infty = \frac{1}{2} Q(-1,q)$$

this m-dissection can be extended to $m \equiv 2 \pmod 6$ and $m \equiv 4 \pmod 6$, since our m-dissection of $Q(z,q)$ is for $\gcd(m,3)=1$.

Extension of a result of Evans and Ramanathan II

Theorem

Let $|q| < 1$ *and* $Q(z, q)$ *be as at* (1).
If m *is a positive integer of the form* $6t + 2$, *then*

$$(q;q)_\infty = \sum_{u=0}^{\frac{m-2}{6}} (-1)^u q^{u(3u-1)/2} Q\left(q^{m(m-6u+1)/6}; q^{m^2}\right) + \sum_{u=1}^{\frac{m-2}{3}} (-1)^u q^{u(3u+1)/2} Q\left(q^{m(m+6u+1)/6}; q^{m^2}\right), \quad (12)$$

with a similar formula if m *is a positive integer of the form* $6t + 4$.

Example

If we set $m = 20$ in the previous theorem we get the 10-dissection of $(q;q)_\infty$ (note however that some quintuple products occur in pairs):

$$\begin{aligned}(q;q)_\infty &= Q(q^{70},q^{400}) - qQ(q^{50},q^{400}) - q^2Q(q^{90},q^{400})\\ &+ q^5Q(q^{30},q^{400}) + q^7Q(q^{110},q^{400}) - q^{12}Q(q^{10},q^{400}) - q^{15}Q(q^{130},q^{400})\\ &+ q^{26}Q(q^{150},q^{400}) - q^{40}Q(q^{170},q^{400}) + q^{57}Q(q^{190},q^{400})\end{aligned}$$

Remark: It is + that this 10-dissection can be derived from the 5-dissection via the following special cases of the main theorem:

$$\begin{aligned}Q(z,q) &= Q\left(\frac{q}{z^2},q^4\right) + qz^3Q\left(\frac{1}{qz^2},q^4\right),\\ &= Q(q^2z^4,q^{16}) + +qz^3Q(q^6z^4,q^{16})\\ &\quad + q^5z^6Q(q^{10}z^4,q^{16}) + q^{12}z^9Q(q^{14}z^4,q^{16}).\end{aligned}$$

Application 2. Proof of Hirschhorn's conjecture on the 2^n dissection of $(q, q)_\infty$

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Hirschhorn's Conjecture I

On page 332 of *The power of q*, Hirschhorn stated the following conjecture, now a theorem, proved by us and also in the following paper (by a different method):

Sarmah, B. K.; Gayan, S. *A proof of Hirschhorn's conjecture on 2^n-dissection of Euler's product.* Bull. Aust. Math. Soc. (First published online 10/8/2024).

Hirschhorn's Conjecture II

Theorem (Hirschhorn's conjecture, continued next slide)

Let $n \geq 1$ *be an integer and let* $m = 2^n$*. Then the* m*-dissection of* $(q;q)_\infty$ *is give by*

$$(q;q)_\infty = \sum_{k=1}^{m} (-1)^{k+\epsilon} q^{c_k} (q^{2(2k-1)m}, q^{8m^2-2(2k-1)m}; q^{8m^2})_\infty \times (q^{2m^2-(2k-1)m}, q^{2m^2+(2k-1)m}, q^{4m^2}; q^{4m^2})_\infty, \quad (13)$$

Hirschhorn's Conjecture II

Theorem (Hirschhorn's conjecture continued)

where $\epsilon = 0$ *(respectively, 1) if* n *is odd (respectively, even),and for* $k = 1, 2, 3, \ldots, 2^n$,

$$c_k = \begin{cases} P\left(\frac{2m-1}{3} - (k-1)\right), & \text{if } n \text{ is odd,} \\ \\ P\left(-\frac{2m-2}{3} + (k-1)\right), & \text{if } n \text{ is even,} \end{cases} \qquad (14)$$

where $P(t) = t(3t-1)/2$.

Hirschhorn's Conjecture III

sketch of proof.

Since $Q(q,q^4)=(q,q^3,q^4;q^4)_\infty(q^2,q^6;q^8)_\infty=(q;q)_\infty$, let $j=1$ and $M=4$ in Corollary 4 to get for $m=2^n\equiv -1 \pmod 3$, or n odd ($m=2^n\equiv 1 \pmod 3$, or n even, is similar):

$$(q;q)_\infty=\sum_{r=0}^{m-1} q^{r(6r+1)}\left(q^{m(2m-12r-1)/3},q^{m(10m+12r+1)/3},q^{4m^2};q^{4m^2}\right)_\infty \times\left(q^{2m(8m-12r-1)/3},q^{2m(4m+12r+1)/3};q^{8m^2}\right)_\infty. \quad (15)$$

The difference between this m-dissection of $(q;q)_\infty$ and Hirschhorn's is that this one has some negative exponents.
When the negative exponents are removed, Hirschhorn's result follows. □

Hirschhorn's Conjecture IV

sketch of proof continued.

Divide the summation interval $0 \le r \le m-1$ into three sub-intervals in which none, exactly one or exactly two of these exponents are negative:

$$0 \le r < \frac{2m-1}{12}, \qquad \frac{2m-1}{12} < r < \frac{8m-1}{12}, \qquad \frac{8m-1}{12} < r \le m-1.$$

If k, $1 \le k \le m$ denotes the summation variable in Hirschhorn's m-dissections, then the three r-subintervals correspond respectively to the following k-subintervals:

even k in the interval $[2(m+1)/3, m]$,

odd k in the interval $[1, m-1]$,

even k in the interval $[2, 2(m-2)/3]$.

Collectively, these cover all k in the interval $1 \le k \le m$. □

Hirschhorn's Conjecture V

Example

The 4-dissection of $(q;q)_\infty$ was stated by Hirschhorn in "The power of q" [6, page 332], when developing the conjecture. Here are the 4- and 8-dissections:

$$(q;q)_\infty = Q(q^{12},q^{64}) - qQ(q^4,q^{64}) - q^2Q(q^{20},q^{64}) + q^7Q(q^{28},q^{64}),$$

$$\begin{aligned} &= Q(q^{40},q^{256}) - qQ(q^{56},q^{256}) - q^2Q(q^{24},q^{256}) \\ &\quad + q^5Q(q^{72},q^{256}) + q^7Q(q^8,q^{256}) - q^{12}Q(q^{88},q^{256}) \\ &\quad + q^{22}Q(q^{104},q^{256}) - q^{35}Q(q^{120},q^{256}) \end{aligned}$$

Application 3. Partition Identities

Partition Identities I

Recall that $b_m(n)$ is the number of m-regular partitions of n (partitions with no parts $\equiv 0 \pmod{m}$).

Theorem

Let $m \geq 5$ be an integer relatively prime to 6 and square-free, and let S be the set of positive integers containing no multiples of m.

Define $D_S(n)$ to be number of partitions of n into an even number of distinct parts from S minus the number of partitions of n into an odd number of distinct parts from S.

Define $r = (m^2 - 1)/24$.

If $m \equiv 1 \pmod{6}$ set $s = (m-1)/6$ and if $m \equiv -1 \pmod{6}$ set $s = (m+1)/6$.

Then

$$D_S(mn + r) = (-1)^s b_m(n), \qquad \text{for all } n \geq 0. \tag{16}$$

Partition Identities II

Partial Proof.

From the m-dissection of $(q;q)_\infty$ proved by Evans [4] and Ramanathan [10] (slightly reformulated): If m is a positive integer of the form $6t+1$ (m of the form $6t-1$ is similar), then

$$(q;q)_\infty = (-1)^{(m-1)/6} q^{(m^2-1)/24} \left(q^{m^2};q^{m^2}\right)_\infty + \sum_{u=0}^{\frac{m-1}{3}} (-1)^u q^{u(3u-1)/2} Q\left(-q^{m(m+6u-1)/6}; q^{m^2}\right) + \sum_{u=1}^{\frac{m-7}{6}} (-1)^u q^{u(3u+1)/2} Q\left(-q^{m(m-6u-1)/6}; q^{m^2}\right). \quad (17)$$

Upon dividing both sides by $(q^m;q^m)_\infty$ the left side becomes (continued next slide)

Partition Identities III

Partial Proof Continued.

$$\frac{(q;q)_\infty}{(q^m;q^m)_\infty} = (q,q^2,\dots,q^{m-1};q^m)_\infty = \sum_{n=0}^{\infty} D_S(n)q^n.$$

Only the first term on the right has powers of q with exponent $\equiv r = (m^2-1)/24 \pmod{m}$, and thus

$$(-1)^s q^r \frac{\left(q^{m^2};q^{m^2}\right)_\infty}{(q^m;q^m)_\infty} = \sum_{n=0}^{\infty} D_S(mn+r)q^{mn+r}.$$

Upon cancelling q^r both sides followed by $q \to q^{1/m}$,one gets

$$\sum_{n=0}^{\infty} D_S(mn+r)q^n = (-1)^s \frac{(q^m;q^m)_\infty}{(q;q)_\infty} = (-1)^s \sum_{n=0}^{\infty} b_m(n)q^n.$$

□

Partition Identities IV

Example

Take $m = 7$, so $s = (7-1)/6 = 1$ and $r = (7^2-1)/24 = 2$.

Let S be the set of positive integers which are not multiples of 7.

If we take $n = 13$, then $mn + r = 7(13) + 2 = 93$.

There are 44530 partitions of 93 into an even number of distinct parts from S, and there are 44620 partitions of 93 into an odd number of distinct parts from S.

Hence

$$D_S(93) = 44530 - 44620 = -90 = (-1)^1 90,$$

in agreement with (16), since $b_7(13) = 90$.

Recall: A Special Case

Recall:

Corollary

Let $|q| < 1$ and let $M > 3$ be an integer and let j is a positive integer satisfying $1 \leq j < M/2$. Let m be a positive integer such that $3 \nmid m$.
(i) If $m \equiv 1 \pmod 3$, then

$$\left(q^j, q^{M-j}, q^M; q^M\right)_\infty \left(q^{M-2j}, q^{M+2j}; q^{2M}\right)_\infty = \sum_{r=0}^{m-1} q^{M(3r-1)r/2+3jr}$$
$$\times \left(q^{mM(m+6r-1)/6+jm}, q^{m^2M-mM(m+6r-1)/6-jm}, q^{m^2M}; q^{m^2M}\right)_\infty$$
$$\times \left(q^{m^2M+2jm+M(m+6r-1)m/3}, q^{m^2M-2jm-mM(m+6r-1)/3}; q^{2m^2M}\right)_\infty. \quad (18)$$

(ii) If $m \equiv 2 \pmod 3$, then there is a similar formula.

Partition Identities V

For a positive integer $M \geq 5$ and a positive integer $a < M/2$ define, for any positive integer n, define $P_{a,M}(n) :=$ the number of partitions of n into parts $\not\equiv \pm a, 0 \pmod{M}$, $\not\equiv M \pm 2a \pmod{2M}$.

Example

Let S be the set of positive integers $\equiv \pm 1, \pm 3, \pm 4 \pmod{10}$. Then

$$\begin{aligned} D_S(5n) &= P_{4,25}(n), \qquad (19)\\ D_S(5n+1) &= -P_{6,25}(n),\\ D_S(5n+2) &= P_{9,25}(n-1),\\ D_S(5n+3) &= -P_{1,25}(n),\\ D_S(5n+4) &= -P_{11,25}(n-2), \end{aligned}$$

Proof: Set $M = m = 5$ and $j = 1$ in Corollary 12 and divide both sides by $(q^5; q^5)_\infty$.

Partition Identities VI

Example (Example continued)

Let S be the set of positive integers $\equiv \pm 1, \pm 3, \pm 4 \pmod{10}$.
Take $n = 15$ so $5n = 75$. Then

$$D_S(5n) = P_{4,25}(n)$$
$$\implies D_S(75) = P_{4,25}(15)$$

There are 895 partitions of 75 into an even number of distinct parts from S and 775 partitions of 75 into an odd number of distinct parts from S.
Hence $D_S(75) = 895 - 775 = 120$.
$P_{4,25}(15)$ is the number of partitions of 15 into parts $\not\equiv 4, 21, 0 \pmod{25}$, $\not\equiv 17, 33 \pmod{50}$, so $P_{4,25}(15)$ equals the number of partitions of 15 with no part equal to 4, which is indeed 120.

Partition Identities VII

Example

Let S denote the set of positive integers with no multiples of 5. Recall that $b_5(n)$ is the number of 5-regular partitions of n, and for $a \in \{1, 2\}$, let $p_{a;5}(n)$ denote the number of partitions of n into parts $\equiv \pm a \pmod 5$, where parts come in two colours. Then

- $D_S(5n) = p_{1;5}(n)$;
- $D_S(5n+1) = -b_5(n)$;
- $D_S(5n+2) = -p_{2;5}(n)$;
- $D_S(5n+3) = D_S(5n+4) = 0$.

Proof: Start with the 5-dissection of $(q;q)_\infty$*:*

$$(q;q)_\infty = -q\left(q^{25};q^{25}\right)_\infty + \left(-q^5,-q^{20},q^{25};q^{25}\right)_\infty\left(q^{15},q^{35};q^{50}\right)_\infty - q^2\left(-q^{10},-q^{15},q^{25};q^{25}\right)_\infty\left(q^5,q^{45};q^{50}\right)_\infty$$

and then divide both sides by $(q^5;q^5)_\infty$.

Example (continued, some explicit examples)

(a) $75 = 5(15)$ has 6140 partitions into an <u>even</u> number of distinct parts from S and 5944 partitions into an <u>odd</u> number of distinct parts from S, so that $D_S(75) = 6140 - 5944 = 196$, in agreement with $p_{1;5}(15) = 196$.

(b) $76 = 5(15) + 1$ has 6506 partitions into an <u>even</u> number of distinct parts from S and 6633 partitions into an <u>odd</u> number of distinct parts from S, so that $D_S(76) = 6506 - 6633 = -127 = -b_5(15)$.

(c) $78 = 5(15) + 3$ has 7755 partitions into an <u>even</u> number of distinct parts from S and also 7755 partitions into an <u>odd</u> number of distinct parts from S, so that $D_S(78) = 7755 - 7755 = 0$

Application 4. Periodicity of Sign Changes in the Series Expansion of Various Eta Quotients

Application 4. Periodicity of Sign Changes in the Series Expansion of Various Eta Quotients

Periodicity of Sign Changes (Sample Result) I

Theorem

Let $p > 3$ be a prime. For $k \geq 1$, write

$$\frac{(q^{2^{k-1}}; q^{2^{k-1}})_\infty}{(q^p; q^p)_\infty} = \sum_{n=0}^{\infty} a_n q^n.$$

Then if $p \equiv 1 \pmod 3$, for each integer r in the indicated intervals there exists a computable integer $\mathcal{L}(r,k)$ such that if $n \geq \mathcal{L}(r,k)$, one has that

$a_n \geq 0$ *if* $n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p$ *with* $0 \leq r < \frac{4(2p+1)-6}{24}$ *or* $\frac{4(5p+1)-6}{24} < r \leq p-1$,

$a_n = 0$ *if* $n \not\equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p$,

$a_n \leq 0$ *if* $n \equiv 3 \cdot 2^k r^2 + 2^{k-1} r \pmod p$ *with* $\frac{4(2p+1)-6}{24} < r < \frac{4(5p+1)-6}{24}$.

A similar statement holds if $p \equiv -1 \pmod 3$

Basic Idea of Proof I

Recall

Corollary

Let $|q| < 1$ and let $M > 3$ be an integer and let j is a positive integer satisfying $1 \leq j < M/2$. Let m be a positive integer such that $3 \nmid m$.
(i) If $m \equiv 1 \pmod 3$, then

$$\left(q^j, q^{M-j}, q^M; q^M\right)_\infty \left(q^{M-2j}, q^{M+2j}; q^{2M}\right)_\infty = \sum_{r=0}^{m-1} q^{M(3r-1)r/2+3jr}$$
$$\times \left(q^{mM(m+6r-1)/6+jm}, q^{m^2M-mM(m+6r-1)/6-jm}, q^{m^2M}; q^{m^2M}\right)_\infty$$
$$\times \left(q^{m^2M+2jm+M(m+6r-1)m/3}, q^{m^2M-2jm-mM(m+6r-1)/3}; q^{2m^2M}\right)_\infty. \quad (20)$$

(ii) If $m \equiv 2 \pmod 3$, then there is a similar formula.

There is a version of this (technically a little messy) with the negative exponents removed, in which we set $j = 2^{k-1}$, $M = 2^{k+1}$ and $m = p \equiv 1 \pmod 3$, p a prime, to get

$$\begin{aligned}
&\left(q^{2^{k-1}}, q^{2^{k+1}-2^{k-1}}, q^{2^{k+1}}; q^{2^{k+1}}\right)_\infty \left(q^{2^{k+1}-2\cdot 2^{k-1}}, q^{2^{k+1}+2\cdot 2^{k-1}}; q^{2^{k+2}}\right)_\infty \\
&= (q^{2^{k-1}}; q^{2^{k-1}})_\infty \\
&= \sum_{r=0}^{p-1} (-1)^{s(r)} q^{\mathcal{L}(r)} (q^{t_1(r)}, q^{2^{k+1}p^2 - t_1(r)}, q^{2^{k+1}p^2}; q^{2^{k+1}p^2})_\infty \\
&\qquad \times (q^{t_2(r)}, q^{2^{k+2}p^2 - t_2(r)}; q^{2^{k+2}p^2})_\infty,
\end{aligned}$$

where the parameters $t_1(r)$, $t_2(r)$, $s(r)$ and $\mathcal{L}(r)$ are related to the technicalities of removing negative exponents.

Note for what comes next that $t_1(r)$ and $t_2(r)$ are positive multiples of p.

Basic Idea of Proof III

After dividing through by $(q^p; q^p)_\infty$, one gets

$$\frac{(q^{2^{k-1}}; q^{2^{k-1}})_\infty}{(q^p; q^p)_\infty} = \sum_{r=0}^{p-1} (-1)^{s(r)} q^{\mathcal{L}(r)}$$
$$\times \frac{(q^{t_1(r)}, q^{2^{k+1}p^2 - t_1(r)}, q^{2^{k+1}p^2}; q^{2^{k+1}p^2})_\infty (q^{t_2(r)}, q^{2^{k+2}p^2 - t_2(r)}; q^{2^{k+2}p^2})_\infty}{(q^p; q^p)_\infty}.$$

After expanding

$$(q^p; q^p)_\infty = \prod_{1 \leq \ell \leq 2^{k+2}p} \frac{1}{(q^{p\ell}; q^{2^{k+2}p^2})_\infty}$$

all the products in the numerator of each term in the sum on the right side cancel, so that the series expansions of the infinite products all have non-negative coefficients, so that all coefficients in the r-th term of the sum have the same sign as $(-1)^{s(r)}$.

Basic Idea of Proof IV

Remark

Note that setting $k = 1$ in

$$\frac{(q^{2^{k-1}}; q^{2^{k-1}})_\infty}{(q^p; q^p)_\infty}$$

recovers the result of Andrews [1, Theorem 2.1] and Borwein [2] on the nonnegativity of $c_n c_{n+p}$, where

$$\sum_{n=0}^{\infty} c_n q^n = \frac{(q; q)_\infty}{(q^p; q^p)_\infty}.$$

Concluding Thoughts

It may be illuminating to find combinatorial proofs of some of the partition identities.

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Thanks

Thank you for listening/watching/attending.