Sequentially Congruent Partitions and Partitions into Squares

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Opening Comments

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  - Discuss the work of Schneider and Schneider and introduce the concept of sequentially congruent partitions
  - Discuss the work of Schneider, Sellers, and Wagner which extends the work of Schneider and Schneider and also incorporates partitions into parts which are squares.

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$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r),\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1.$

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$\ell(\lambda) := r$ denotes the *length* or number of parts of the partition.

$|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r$ denotes the *size* or *weight* of the partition.

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In that case, we will write $\lambda = (1^{m_1}\ 2^{m_2}\ 3^{m_3}\ 4^{m_4} \dots)$ where $m_i = m_i(\lambda) :=$ *frequency* or *multiplicity* of part i in λ.

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We next provide one of the most important definitions for this talk.

Definitions and Notation

Definition: The partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ is *sequentially congruent* if:

$$
\begin{aligned}
\lambda_1 &\equiv \lambda_2 \pmod{1}, \\
\lambda_2 &\equiv \lambda_3 \pmod{2}, \\
\lambda_3 &\equiv \lambda_4 \pmod{3}, \\
&\vdots \\
\lambda_{r-1} &\equiv \lambda_r \pmod{r-1},
\end{aligned}
$$

and for the smallest part, $\lambda_r \equiv 0 \pmod{r}$.

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We let $\mathcal{S} \subset \mathcal{P}$ be the set of sequentially congruent partitions.

We let $\mathcal{S}_{lg=n}$ be the set of sequentially congruent partitions with **largest part** equal to n.

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Thanks to recent work of Schneider and Schneider (*Annals of Combinatorics*, 2019), we see that these partitions actually do fit quite nicely!

Work of Schneider and Schneider

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Theorem (Schneider and Schneider):

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Schneider and Schneider provide a wonderful constructive / bijective proof of the theorem above.

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Proof: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of weight n.

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Note that $\lambda_r' \equiv 0 \pmod{r}$ since the sum in the formula above is empty when $i = r$. The other congruences between successive parts of λ' are immediate from the formula above since, for $i < r$,

$$\lambda_i' - \lambda_{i+1}' = i(\lambda_i - \lambda_{i+1}).$$

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Lastly, note that

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I note, in passing, that Schneider and Schneider (2019) also shared a generating function argument for the above theorem.

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Definition: A partition is *frequency congruent* if it has the property that each part divides its frequency.

Let $\mathcal{F}$ be the set of frequency congruent partitions, and let $\mathcal{F}_{\ell=n}$ be the set of frequency congruent partitions of **length** n.

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Theorem (Schneider and Schneider, 2019): For each n,

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Proof: Conjugation!

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Example: Earlier, we noted that $(20, 17, 15, 9, 5)$ is a partition in $\mathcal{S}_{lg=20}$. The conjugate of this partition, written in frequency notation, is $(1^3\ 2^2\ 3^6\ 4^4\ 5^5)$ which is clearly a frequency congruent partition and has exactly 20 parts, so that it is an element of $\mathcal{F}_{\ell=20}$.

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Corollary: The number of partitions in $\mathcal{F}_{\ell=n}$ equals $p(n)$.

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It seemed natural (even if naive) to ask the question: For a specific value of n, how many sequentially congruent partitions of **weight** n are there?

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I shared what I found with Robert that evening at dinner, and what I will share with you now was the work that followed (between me, Robert, and Ian Wagner).

Work of Schneider, Sellers, and Wagner

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Theorem (Schneider, Sellers, Wagner): For each $n \geq 1$,

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We begin with a frequency congruent partition λ^* of weight n.

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Thus, we know that each part in the partition divides its frequency. So we can write λ^* as

$$\lambda^* = (1^{1 \cdot e_1} \; 2^{2 \cdot e_2} \dots \; i^{i \cdot e_i} \; \dots)$$

where each $e_i \geq 0$.

Work of Schneider, Sellers, and Wagner

Next, we define a map $\pi : \mathcal{F} \to \mathcal{P}_{\square}$ taking $\lambda^* \in \mathcal{F}$ to the partition

$$\pi(\lambda^*) = ((1^2)^{e_1}\ (2^2)^{e_2} \dots\ (i^2)^{e_i} \dots) \in \mathcal{P}_{\square}.$$

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Note also that $|\lambda^*| = |\pi(\lambda^*)| = n$ since, for each $i \geq 1$, the sum of corresponding parts is preserved:

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Note that $\pi(\lambda^*)$ is indeed a partition into square parts.

Note also that $|\lambda^*| = |\pi(\lambda^*)| = n$ since, for each $i \geq 1$, the sum of corresponding parts is preserved:

$$\underbrace{i + i + \cdots + i}_{i \cdot e_i\ times} = i \cdot (\underbrace{i + i + \cdots + i}_{e_i\ times}) = \underbrace{i^2 + i^2 + \cdots + i^2}_{e_i\ times}$$

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Let's consider an example of this bijection.

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Using the "squaring" map π defined above, λ^* is mapped to the partition

$$(1^2\ 4^3\ 16^2\ 25^1) \in \mathcal{P}_{\Box}.$$

Work of Schneider, Sellers, and Wagner

Composing these two maps means that the sequentially congruent partition $(21, 19, 13, 13, 5)$ gets mapped to the partition into squares $(1^2\ 4^3\ 16^2\ 25^1)$, both of which are partitions of weight 71.

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This bijection is visually evident in the Young diagram of the sequentially congruent partition, which we shade to highlight that it breaks down into a concatenation of perfect squares:

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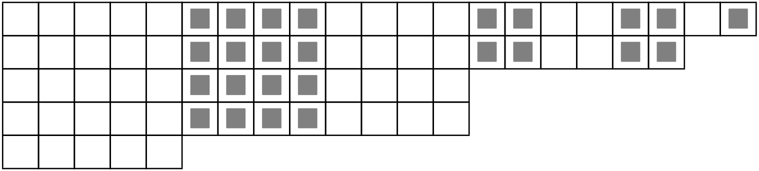

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1. $\lambda_i - \lambda_{i+1} = j \cdot i^k$ for $1 \leq i \leq r-1$; and
2. $\lambda_r = j \cdot r^k$.

Theorem: The partitions $\lambda \in \mathcal{S}(j,k)$ of weight n are in bijection with partitions of n into $(k+1)$st powers where each part occurs exactly j times.

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Proof: A very similar map to π, which we will denote by $\pi_{j,k}$, can be employed to give this result.

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Notice that the conjugate λ^* of a partition $\lambda \in \mathcal{S}(j,k)$ is a partition where $m_i(\lambda^*) = j \cdot i^k \epsilon_i$, with $\epsilon_i = \epsilon_i(\lambda^*) = 1$ if i is a part of λ^* and $\epsilon_i = 0$ otherwise.

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Notice that the conjugate λ^* of a partition $\lambda \in \mathcal{S}(j,k)$ is a partition where $m_i(\lambda^*) = j \cdot i^k \epsilon_i$, with $\epsilon_i = \epsilon_i(\lambda^*) = 1$ if i is a part of λ^* and $\epsilon_i = 0$ otherwise.

Then we define the map $\pi_{j,k}$ as follows:

Closing Thoughts and Questions

$$\pi_{j,k}(\lambda^*) = \pi_{j,k}((1^{j\cdot 1^k \epsilon_1}\ 2^{j\cdot 2^k \epsilon_2}\ 3^{j\cdot 3^k \epsilon_3} \dots\ i^{j\cdot i^k \epsilon_i} \dots))$$
$$= ((1^{k+1})^{j\epsilon_1}\ (2^{k+1})^{j\epsilon_2}\ (3^{k+1})^{j\epsilon_3} \dots\ (i^{k+1})^{j\epsilon_i} \dots).$$

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Noting that the map $\pi_{j,k}$ is both weight-preserving and reversible, much like the map π, completes the proof of the bijection. □

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