Congruences for k-Elongated Partition Diamonds

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Acknowledgements

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- Describe work that Robson and I have since completed to prove infinitely many additional congruences for these objects using truly elementary methods
- Close with some thoughts on possible future work

Introductory Thoughts

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In that paper, Andrews and Paule introduced the idea of k-elongated partition diamonds.

Recently, Andrews and Paule revisited the topic of k–elongated partition diamonds.

Introductory Thoughts

Using partition analysis and the Omega operator, Andrews and Paule proved the generating function for the partition numbers $d_k(n)$ produced by summing the links of k–elongated plane partition diamonds of length n.

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$$\sum_{n=0}^{\infty} d_k(n)q^n = \frac{f_2^k}{f_1^{3k+1}}$$

where $f_r = (q^r;q^r)_\infty$ is the usual q–Pochhammer symbol.

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where $f_r = (q^r;q^r)_\infty$ is the usual q–Pochhammer symbol.

They then proceeded to prove several (individual) congruence properties satisfied by d_1, d_2 and d_3 using modular forms and Nicolas Smoot's Mathematica package as their primary proof tools.

Introductory Thoughts

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Theorem (Smoot): For all $\alpha \geq 0$ and all $n \geq 0$ such that $8n \equiv 1 \pmod{3^\alpha}$,

$$d_2(n) \equiv 0 \pmod{3^{2\lfloor \alpha/2 \rfloor + 1}}.$$

Introductory Thoughts

Our goal is to extend some of the results proven by Andrews and Paule by proving infinitely many congruence properties satisfied by the functions d_k for an **infinite** set of values of k.

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The proof techniques employed below are all elementary, relying on generating function manipulations and classical q-series results.

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Our goal is to extend some of the results proven by Andrews and Paule by proving infinitely many congruence properties satisfied by the functions d_k for an **infinite** set of values of k.

The proof techniques employed below are all elementary, relying on generating function manipulations and classical q-series results.

We require a number of well–known lemmas in order to complete our proofs:

Introductory Thoughts

Lemma:

$$\frac{f_2^2}{f_1} = \sum_{m \geq 0} q^{m(m+1)/2}$$

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Lemma:

$$f_1 = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}.$$

Introductory Thoughts

Lemma:

$$\frac{f_1^5}{f_2^2} = \sum_{m=-\infty}^{\infty} (6m+1) q^{m(3m+1)/2}.$$

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Lemma:

$$\begin{aligned}
\frac{f_1^2}{f_2} &= \sum_{j=-\infty}^{\infty} q^{j^2}, \\
&= \frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}, \\
&= \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}.
\end{aligned}$$

Introductory Thoughts

Lemma:

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}.$$

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$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}.$$

Lemma:

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}.$$

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Theorem: (Andrews and Paule, Corollary 5) For all $n \geq 0$, $d_2(3n+2) \equiv 0 \pmod 3$.

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As a first example, we note the following:

Theorem: (Andrews and Paule, Corollary 5) For all $n \geq 0$, $d_2(3n+2) \equiv 0 \pmod{3}$.

As we mentioned earlier, Andrews and Paule used significant tools based on the work of Smoot, which are derived from modular forms, in order to prove their congruences for d_2 and d_3.

Elementary Proofs of Several Congruences from Andrews and Paule

Their proof of this theorem is that

$$g_1 \cdot \sum_{m=0}^{\infty} d_2(3m+2)q^m = 3(8+t)(1728+288t+11t^2),$$

where

$$g_1 = \frac{1}{q^3}\frac{(q;q)_\infty^{19}(q^2;q^2)_\infty(q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^{21}}$$

and

$$t = \frac{1}{q}\frac{(q;q)_\infty^5(q^3;q^3)_\infty}{(q^2;q^2)_\infty(q^6;q^6)_\infty^5}.$$

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Our proof of this result is very different.

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Proof:

$$\begin{aligned}
\sum_{n=0}^{\infty} d_2(n)q^n &= \frac{f_2^2}{f_1^7} \\
&= \frac{f_2^2}{f_1}\frac{1}{f_1^6} \\
&\equiv \frac{1}{f_3^2}\left(\sum_{m\geq 0} q^{m(m+1)/2}\right) \pmod{3}.
\end{aligned}$$

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Now we simply need to determine whether

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This congruence never holds because 2 is a quadratic nonresidue modulo 3. $\square$

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Theorem: (Andrews and Paule, Corollary 10) For all $n \geq 0$, $d_3(2n+1) \equiv 0 \pmod{2}$.

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Proof: Note that

$$\begin{aligned}
\sum_{n=0}^{\infty} d_3(n) q^n &= \frac{f_2^3}{f_1^{10}} \\
&\equiv \frac{f_2^3}{f_2^5} \pmod{2} \\
&\equiv \frac{1}{f_2^2} \pmod{2}.
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Since $\dfrac{1}{f_2^2}$ is an even function of q, the result follows. $\square$

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Proof: Thanks to the proof of the previous result, we know

$$\begin{aligned}\sum_{n=0}^{\infty} d_3(n)q^n &\equiv \frac{1}{f_2^2} \pmod{2} \\ &\equiv \frac{1}{f_4} \pmod{2}\end{aligned}$$

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Since $\frac{1}{f_4}$ is a function of q^4, the result follows. $\square$

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Proof: We have

$$\begin{aligned}\sum_{n=0}^{\infty} d_3(n)q^n &= \frac{f_2^3}{f_1^{10}} = \frac{f_2^4}{f_1^{12}}\frac{f_1^2}{f_2} \\ &\equiv \frac{f_2^4}{f_2^6}\frac{f_1^2}{f_2} \pmod{4} \\ &= \frac{1}{f_2^2}\left(\frac{f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_{16}^2}{f_8}\right).\end{aligned}$$

Elementary Proofs of Several Congruences from Andrews and Paule

Extracting the odd parts, dividing by q and replacing q^2 by q, we are left with

$$\sum_{n=0}^{\infty} d_3(2n+1)q^n \equiv 2\frac{f_8^2}{f_1^2 f_4} \equiv 2f_2^5 \pmod{4}.$$

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Since f_2 is a function of q^2 the result follows. $\square$

Elementary Proofs of Several Congruences from Andrews and Paule

Theorem: (Andrews and Paule, Corollary 14) For all $n \geq 0$, $d_3(5n+1) \equiv 0 \pmod{5}$.

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Proof: We have

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\sum_{n=0}^{\infty} d_3(n)q^n &= \frac{f_2^3}{f_1^{10}} \\
&\equiv \frac{f_2^3}{f_5^2} \pmod{5} \\
&= \frac{1}{f_5^2}\left(\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)}\right).
\end{aligned}$$

Elementary Proofs of Several Congruences from Andrews and Paule

We now need to ask whether $5n+1$ can be represented as $m(m+1)$, and this is equivalent to asking whether $4(5n+1)+1$ or $20n+5$ can be represented as $(2m+1)^2$.

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If this is the case, then we know

$$(2m+1)^2 = 20n+5 \equiv 0 \pmod{5}$$

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Thanks to the presence of the coefficient of $2m+1$ in front of the term $q^{m(m+1)}$ in the series above, and the fact that this $2m+1$ must be divisible by 5, we know that our result follows. $\square$

Elementary Proofs of Several Congruences from Andrews and Paule

Theorem: (Andrews and Paule, Corollary 15) For all $n \geq 0$,

$$\begin{aligned} d_3(5n+3) &\equiv 0 \pmod{5}, \\ d_3(5n+4) &\equiv 0 \pmod{5}. \end{aligned}$$

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Theorem: (Andrews and Paule, Corollary 15) For all $n \geq 0$,

$$\begin{aligned} d_3(5n+3) &\equiv 0 \pmod 5, \\ d_3(5n+4) &\equiv 0 \pmod 5. \end{aligned}$$

Proof: In the proof of the previous result, we noted that

$$\begin{aligned} &\sum_{n=0}^{\infty} d_3(n) q^n \\ \equiv\ & \frac{1}{f_5^2} \left(\sum_{m=0}^{\infty} (-1)^m (2m+1) q^{m(m+1)} \right) \pmod 5. \end{aligned}$$

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We now need to ask whether $5n + 3$ can be represented as $m(m + 1)$, and this is equivalent to asking whether $4(5n + 3) + 1$ or $20n + 13$ can be represented as $(2m + 1)^2$.

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This would mean that $(2m+1)^2 \equiv 3 \pmod{5}$.

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However, since 3 is a quadratic nonresidue modulo 5, we know that this cannot be the case.

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This would mean that $(2m + 1)^2 \equiv 3 \pmod 5$.

However, since 3 is a quadratic nonresidue modulo 5, we know that this cannot be the case.

Similarly, note that

$$4(5n + 4) + 1 = 20n + 17 \equiv 2 \pmod 5$$

and 2 is the other quadratic nonresidue modulo 5. $\square$

New “Individual” Congruences

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We begin with a theorem connecting $d_p(n)$ and $p(n)$, for each prime p, where $p(n)$ denotes the number of unrestricted partitions of n.

Theorem: Let p be a prime and let a and b be integers such that $a \nmid b$ and $p(an+b) \equiv 0 \pmod{p}$, for all $n \geq 0$. Then, for all $n \geq 0$,

$$d_p(an+b) \equiv 0 \pmod{p}.$$

New “Individual” Congruences

Proof: The generating function for $d_p(n)$ satisfies

$$\sum_{n=0}^{\infty} d_p(n)q^n = \frac{f_2^p}{f_1^{3p+1}} \equiv \frac{f_{2p}}{f_p^3}\frac{1}{f_1} \pmod{p}$$
$$= \frac{f_{2p}}{f_p^3}\sum_{n=0}^{\infty} p(n)q^n.$$

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Since $\dfrac{f_{2p}}{f_p^3}$ is a function of q^p and $p(an+b) \equiv 0 \pmod{p}$, the result follows. $\square$

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$$= \frac{f_{2p}}{f_p^3}\sum_{n=0}^{\infty} p(n)q^n.$$

Since $\frac{f_{2p}}{f_p^3}$ is a function of q^p and $p(an+b) \equiv 0 \pmod{p}$, the result follows. $\square$

Corollary: For all $n \geq 0$,

$$\begin{aligned} d_5(5n+4) &\equiv 0 \pmod{5}, \\ d_7(7n+5) &\equiv 0 \pmod{7}, \\ d_{11}(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

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Theorem: For all $n \geq 0$,

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\begin{aligned}
d_7(4n+2) &\equiv 0 \pmod{4}, \\
d_7(8n+5) &\equiv 0 \pmod{4}, \\
d_7(16n+9) &\equiv 0 \pmod{4}, \\
d_7(4n+3) &\equiv 0 \pmod{8}, \\
d_7(8n+4) &\equiv 0 \pmod{8}.
\end{aligned}
$$

New "Individual" Congruences

Theorem: Let $p \geq 5$ be a prime and let r, $1 \leq r \leq p-1$, be such that $3r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,

$$d_7(2pn+2r+1) \equiv 0 \pmod{4}.$$

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$$d_7(2pn + 2r + 1) \equiv 0 \pmod{4}.$$

Theorem: For all $n \geq 0$,

$$\begin{aligned} d_8(3n+2) &\equiv 0 \pmod{9}, \\ d_8(9n+3) &\equiv 0 \pmod{9}. \end{aligned}$$

New Infinite Families of Congruences

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Said differently (and, potentially, more clearly), we want to demonstrate families of congruences for "fixed" moduli where the subscripts k range over an infinite set (and the arithmetic progressions in question are "fixed" or follow a nice pattern).

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We begin with a somewhat surprising result, primarily because the moduli in question range across **all** primes $p \geq 5$.

New Infinite Families of Congruences

Theorem: Let $p \geq 5$ be a prime and let r, $1 \leq r \leq p-1$, be such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,

$$d_{p-2}(pn+r) \equiv 0 \pmod{p}.$$

New Infinite Families of Congruences

Theorem: Let $p \geq 5$ be a prime and let r, $1 \leq r \leq p-1$, be such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,

$$d_{p-2}(pn+r) \equiv 0 \pmod{p}.$$

Proof: The generating function for $d_{p-2}(n)$ satisfies

$$\sum_{n=0}^{\infty} d_{p-2}(n)q^n = \frac{f_2^{p-2}}{f_1^{3p-5}} \equiv \frac{f_1^5}{f_2^2}\frac{f_{2p}}{f_p^3} \pmod{p}.$$

New Infinite Families of Congruences

Theorem: Let $p \geq 5$ be a prime and let r, $1 \leq r \leq p-1$, be such that $24r+1$ is a quadratic nonresidue modulo p. Then, for all $n \geq 0$,

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Thanks to the lemma above from Ramanujan,

$$\sum_{n=0}^{\infty} d_{p-2}(n)q^n \equiv \frac{f_{2p}}{f_p^3}\sum_{m=-\infty}^{\infty}(6m+1)q^{m(3m+1)/2} \pmod{p}.$$

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Therefore $d_{p-2}(pn + r) \equiv 0 \pmod{p}$. $\square$

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We explain this additional "special" congruence for each prime $p \geq 5$ here.

Theorem: Let $p \geq 5$ be a prime and let t, $1 \leq t \leq p-1$, be the unique value such that $24t+1 \equiv 0 \pmod{p}$. Then, for all $n \geq 0$,

$$d_{p-2}(pn+t) \equiv 0 \pmod{p}.$$

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We now consider a second infinite family of congruences satisfied by these functions.

New Infinite Families of Congruences

Theorem: Let $p \geq 5$ be a prime and let r, $1 \leq r \leq p-1$, be a quadratic nonresidue modulo p. Then, for all $n \geq 0$,

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Proof: For prime $p \geq 5$,

$$\begin{aligned}\sum_{n=0}^{\infty} d_{p-1}(n)q^n &= \frac{f_2^{p-1}}{f_1^{3p-2}} = \frac{f_2^p}{f_1^{3p}}\frac{f_1^2}{f_2} \\ &\equiv \frac{f_{2p}}{f_p^3}\frac{f_1^2}{f_2} \pmod{p} \\ &= \frac{f_{2p}}{f_p^3}\left(\sum_{j=-\infty}^{\infty} q^{j^2}\right).\end{aligned}$$

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The result immediately follows by recognizing that $\frac{f_{2p}}{f_p^3}$ is a function of q^p and that r has been defined to be a quadratic nonresidue modulo p. $\square$

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The result is reminiscent of a result of Garvan and Sellers (2014) involving generalized Frobenius partitions with an arbitrarily large number of colors k.

Theorem: Let p be a prime, $k \geq 1$ and $j \geq 0$. Moreover, assume $p \mid a$ and $b \not\equiv 0 \pmod{a}$. If, for all $n \geq 0$,

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then for all $n \geq 0$,

$$d_{pj+k}(an+b) \equiv 0 \pmod{p}.$$

New Infinite Families of Congruences

Proof: From the generating function, we know

$$\sum_{n=0}^{\infty} d_{pj+k}(n)q^n = \frac{f_2^{pj+k}}{f_1^{3pj+3k+1}} = \frac{f_2^k}{f_1^{3k+1}} \frac{f_2^{pj}}{f_1^{3pj}}$$

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And $\dfrac{f_{2p}^j}{f_p^{3j}}$ is a function of q^p.

Since $d_k(an+b) \equiv 0 \pmod{p}$, it follows that

$$d_{pj+k}(an+b) \equiv 0 \pmod{p}.$$

□

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In what follows, the notation

$$An + B_1, B_2, \ldots, B_t$$

means we are considering the set of arithmetic progressions

$$An + B_1, An + B_2, \ldots, An + B_t.$$

New Infinite Families of Congruences

Corollary: For all $j \geq 0$ and $n \geq 0$,

$$\begin{aligned}
d_{2j+1}(2n+1) &\equiv 0 \pmod{2},\\
d_{2j+3}(4n+2) &\equiv 0 \pmod{2},\\
d_{3j+2}(3n+2) &\equiv 0 \pmod{3},\\
d_{5j+3}(5n+1,3,4) &\equiv 0 \pmod{5},\\
d_{5j+4}(5n+2,3) &\equiv 0 \pmod{5},\\
d_{7j+5}(7n+2,3,4,6) &\equiv 0 \pmod{7},\\
d_{7j+6}(7n+3,5,6) &\equiv 0 \pmod{7},\\
d_{11j+9}(11n+3,5,6,8,9,10) &\equiv 0 \pmod{11},\\
d_{11j+10}(11n+2,6,7,8,10) &\equiv 0 \pmod{11},\\
d_{13j+11}(13n+3,4,6,7,8,10,11) &\equiv 0 \pmod{13},\\
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d_{5j+4}(5n+2,3) &\equiv 0 \pmod{5}, \\
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d_{13j+11}(13n+3,4,6,7,8,10,11) &\equiv 0 \pmod{13}, \\
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The corollary above does not provide an exhaustive list of congruences satisfied by these functions.

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Our goal in writing these here is to provide a representative set of the kinds of congruences that arise within this family of partition functions.

Closing Questions/Thoughts

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- $d_7(n)$ modulo powers of 2
  - Remember all of those congruences I highlighted above for d_7 modulo small powers of 2.

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  - Remember all of those congruences I highlighted above for d_7 modulo small powers of 2.
- $d_{15}(n)$ modulo powers of 2

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Indeed, for $k = 2^j - 1$ for some j, it is clear that the generating function for $d_k(n)$ will have a structure that allows for a number of congruences to hold for small powers of 2.

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Let me also return to one of the first congruences I mentioned in this talk:

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One must wonder whether an **infinite** family of congruences, modulo powers of 2, like the family in the work of Nicolas Smoot, holds for these special values of k.

Let me also return to one of the first congruences I mentioned in this talk:

Theorem: (Andrews and Paule, Corollary 5) For all $n \geq 0$, $d_2(3n+2) \equiv 0 \pmod{3}$.

Closing Questions/Thoughts

Andrews and Paule's proof of this theorem is that

$$g_1 \cdot \sum_{m=0}^{\infty} d_2(3m+2)q^m = 3(8+t)(1728+288t+11t^2),$$

where

$$g_1 = \frac{1}{q^3} \frac{(q;q)_\infty^{19}(q^2;q^2)_\infty (q^3;q^3)_\infty^6}{(q^6;q^6)_\infty^{21}}$$

and

$$t = \frac{1}{q} \frac{(q;q)_\infty^5 (q^3;q^3)_\infty}{(q^2;q^2)_\infty (q^6;q^6)_\infty^5}.$$

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A few minutes ago, I noted that this result is the first case of a much larger family of congruences, namely,

Closing Questions/Thoughts

Theorem: (da Silva and JAS) For all $j \geq 0$ and $n \geq 0$, $d_{3j+2}(3n+2) \equiv 0 \pmod{3}$.

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This makes me wonder: What would the Andrews and Paule proof of this result for $d_5(3n+2)$ or $d_8(3n+2)$ or, for that matter, $d_{3j+2}(3n+2)$ look like?

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Closing Questions/Thoughts

Would there be a pattern to the proofs, so that one unified proof could be written down to prove the result for all $j \geq 0$ simultaneously?

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Lastly, what can be said about similar proofs (based on modular forms) for all of the other families of congruences I have demonstrated today?

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Lastly, what can be said about similar proofs (based on modular forms) for all of the other families of congruences I have demonstrated today?

And with that I will close. Thanks very much for attending today!

Congruences for k-Elongated Partition Diamonds

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