Arithmetic Properties of d–fold Partition Diamonds

James Sellers
University of Minnesota Duluth

jsellers@d.umn.edu

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- Our manuscript can be found at

https://arxiv.org/abs/2307.02579

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The partition function $p(n)$ counts the number of partitions of n.

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$$(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1)$$

Note that there are five partitions of $n = 4$, so we have $p(4) = 5$.

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$$\begin{tikzcd} a_0 \arrow[r] & a_1 \arrow[r] & a_2 \arrow[r] & a_3 \arrow[r] & a_4 \arrow[r] & a_5 \cdots \end{tikzcd}$$

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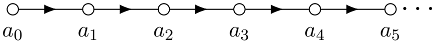

Here an arrowhead between two parts simply indicates the inequality of the parts in the partition.

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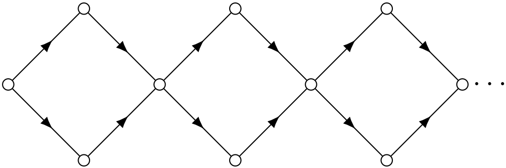
A plane partition diamond is a partition whose parts are nonnegative integers which are placed at the nodes of the graph below and sum to n.



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$$\sum_{n=0}^{\infty} d(n)q^n = \prod_{n=1}^{\infty} \frac{1+q^{3n-1}}{1-q^n}.$$

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The generating function for $s(n)$ is given by

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Definition: A d-fold partition diamond is a partition whose parts are non-negative integers a_i and $b_{i,j}$ which are placed at the nodes of this graph and sum to n. The corresponding counting function will be denoted $r_d(n)$.

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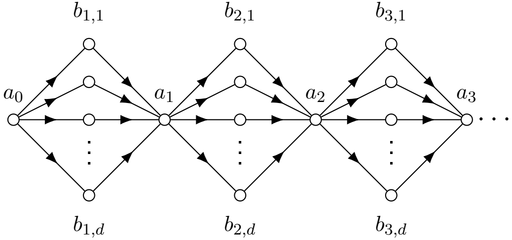

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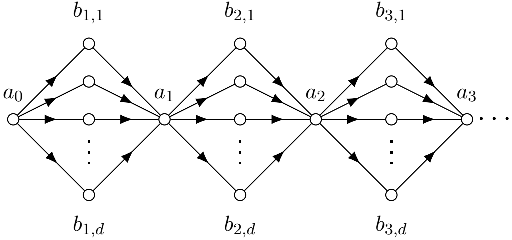
Definition: A Schmidt–type d-fold partition diamond is the same as above, except $\sum a_i = n$. The corresponding counting function will be denoted $s_d(n)$.



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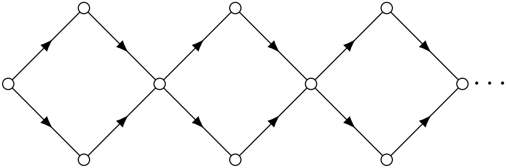
Andrews and Paule considered the Schmidt–type diamond case in their 2022 *JNT* paper.



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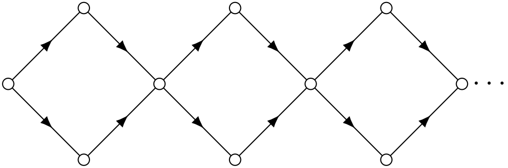
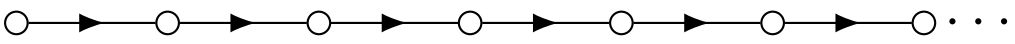


In their paper, among many other things, Andrews and Paule prove that the generating function in question is indeed

$$\prod_{n=1}^{\infty} \frac{1+q^n}{(1-q^n)^3}.$$

Background

Here are the first few examples for the Schmidt–type d-fold partition diamonds:



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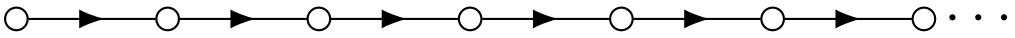

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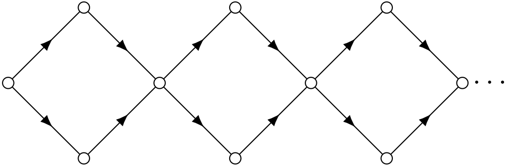

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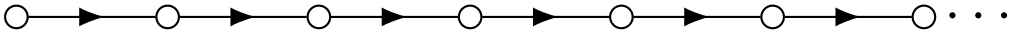

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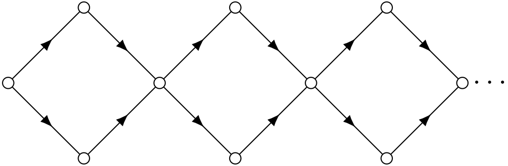

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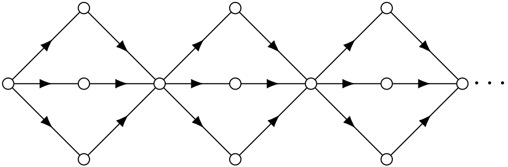

$$\prod_{n=1}^{\infty} \frac{1+4q^n+q^{2n}}{(1-q^n)^4}$$

Generating Function Results

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Theorem: (Dockery-Jameson-Sellers-Wilson) For $d \geq 1$,

$$\sum_{n=0}^{\infty} r_d(n)q^n = \prod_{n=1}^{\infty} \frac{F_d(q^{(n-1)(d+1)+1}, q)}{1-q^n}$$

$$\sum_{n=0}^{\infty} s_d(n)q^n = \prod_{n=1}^{\infty} \frac{A_d(q^n)}{(1-q^n)^{d+1}}$$

where F_d and A_d are recursively defined polynomials.

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$$A_1(x) = 1$$

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Also, $A_d(x) = (1 + (d-1)x)A_{d-1}(x) + x(1-x)A'_{d-1}(x)$.

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Theorem (Dockery-Jameson-Sellers-Wilson) For all $d \geq 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} s_d(n)q^n &= \prod_{n=1}^{\infty}\left(\sum_{j=0}^{\infty}(j+1)^d q^{jn}\right) \\ &= \prod_{n=1}^{\infty}\left(1+2^d q^n + 3^d q^{2n} + 4^d q^{3n} + \dots\right).\end{aligned}$$

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Proof: Euler!

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Theorem: (Jacobi)

$$\prod_{m=1}^{\infty}(1-q^m)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$$

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In the penultimate line above, we used the fact that $2^d \mid (2j)^d$ for all $j \geq 1$.

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The last expression above is a function of q^2, and this immediately implies our result. $\square$

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We note, in passing, that the $d = 2$ case of the above theorem was proven by Andrews and Paule in their 2022 *JNT* paper.

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We note, in passing, that the $d = 2$ case of the above theorem was proven by Andrews and Paule in their 2022 *JNT* paper.

We next prove the following overarching lemma that will provide us with the machinery needed to prove infinite families of divisibility properties satisfied by $s_d(n)$.

Arithmetic Properties

Lemma: Let k and r be nonnegative integers and let $p \geq 2$ be prime. For all $n \geq 0$,

$$s_{(p-1)k+r}(n) \equiv s_r(n) \pmod{p}.$$

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The penultimate line above follows from Fermat's Little Theorem. □

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&\equiv \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1-q^{5n})} \pmod 5
\end{aligned}$$

Arithmetic Properties

Thus, using the result of Jacobi mentioned above, we know

$$\sum_{n=0}^{\infty} s_1(n)q^n \equiv \left(\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)/2}\right)\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n})} \pmod 5.$$

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Since $\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n})}$ is a function of q^5, the above product will satisfy a congruence modulo 5 in an arithmetic progression of the form $5n+r$, for $0 \le r \le 4$, if and only if such a congruence is satisfied by

$$\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)/2}.$$

Arithmetic Properties

Since $5n+2$ and $5n+4$ are never triangular numbers, this immediately tells us that, for all $n \geq 0$,

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Moreover, we know that $5n+3 = j(j+1)/2$ if and only if $j \equiv 2 \pmod 5$, and in such cases, $2j+1 \equiv 0 \pmod 5$.

Arithmetic Properties

Since $5n+2$ and $5n+4$ are never triangular numbers, this immediately tells us that, for all $n \geq 0$,

$$s_1(5n+2) \equiv s_1(5n+4) \equiv 0 \pmod{5}.$$

Moreover, we know that $5n+3 = j(j+1)/2$ if and only if $j \equiv 2 \pmod{5}$, and in such cases, $2j+1 \equiv 0 \pmod{5}$.

Because of the presence of the factor $2j+1$ in the sum above, we then see that, for all $n \geq 0$,

$$s_1(5n+3) \equiv 0 \pmod{5}$$

since $2 \cdot 2 + 1 = 5 \equiv 0 \pmod{5}$.

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We note, in passing, that Baruah and Sarmah also provided a proof of the $k = 0$ case of the above theorem in their 2013 paper in the *Indian Journal of Pure and Applied Mathematics*.

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Theorem: For all $k \geq 0$ and all $n \geq 0$,

$$s_{4k+2}(25n + 23) \equiv 0 \pmod{5}.$$

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Remark: As above, it suffices to prove this result for s_2 as the "internal congruences" lemma can be invoked to handle all of the cases when $k \geq 1$.

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Proof: Note that

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\equiv & \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^n)}{(1-q^{5n})} \pmod{5}.
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Arithmetic Properties

Thanks to Euler's Pentagonal Number Theorem, this means

$$\sum_{n=0}^{\infty} s_2(n)q^n \equiv \left(\sum_{j,k=-\infty}^{\infty} q^{2(j(3j+1)/2)+k(3k+1)/2}\right)\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n})} \pmod 5.$$

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At this point, we consider those terms of the form q^{25n+23} in the power series representation of the last expression above.

There are no terms of the form $q^{25n+8}, q^{25n+13}, q^{25n+18}$, or q^{25n+23} in the double sum above, although there are terms of the form q^{25n+3}.

Arithmetic Properties

Thus, once we multiply the double sum above by

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^{5n})} = \sum_{n=0}^{\infty} p(n)q^{5n}, \tag{1}$$

we see that the only way to obtain a term of the form q^{25n+23} which can contribute to $s_2(25n+23)$ modulo 5 is to multiply by a term of the form $q^{5(5m+4)}$ from the power series representation of (1).

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This will then contribute, modulo 5, to the value of $s_2(25n+23)$ by multiplying by the value $p(5m+4)$.

And Ramanujan's famous mod 5 result for $p(5m+4)$ implies that, for all $n \geq 0$, $s_2(25n+23) \equiv 0 \pmod{5}$. $\square$

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We begin with the following generating function manipulations:

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&= \prod_{n=1}^{\infty} \frac{1+(-q^n)+(-q^n)^2}{(1-q^n)^4}
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&= \prod_{n=1}^{\infty} \frac{1+(-q^n)+(-q^n)^2}{(1-q^n)^4} \\
&= \prod_{n=1}^{\infty} \frac{1-(-q^n)^3}{(1-(-q^n))(1-q^n)^4}
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We now consider the power series representation of

$$F(q) := \prod_{n=1}^{\infty} \frac{(1-q^{6n})(1-q^n)^2}{(1-q^{3n})(1-q^{2n})}.$$

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Notice that the two exponents in the sums above are triangular numbers.

However, we shared earlier that $5n+2$ and $5n+4$ can never be triangular numbers.

Therefore, we can conclude that $s_3(5n+2) \equiv 0 \pmod{5}$ and $s_3(5n+4) \equiv 0 \pmod{5}$. $\square$

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Related to the above, we can also prove the following additional congruence family modulo 5.

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The proof of this result uses tools similar to those already mentioned above. $\square$

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Lastly, let me note the following family of congruences modulo 11.

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Theorem: For all $n \geq 0$,

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The full result then follows from the "internal congruences" lemma. □

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Next, it would be interesting to see elementary proofs of the following conjectured results:

Conjecture: For all $k \geq 0$ and $n \geq 0$,

$$\begin{aligned}
s_{6k+1}(49n+17) &\equiv s_{6k+2}(49n+17) \equiv 0 \pmod{7},\\
s_{6k+1}(49n+31) &\equiv s_{6k+2}(49n+31) \equiv 0 \pmod{7},\\
s_{6k+1}(49n+38) &\equiv s_{6k+2}(49n+38) \equiv 0 \pmod{7},\\
s_{6k+1}(49n+45) &\equiv s_{6k+2}(49n+45) \equiv 0 \pmod{7}.
\end{aligned}$$

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Lastly, we note that the above results provide three Ramanujan congruences for $s_3(n)$, namely

$$s_3(2n+1) \equiv 0 \pmod{2},$$
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We would like to know if there are any other Ramanujan congruences for $s_3(n)$ and if any exist for $d \geq 4$.

We are especially interested in knowing if $s_d(n)$ satisfies only finitely many Ramanujan congruences for fixed $d \geq 3$.

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And with that I will close.

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Thanks very much!

Arithmetic Properties of d–fold Partition Diamonds

James Sellers
University of Minnesota Duluth

jsellers@d.umn.edu

December 2023