Elementary Proofs of Two Congruences for Partitions with Odd Parts Repeated at Most Twice

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February 2025

Opening Comments

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- Thanks to each of you for attending!

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- I will begin by setting the stage for the talk – a very recent paper of Mircea Merca.
- In particular, I will focus our attention on two congruences in that paper which Merca proved in "automated" fashion.
- I will then provide elementary / classical proofs of these two congruences (fulfilling a request of Merca).
- The hope is that these proofs shed some light on the structure of the congruences in question.

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For example, the number of partitions of the integer $n = 4$ is 5, and the partitions counted in that instance are as follows:

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

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$$4, \quad \overline{4}, \quad 3+1, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{3}+\overline{1},$$

$$2+2, \quad \overline{2}+2, \quad 2+1+1, \quad \overline{2}+1+1, \quad 2+\overline{1}+1, \quad \overline{2}+\overline{1}+1,$$

$$1+1+1+1, \quad \overline{1}+1+1+1$$

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From this definition, it is clear that the generating function for $a(n)$ is given by

$$\sum_{n\geq 0} a(n)q^n = \prod_{i\geq 1} \frac{(1-q^{6i-3})}{(1-q^i)}.$$

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We can then rewrite this generating function as the following:

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This is easily seen via generating function manipulations.

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$$\sum_{n\geq 0} a(n)q^n \quad = \quad \frac{(1+q+q^2)(1+q^3+q^6)(1+q^5+q^{10})\dots}{(1-q^2)(1-q^4)(1-q^6)\dots}$$

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Theorem: We have

$$\sum_{n\geq 0} a(4n+2)q^n = 2\frac{f_3 f_4^3 f_6^2 f_{24}}{f_1^2 f_2^3 f_8 f_{12}^2} + 4q\frac{f_6^5 f_{24}^2}{f_1^3 f_2^2 f_{12}^3} - 2q^2\frac{f_3 f_4^3 f_{24}^4}{f_1^2 f_2^2 f_6 f_8^2 f_{12}^2} + 4q^3\frac{f_6^2 f_{24}^5}{f_1^3 f_2^5 f_8 f_{12}^3} - 8q^5\frac{f_{24}^8}{f_1^3 f_6 f_8^2 f_{12}^3}.$$

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$$\sum_{n\geq 0} a(4n+3)q^n = 2\frac{f_4^8 f_6^6 f_{24}^3}{f_1 f_2^7 f_8^3 f_{12}^7} + 8q\frac{f_4^5 f_6^9 f_{24}^4}{f_1^2 f_2^6 f_3 f_8^2 f_{12}^8} - 2q^2\frac{f_4^8 f_6^3 f_{24}^6}{f_1 f_2^6 f_8^4 f_{12}^7} - 16q^3\frac{f_4^5 f_6^6 f_{24}^7}{f_1^2 f_2^5 f_3 f_8^3 f_{12}^8} + 8q^5\frac{f_4^5 f_6^3 f_{24}^{10}}{f_1^2 f_2^4 f_3 f_8^4 f_{12}^8}.$$

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While these proofs are indeed correct, they do not provide any insights into the results themselves. (In particular, the technique provides no guidance as to whether simpler generating function identities exist.)

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While these proofs are indeed correct, they do not provide any insights into the results themselves. (In particular, the technique provides no guidance as to whether simpler generating function identities exist.)

In this case, the "power" of these techniques is, in essence, unnecessary.

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As corollaries of the above theorems, Merca then noted the following:

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Corollary: For all $n \geq 0$,

$$a(4n+2) \equiv 0 \pmod{2}, \quad \text{and}$$
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At the end of Section 4 of his paper, Merca indicates that a classical proof of these congruences would be very interesting.

The goal of this talk is to fulfill Merca's request by providing two truly elementary (classical) proofs of this corollary.

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To do so, we want to 2–dissect the generating function for $a(n)$ twice.

In order to complete that task, we only require two elementary dissection lemmas.

Necessary Tools

Lemma: We have

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}.$$

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An excellent, step–by–step, elementary proof of this result can be found in Chapter 30 of Hirschhorn's *Power of* q.

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An excellent, step–by–step, elementary proof of this result can be found in Chapter 30 of Hirschhorn's *Power of* q.

As an aside, Mike and I needed the above 2–dissection to complete some of our congruence work on **partitions** into four squares.

Necessary Tools

Lemma: We have

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}.$$

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It begins with the identity $\varphi(q) = \varphi(q^4) + 2q\psi(q^8)$ where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2} \quad \text{and} \quad \psi(q) = \sum_{n \geq 0} q^{n(n+1)/2} = \frac{f_2^2}{f_1}$$

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These two lemmas are all that we need to prove our results.

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Theorem: We have

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Now to the proofs.

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We now perform two additional 2–dissections.

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We then see that

$$\sum_{n\geq 0} a(4n+2)q^n = 2\frac{f_2 f_6^2 f_8^2}{f_1^4 f_3 f_{12}}$$

and this is our first generating function result.

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Hence,

$$\sum_{n\geq 0} a(4n+3)q^n = 2\frac{f_2^4 f_8^2 f_{12}}{f_1^5 f_4^2 f_6}$$

and this proves our second generating function theorem.

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Thanks to the above, we've clearly provided classical proofs of these parity results (as requested by Merca).

Before we close, we quickly provide alternative proofs of these two congruences!

One of the first things I noted when I read Merca's paper was that the two arithmetic progressions in question, $4n+2$ and $4n+3$, have the property that they contain no squares.

This led me to find a proof of these congruences that reflected this fact.

Our Proofs

Consider

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(The exponents above are generalized octagonal numbers.)

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Since f_{12} is a function of q^4, and since we are interested in $a(4n+2)$ and $a(4n+3)$, we can effectively ignore this instance of $\frac{1}{f_{12}}$ if we can show that, for all $n \geq 0$, $g(4n+2) = g(4n+3) = 0$.

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So in both of the cases in which we are interested, we see that $g(4n + 2) = g(4n + 3) = 0$ for all n.

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So in both of the cases in which we are interested, we see that $g(4n+2) = g(4n+3) = 0$ for all n. This yields an alternative proof of our congruences modulo 2.

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Thanks again for attending today.

Elementary Proofs of Two Congruences for Partitions with Odd Parts Repeated at Most Twice

James Sellers
University of Minnesota Duluth

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February 2025