Overpartitions and Kaur, Rana, and Eyyunni's Mex Sequences

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- Hopkins, B. and Sellers, J. A., Overpartitions and Kaur, Rana, and Eyyunni's Mex Sequences, to appear in *Annals of Combinatorics*

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- Our combinatorial proofs of the K-R-E results (which satisfy a request of K, R, and E)

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For example,

$$P(4) = \{(4), \quad (3,1), \quad (2,2), \quad (2,1,1), \quad (1,1,1,1)\}$$

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Several authors have studied the mex statistic from a wide variety of perspectives, including Hopkins and Sellers (in 2020 and 2024), Hopkins-Sellers-Stanton, Hopkins-Sellers-Yee, Konan, and Yao.

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Let's look at several examples of mex and mex sequences.

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With the above definition of mex sequence in mind, Kaur, Rana, and Eyyunni defined the following family of partitions.

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For any integer $r \geq 1$, let $P_r^{\mathrm{mex}}(n)$ be the set of partitions of n whose mex sequences have length **at least** r with $p_r^{\mathrm{mex}}(n) := |P_r^{\mathrm{mex}}(n)|$.

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For all $r \geq 1$ and any $n \geq 0$, it is clear that $P_{r+1}^{\mathrm{mex}}(n) \subseteq P_r^{\mathrm{mex}}(n)$.

Moreover, $P_1^{\mathrm{mex}}(n) = P(n)$ for all n since the mex sequence of any partition of n has length at least 1 (the mex itself).

The Work of Kaur, Rana, and Eyyunni

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In their *Rocky Mountain J. Math.* paper, Kaur, Rana, and Eyyunni use elementary generating function manipulations, as well as a version of Heine's transformation of a ${}_2\phi_1$ summation formula, to obtain a generating function identity for $p_r^{\mathrm{mex}}(n)$.

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It incorporates the standard Pochhammer notation: Assuming $|q| < 1$,

$$\begin{aligned}(a;q)_0 &:= 1,\\ (a;q)_n &:= (1-a)(1-aq)\cdots(1-aq^{n-1}),\\ (a;q)_\infty &= \lim_{n\to\infty}(a;q)_n.\end{aligned}$$

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Theorem: (K-R-E, Theorem 10) For r a positive integer,

$$\sum_{n\geq 0} p_r^{\text{mex}}(n)q^n = \frac{1}{(q;q^2)_\infty (q^{r+1};q^2)_\infty}.$$

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Let $P_e^{>r}(n)$ be the set of partitions of n in which any even part is at least r with $p_e^{>r}(n) := |P_e^{>r}(n)|$.

Let $P_{o,2}^{>r}(n)$ be the set of partitions of n into odd parts where parts greater than r come in two colors with $p_{o,2}^{>r}(n) := |P_{o,2}^{>r}(n)|$.

The Work of Kaur, Rana, and Eyyunni

For example, using subscripts for colors, $p^{>2}_{o,2}(6) = 8$ from

$$\{(5_1, 1), (5_2, 1), (3_1, 3_1), (3_1, 3_2), (3_2, 3_2), (3_1, 1, 1, 1), (3_2, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}.$$

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Also, the parts 1 can be considered as 1_1, i.e., having the first color.

The following result then follows from elementary interpretations of the generating function identity in the above theorem.

The Work of Kaur, Rana, and Eyyunni

Corollary: (K-R-E, Corollary 11) For each integer $r \geq 1$,

$$p_r^{\text{mex}}(n) = \begin{cases} p_e^{>r}(n) & \text{if } r \text{ is odd,} \\ p_{o,2}^{>r}(n) & \text{if } r \text{ is even.} \end{cases}$$

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In their concluding remarks, K-R-E write, "It would be highly desirable to get a bijective proof of the identities for $p_r^{\mathrm{mex}}(n)$ in Corollary 11."

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The goal of the rest of this talk is to provide such proofs after establishing a connection between the mex sequence and overpartitions.

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$$\overline{P}(4) = \{(4), (\overline{4}), (3,1), (3,\overline{1}), (\overline{3},1), (\overline{3},\overline{1}), (2,2), (\overline{2},2), (2,1,1), (2,\overline{1},1), (\overline{2},1,1), (\overline{2},\overline{1},1), (1,1,1,1), (\overline{1},1,1,1)\}$$

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We will sometimes write an overpartition as (λ, μ) where λ is a partition with distinct parts (corresponding to the overlined parts) and $|\lambda| + |\mu| = n$ so that, for example, $(\overline{2}, \overline{1}, 1)$ corresponds to $((2,1),(1))$.

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Overpartitions were named by Corteel and Lovejoy (*Trans. Amer. Math. Soc.*, 2004) although, as they discuss, equivalent sets of partitions had previously been considered in various contexts.

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Both of these are discussed in the book of Andrews and Eriksson entitled *Integer Partitions*.

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One consequence is the equal count of two types of restricted partitions.

Proposition: The number of partitions of n into distinct parts equals the number of partitions of n where each part size from 1 to λ_1 appears at least once.

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Although not stated directly, this follows from a discussion in Sylvester and Franklin's major work (*Amer. J. Math.*, 1882).

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Note that partitions where each part size from 1 to λ_1 appears at least once, also called partitions with no gaps, have mex $\lambda_1 + 1$ and an infinite mex sequence.

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It is easily described as follows:

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Here's a quick reminder of that result.

Corollary: (K-R-E, Corollary 11) For each integer $r \geq 1$,

$$p_r^{\mathrm{mex}}(n) = \begin{cases} p_e^{>r}(n) & \text{if } r \text{ is odd,} \\ p_{o,2}^{>r}(n) & \text{if } r \text{ is even.} \end{cases}$$

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Proposition: There are $p(n)$ overpartitions of n where every non-overlined part is even, i.e., overpartitions (λ, μ) where μ consists of only even parts.

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Clearly, overpartitions with no overlined parts are simply partitions.

We show that the number of overpartitions of n where any non-overlined parts are even also equals $p(n)$.

Proposition: There are $p(n)$ overpartitions of n where every non-overlined part is even, i.e., overpartitions (λ, μ) where μ consists of only even parts.

To prove this proposition, we establish a bijection between $P(n)$ and the described subset of $\overline{P}(n)$.

Our Fulfillment of the K-R-E Request

Proof:

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Because Glaisher's map establishes a bijection, the maps described here are inverses. □

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$P(6)$	6	51	42	411	33	321	31^3	2^3	2211	21^4	1^6
overpartitions	6	$\overline{5}\overline{1}$	42	$4\overline{2}$	$\overline{6}$	$\overline{3}2\overline{1}$	$\overline{3}\overline{2}\overline{1}$	2^3	$\overline{2}22$	$\overline{4}2$	$\overline{4}\overline{2}$

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Recall the observation that $P_1^{\mathrm{mex}}(n) = P(n)$.

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The proposition above then suggests a connection between partitions with longer mex sequences and more restricted overpartitions.

Our Fulfillment of the K-R-E Request

Definition: For any integer $r \geq 1$, let $\overline{P}_r(n)$ be the set of overpartitions of n where non-overlined parts must be greater than r and have the same parity as $r+1$, and let $\overline{p}_r(n) := |\overline{P}_r(n)|$.

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With this notation, the above proposition establishes the fact that $\overline{p}_1(n) = p_1^{\mathrm{mex}}(n)$ for all n.

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Theorem: For any integer $r \geq 1$, $\overline{p}_r(n) = p_r^{\mathrm{mex}}(n)$ for all n.

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We provide both a generating function proof and, in the spirit of the remaining results, a combinatorial proof of this connection.

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We will also allow some temporary parts 0 at the end of partitions to avoid having to detail special cases.

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Let's look first at our generating function proof (which is rather brief).

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$$\sum_{n\geq 0} p_r^{\text{mex}}(n)q^n = \frac{1}{(q;q^2)_\infty (q^{r+1};q^2)_\infty} = \frac{(-q;q)_\infty}{(q^{r+1};q^2)_\infty}.$$

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This is the generating function for the overpartitions $\overline{P}_r(n)$ (with the restriction on non-overlined parts apparent in the denominator). □

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This is the generating function for the overpartitions $\overline{P}_r(n)$ (with the restriction on non-overlined parts apparent in the denominator). □

Now let's turn to our combinatorial proof of this result (where we establish a bijection between $\overline{P}_r(n)$ and $P_r^{\mathrm{mex}}(n)$).

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(a) Given $\kappa \in P_r^{\mathrm{mex}}(n)$, if the mex sequence is infinite, then κ has no gaps and κ' (the conjugate of κ) is a distinct part partition by the proposition mentioned above.

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Proof: (via combinatorial techniques)

(a) Given $\kappa \in P_r^{\text{mex}}(n)$, if the mex sequence is infinite, then κ has no gaps and κ' (the conjugate of κ) is a distinct part partition by the proposition mentioned above.

In this case, send κ to the overpartition $(\kappa', \varnothing)$ which, with only overlined parts, is trivially in $\overline{P}_r(n)$.

Our Fulfillment of the K-R-E Request

If the mex sequence of κ is finite, write

$$\kappa = (\kappa_1, \ldots, \kappa_i, \sigma_{i+1}, \ldots, \sigma_j)$$

where $\kappa_i > \mathrm{mex}(\kappa)$ and $\sigma_{i+1} = \mathrm{mex}(\kappa) - 1$ (possibly 0).

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We describe how to write κ as

$$\kappa = (\kappa_1 - \sigma_1, \ldots, \kappa_i - \sigma_i) \oplus (\sigma_1, \ldots, \sigma_j)$$

where $(\kappa_1 - \sigma_1, \ldots, \kappa_i - \sigma_i)$ is a partition with each part at least r and the same parity as $r + 1$, and $(\sigma_1, \ldots, \sigma_j)$ is a partition with no gaps.

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Since $\kappa \in P_r^{\mathrm{mex}}(n)$, in fact $\kappa_i \geq \sigma_{i+1} + r + 1$.

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There are two possibilities for σ_i in order for $(\sigma_i, \ldots, \sigma_j)$ to have no gaps: Either $\sigma_i = \sigma_{i+1}$ or $\sigma_i = \sigma_{i+1} + 1$.

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Exactly one of the two possibilities for σ_i will make $\kappa_i - \sigma_i$ have the same parity as $r + 1$.

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With σ_i set, we proceed to σ_{i-1}, etc.

Our Fulfillment of the K-R-E Request

More precisely, working from $\ell = i$ down to $\ell = 1$, let

$$\sigma_\ell = \begin{cases} \sigma_{\ell+1} & \text{if } \kappa_\ell - \sigma_{\ell+1} \equiv r + 1 \bmod 2, \\ \sigma_{\ell+1} + 1 & \text{if } \kappa_\ell - (\sigma_{\ell+1} + 1) \equiv r + 1 \bmod 2; \end{cases}$$

$$= \begin{cases} \sigma_{\ell+1} & \text{if } \kappa_\ell - \sigma_{\ell+1} \not\equiv r \bmod 2, \\ \sigma_{\ell+1} + 1 & \text{if } \kappa_\ell - \sigma_{\ell+1} \equiv r \bmod 2. \end{cases}$$

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Clearly the resulting $(\sigma_1, \ldots, \sigma_j)$ is a partition with no gaps and each $\kappa_\ell - \sigma_\ell \equiv r + 1 \bmod 2$.

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Clearly the resulting $(\sigma_1, \dots, \sigma_j)$ is a partition with no gaps and each $\kappa_\ell - \sigma_\ell \equiv r+1 \bmod 2$.

It remains to show that each $\kappa_\ell - \sigma_\ell \geq r$.

Our Fulfillment of the K-R-E Request

Note that $\kappa_i - \sigma_i \geq \sigma_{i+1} + r + 1 - (\sigma_{i+1} + 1) = r$, as desired.

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Continuing in this way, we see that each $\kappa_\ell - \sigma_\ell$ is at least r.

Our Fulfillment of the K-R-E Request

With the above in hand, assign κ to the overpartition

$$((\sigma_1, \ldots, \sigma_j)', (\kappa_1 - \sigma_1, \ldots, \kappa_i - \sigma_i)).$$

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Since $(\sigma_1, \ldots, \sigma_j)$ is a partition with no gaps, by the "no gaps" proposition above, its conjugate has distinct parts.

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With $(\kappa_1 - \sigma_1, \ldots, \kappa_i - \sigma_i)$ established as a partition with each part at least r and congruent to $r + 1$ modulo 2, the overpartition image of κ is in $\overline{P}_r(n)$.

Our Fulfillment of the K-R-E Request

(b) Given $(\lambda, \mu) \in \overline{P}_r(n)$, send the overpartition to $\kappa = \lambda' \oplus \mu \in P(n)$.

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Writing $\lambda = (\lambda_1, \ldots, \lambda_j)$, by the "no gaps" proposition, the conjugate λ' has first part j, length λ_1, and no gaps.

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Now suppose μ consists of m parts (which are all greater than r and have the same parity as $r+1$).

If $m = 0$, then $\kappa = \lambda'$ has mex $j+1$ and an infinite mex sequence.

Our Fulfillment of the K-R-E Request

If $1 \leq m < \lambda_1$, then $\kappa_m > \lambda'_m + r$ and $\kappa_{m+1} = \lambda'_{m+1}$ which is either λ'_m or $\lambda'_m - 1$ and every smaller part size appears at least once since λ' has no gaps.

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If $1 \leq m < \lambda_1$, then $\kappa_m > \lambda'_m + r$ and $\kappa_{m+1} = \lambda'_{m+1}$ which is either λ'_m or $\lambda'_m - 1$ and every smaller part size appears at least once since λ' has no gaps.

Therefore κ has finite mex sequence of length at least r (with $\mathrm{mex}(\kappa) = \kappa_{m+1} + 1 > 1$).

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Therefore κ has finite mex sequence of length at least r (with $\mathrm{mex}(\kappa) = \kappa_{m+1} + 1 > 1$).

Finally, if $m \geq \lambda_1$, then $\kappa = \lambda' \oplus \mu$ has each part greater than r and finite mex sequence of length at least r (with $\mathrm{mex}(\kappa) = 1$).

Our Fulfillment of the K-R-E Request

In each case, $\kappa \in P_r^{\mathrm{mex}}(n)$.

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It is straightforward to verify that the maps are inverses, establishing a bijection and completing the combinatorial proof. $\square$

Let's look at a few examples of the maps in the combinatorial proof above.

Our Fulfillment of the K-R-E Request

(a) The partition $\kappa = (8, 6, 5, 1, 1)$ has mex sequence $(2, 3, 4)$.

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- Below the mex the parts are $\sigma_4 = \sigma_5 = 1$.
- Since $5 - 1 \not\equiv 3 \bmod 2$, we let $\sigma_3 = \sigma_4 + 1 = 2$ so that $\kappa_3 - \sigma_3 = 3$.
- Next, since $6 - 2 \not\equiv 3 \bmod 2$, we let $\sigma_2 = 3$ so that $\kappa_2 - \sigma_2 = 3$.

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- Next, since $6 - 2 \not\equiv 3 \bmod 2$, we let $\sigma_2 = 3$ so that $\kappa_2 - \sigma_2 = 3$.
- Finally, since $8 - 3 \equiv 3 \bmod 2$, we let $\sigma_1 = 3$ so that $\kappa_1 - \sigma_1 = 5$.

Our Fulfillment of the K-R-E Request

(a) The partition $\kappa = (8, 6, 5, 1, 1)$ has mex sequence $(2, 3, 4)$.

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- Below the mex the parts are $\sigma_4 = \sigma_5 = 1$.
- Since $5 - 1 \not\equiv 3 \bmod 2$, we let $\sigma_3 = \sigma_4 + 1 = 2$ so that $\kappa_3 - \sigma_3 = 3$.
- Next, since $6 - 2 \not\equiv 3 \bmod 2$, we let $\sigma_2 = 3$ so that $\kappa_2 - \sigma_2 = 3$.
- Finally, since $8 - 3 \equiv 3 \bmod 2$, we let $\sigma_1 = 3$ so that $\kappa_1 - \sigma_1 = 5$.
- Therefore

$$\begin{aligned}(8, 6, 5, 1, 1) &= (3, 3, 2, 1, 1) \oplus (5, 3, 3) \\ &\mapsto ((3, 3, 2, 1, 1)', (5, 3, 3)) = ((5, 3, 2), (5, 3, 3)) \\ &\sim (\overline{5}, 5, \overline{3}, 3, 3, \overline{2}).\end{aligned}$$

Our Fulfillment of the K-R-E Request

(b) For $r = 3$, the same partition $\kappa = (8, 6, 5, 1, 1)$ has a different image in $\overline{P}_3(21)$.

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Our Fulfillment of the K-R-E Request

(b) For $r = 3$, the same partition $\kappa = (8, 6, 5, 1, 1)$ has a different image in $\overline{P}_3(21)$.

- Since $5 - 1 \equiv 4 \bmod 2$, we let $\sigma_3 = 1$ so that $\kappa_3 - \sigma_3 = 4$.
- Next, since $6 - 1 \not\equiv 4 \bmod 2$, we let $\sigma_2 = 2$ so that $\kappa_2 - \sigma_2 = 4$.
- Finally, since $8 - 2 \equiv 4 \bmod 2$, we let $\sigma_1 = 2$ so that $\kappa_1 - \sigma_1 = 6$.

Our Fulfillment of the K-R-E Request

(b) For $r = 3$, the same partition $\kappa = (8, 6, 5, 1, 1)$ has a different image in $\overline{P}_3(21)$.

- Since $5 - 1 \equiv 4 \bmod 2$, we let $\sigma_3 = 1$ so that $\kappa_3 - \sigma_3 = 4$.
- Next, since $6 - 1 \not\equiv 4 \bmod 2$, we let $\sigma_2 = 2$ so that $\kappa_2 - \sigma_2 = 4$.
- Finally, since $8 - 2 \equiv 4 \bmod 2$, we let $\sigma_1 = 2$ so that $\kappa_1 - \sigma_1 = 6$.
- Therefore

$$\begin{aligned}(8,6,5,1,1) &= (2,2,1,1,1) \oplus (6,4,4) \\ &\mapsto ((2,2,1,1,1)', (6,4,4)) = ((5,2),(6,4,4)) \\ &\sim (6, \overline{5}, 4, 4, \overline{2}).\end{aligned}$$

Our Fulfillment of the K-R-E Request

(c) The partition $\nu = (8, 5, 4, 4)$ has mex sequence $(1, 2, 3)$ so we temporarily set $\sigma_5 = 0$.

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- Likewise, $\sigma_3 = 0$ and $\nu_3 - \sigma_3 = 4$.

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- Likewise, $\sigma_3 = 0$ and $\nu_3 - \sigma_3 = 4$.
- Next, since $5 - 0 \not\equiv 4 \bmod 2$, we let $\sigma_2 = 1$ so that $\nu_2 - \sigma_2 = 4$.

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- Finally, since $8 - 1 \not\equiv 4 \bmod 2$, we let $\sigma_1 = 2$ so that $\nu_1 - \sigma_1 = 6$.

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- Finally, since $8 - 1 \not\equiv 4 \bmod 2$, we let $\sigma_1 = 2$ so that $\nu_1 - \sigma_1 = 6$.
- Therefore the corresponding overpartition in $\overline{P}_3(21)$ is

$$\begin{aligned}(8, 5, 4, 4) &= (2, 1, 0, 0, 0) \oplus (6, 4, 4, 4) \\ &\mapsto ((2, 1)', (6, 4, 4, 4)) = ((2, 1), (6, 4, 4, 4)) \\ &\sim (6, 4, 4, 4, \overline{2}, \overline{1}).\end{aligned}$$

Our Fulfillment of the K-R-E Request

(d) For $r = 2$, the overpartition $(\overline{5}, 5, \overline{3}, 3, 3, \overline{2}) \in \overline{P}_3(21)$ has $\lambda = (5, 3, 2)$ with conjugate $\lambda' = (3, 3, 2, 1, 1)$ and $\mu = (5, 3, 3)$.

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Thus, it maps to

$$(3, 3, 2, 1, 1) \oplus (5, 3, 3) = (8, 6, 5, 1, 1) \in P(21)$$

which has mex sequence $(2, 3, 4)$.

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This is an element of $P_2^{\text{mex}}(21)$.

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This is an element of $P_2^{\mathrm{mex}}(21)$.

Along with Example (a) above, this is an example of the maps being inverses.

Our Fulfillment of the K-R-E Request

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For example, $(\overline{5}, 4, \overline{3}, 3, 3, \overline{2}, \overline{1})$, which is not in $\overline{P}_2(21)$ since $4 \not\equiv 3 \bmod 2$, would be associated with

$$(5, 3, 2, 1)' \oplus (4, 3, 3) = (4, 3, 2, 1, 1) \oplus (4, 3, 3) = (8, 6, 5, 1, 1),$$

the same image in $P_2^{\mathrm{mex}}(21)$ associated with $(\overline{5}, 5, \overline{3}, 3, 3, \overline{2})$ in example (d) above.

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One additional remark is in order. The map described in part (b) of the proof of the theorem above can be applied to a larger set of overpartitions, but then it is not injective.

For example, $(\overline{5}, 4, \overline{3}, 3, 3, \overline{2}, \overline{1})$, which is not in $\overline{P}_2(21)$ since $4 \not\equiv 3 \mod 2$, would be associated with

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the same image in $P_2^{\mathrm{mex}}(21)$ associated with $(\overline{5}, 5, \overline{3}, 3, 3, \overline{2})$ in example (d) above.

However, we know from example (a) that $(8, 6, 5, 1, 1)$ is uniquely associated with $(\overline{5}, 5, \overline{3}, 3, 3, \overline{2}) \in \overline{P}_2(21)$.

Our Fulfillment of the K-R-E Request

Here's a table indicating the $n = 7$ cases of the correspondence of the above theorem for $r = 2$ and $r = 3$.

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$P_2^{\text{mex}}(7)$	7	61	511	43	41^3	3211	2^31	221^3	21^5	1^7
$\overline{P}_2(7)$	7	$5\overline{2}$	$\overline{3}3\overline{1}$	$33\overline{1}$	$\overline{4}3$	$\overline{421}$	$\overline{43}$	$\overline{52}$	$\overline{61}$	$\overline{7}$
$\overline{P}_3(7)$	$\overline{7}$	$6\overline{1}$	$\overline{61}$	$\overline{52}$	$4\overline{3}$	$\overline{43}$	$4\overline{21}$	$\overline{421}$		
$P_3^{\text{mex}}(7)$	1^7	7	21^5	221^3	511	2^31	61	3211		

Our Fulfillment of the K-R-E Request

With this unified interpretation of $p_r^{\text{mex}}(n)$ in terms of overpartitions, we can provide combinatorial proofs of our corollary requested by Kaur, Rana, and Eyyunni.

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Given the very different nature of their combinatorial interpretations depending on the parity of r, we give two bijections.

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Given the very different nature of their combinatorial interpretations depending on the parity of r, we give two bijections.

Proposition: For each odd $r \geq 1$, $\overline{p}_r(n) = p_e^{>r}(n)$ for all n.

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Proposition: For each odd $r \geq 1$, $\overline{p}_r(n) = p_e^{>r}(n)$ for all n.

The $r = 1$ case is one of the propositions above.

In fact, as detailed next, the result for any odd r follows from the same map applied to the appropriate subsets.

Our Fulfillment of the K-R-E Request

Proof: Given $r \geq 1$ odd, we establish a bijection between $P_e^{>r}(n)$ and $\overline{P}_r(n)$.

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We first build a map from $P_e^{>r}(n)$ to $\overline{P}_r(n)$.

Given a partition $\kappa \in P_e^{>r}(n)$, apply Glaisher's map to any odd parts and overline the resulting distinct parts while leaving any even parts of κ (necessarily greater than r) non-overlined.

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This produces an overpartition of n where any non-ovelined parts are even and greater than r, i.e., an element of $\overline{P}_r(n)$.

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We next build a map from $\overline{P}_r(n)$ to $P_e^{>r}(n)$.

Given an overpartition $(\lambda, \mu) \in \overline{P}_r(n)$, note that μ consists entirely of even parts greater than r.

Apply Glaisher's map to λ and take the union of the resulting odd parts and μ to make a partition in $P_e^{>r}(n)$.

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Apply Glaisher's map to λ and take the union of the resulting odd parts and μ to make a partition in $P_e^{>r}(n)$.

As explained above, these maps are inverses. $\square$

Our Fulfillment of the K-R-E Request

The following table includes examples of the maps in the proof of the above proposition.

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Here we have $n = 8$, $r = 3$ which includes the $r = 5$ and $r = 7$ cases.

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Here we have $n = 8$, $r = 3$ which includes the $r = 5$ and $r = 7$ cases.

$P_e^{>3}(8)$	8	71	611	53	51^3	44	431	41^4	3311	31^5	1^8
$\overline{P}_3(8)$	8	$\overline{71}$	$6\overline{2}$	$\overline{53}$	$\overline{521}$	44	$4\overline{31}$	$\overline{4}4$	$\overline{62}$	$\overline{431}$	$\overline{8}$

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$P_e^{>3}(8)$	8	71	611	53	51^3	44	431	41^4	3311	31^5	1^8
$\overline{P}_3(8)$	8	$\overline{71}$	$6\overline{2}$	$\overline{53}$	$\overline{521}$	44	$4\overline{31}$	$\overline{4}4$	$\overline{62}$	$\overline{431}$	$\overline{8}$

Combined with our theorem above, this proposition gives a combinatorial proof of the K-R-E corollary for the odd r case, as desired.

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Next, we proceed to providing a bijection for the even r case.

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Proposition: For each even $r \geq 2$, $\overline{p}_r(n) = p_{o,2}^{>r}(n)$ for all n.

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Next, we proceed to providing a bijection for the even r case.

Proposition: For each even $r \geq 2$, $\overline{p}_r(n) = p_{o,2}^{>r}(n)$ for all n.

Proof: Given $r \geq 2$ even, we establish a bijection between $P_{o,2}^{>r}(n)$ and $\overline{P}_r(n)$.

Our Fulfillment of the K-R-E Request

Next, we proceed to providing a bijection for the even r case.

Proposition: For each even $r \geq 2$, $\overline{p}_r(n) = p_{o,2}^{>r}(n)$ for all n.

Proof: Given $r \geq 2$ even, we establish a bijection between $P_{o,2}^{>r}(n)$ and $\overline{P}_r(n)$.

We begin by finding a map from $P_{o,2}^{>r}(n)$ to $\overline{P}_r(n)$.

Our Fulfillment of the K-R-E Request

Given $\nu \in P_{o,2}^{>r}(n)$, write $\nu = (\pi, \rho)$ where π consists of any odd parts of the first color (including any odd parts less than r) and ρ consists of any odd parts of the second color (each at least r).

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Apply Glaisher's map to π, overline the resulting distinct parts, and leave any parts of ρ non-overlined.

This produces an overpartition of n where any non-overlined parts are odd and greater than r, i.e., an element of $\overline{P}_r(n)$.

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Next, we find a map from $\overline{P}_r(n)$ to $P_{o,2}^{>r}(n)$.

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Next, we find a map from $\overline{P}_r(n)$ to $P_{o,2}^{>r}(n)$.

Given an overpartition $(\lambda, \mu) \in \overline{P}_r(n)$, note that μ consists entirely of odd parts greater than r (if it is nonempty).

Apply Glaisher's map to λ, assign the resulting odd parts the first color, and assign the parts of μ the second color.

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Apply Glaisher's map to λ, assign the resulting odd parts the first color, and assign the parts of μ the second color.

The union of these colored odd parts is an element of $P_{o,2}^{>r}(n)$.

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Apply Glaisher's map to λ, assign the resulting odd parts the first color, and assign the parts of μ the second color.

The union of these colored odd parts is an element of $P_{o,2}^{>r}(n)$.

Since Glaisher's map is a bijection, it is clear that these maps are inverses. $\square$

Our Fulfillment of the K-R-E Request

Here is a table that provides an example of this result for $n = 6$ and $r = 2$.

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$P_{o,2}^{>2}(6)$	$5_1 1_1$	$5_2 1_1$	$3_1 3_1$	$3_1 3_2$	$3_2 3_2$	$3_1 (1_1)^3$	$3_2 (1_1)^3$	$(1_1)^6$
$\overline{P}_2(6)$	$\overline{51}$	$5\overline{1}$	$\overline{6}$	$\overline{3}3$	33	$\overline{321}$	$3\overline{21}$	$\overline{42}$

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$P_{o,2}^{>2}(6)$	$5_1 1_1$	$5_2 1_1$	$3_1 3_1$	$3_1 3_2$	$3_2 3_2$	$3_1(1_1)^3$	$3_2(1_1)^3$	$(1_1)^6$
$\overline{P}_2(6)$	$\overline{5}\,\overline{1}$	$5\overline{1}$	$\overline{6}$	$\overline{3}3$	33	$\overline{3}\,\overline{2}\,\overline{1}$	$3\overline{2}\,\overline{1}$	$\overline{4}\,\overline{2}$

Combined with our theorem above, this proposition gives a combinatorial proof of the K-R-E corollary for the even r case, as desired.

Our Fulfillment of the K-R-E Request

Here is a table that provides an example of this result for $n = 6$ and $r = 2$.

$P_{o,2}^{>2}(6)$	$5_1 1_1$	$5_2 1_1$	$3_1 3_1$	$3_1 3_2$	$3_2 3_2$	$3_1(1_1)^3$	$3_2(1_1)^3$	$(1_1)^6$
$\overline{P}_2(6)$	$\overline{51}$	$5\overline{1}$	$\overline{6}$	$\overline{3}3$	33	$\overline{321}$	$3\overline{21}$	$\overline{42}$

Combined with our theorem above, this proposition gives a combinatorial proof of the K-R-E corollary for the even r case, as desired.

That means we have completely fulfilled the request of K-R-E.

Our Fulfillment of the K-R-E Request

Please allow me to share some closing thoughts.

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Given the very different nature of $P_e^{>r}(n)$, a subset of ordinary partitions, and $P_{o,2}^{>r}(n)$, a restricted partition with parts of two colors, it is pleasantly surprising that a single type of overpartition allows bijections to both sets, enabling us to satisfy the request of Kaur, Rana, and Eyyunni for combinatorial proofs of their corollary in a fairly unified way.

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This suggests that the overpartitions $\overline{P}_r(n)$ provide a more natural combinatorial interpretation of $P_r^{\text{mex}}(n)$, the partitions with mex sequence of length at least r.

Our Fulfillment of the K-R-E Request

And with that I will close.

Our Fulfillment of the K-R-E Request

And with that I will close.

Thanks very much for your attention!

Overpartitions and Kaur, Rana, and Eyyunni's Mex Sequences

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