A New View of Odd-Part Partitions with Designated Summands

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Opening Comments

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- Thanks to my co–author Shishuo Fu (Chongqing University, China) for our very fruitful collaboration!

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- In particular, I will focus our attention on a curious identity satisfied by PDO.
- I will look at past work of P. A. MacMahon which has gained a great deal of attention recently.
- I will connect these results of MacMahon (and others) to $\mathrm{PDO}(n)$ and transition to a conversation about a 2–parameter refinement of the above–mentioned identity, highlighting how we prove this result.

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These are built by taking unrestricted integer partitions and designating exactly one of each occurrence of a part.

For example, there are 10 partitions with designated summands of weight 4:

$$4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1, \quad 2'+1+1'$$

$$1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'$$

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Using this notation and the example above, we know $\mathrm{PD}(4) = 10$.

In the same paper, Andrews, Lewis, and Lovejoy also considered the restricted partitions with designated summands wherein all parts must be odd, and they denoted the corresponding enumeration function by $\mathrm{PDO}(n)$.

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- S. Chern and J. A. Sellers, An infinite family of internal congruences modulo powers of 2 for partitions into odd parts with designated summands, *Acta Arith.* **215**, no. 1 (2024), 43–64.

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Theorem: For all $n \geq 0$,

$$\mathrm{PDO}(2n) = \sum_{k=0}^{n} \mathrm{PDO}(k)\mathrm{PDO}(n-k).$$

From a generating function perspective, this is equivalent to proving that

$$\sum_{n\geq 0} \mathrm{PDO}(2n)q^n = \left(\sum_{n\geq 0} \mathrm{PDO}(n)q^n\right)^2.$$

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It is explicitly called out in the final section of the 2015 *Integers* paper of Baruah and Ojah where the authors note that it would be "interesting to find a combinatorial proof of this identity".

In all of the above–mentioned works, this curious identity was proved via elementary 2–dissection of the generating function for $\mathrm{PDO}(n)$.

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Theorem: The generating function for $\mathrm{PDO}(n)$ is given by

$$\sum_{n=0}^{\infty} \mathrm{PDO}(n) q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}$$

where $f_r = (1-q^r)(1-q^{2r})(1-q^{3r})(1-q^{4r})\dots$ is the usual q–Pochhammer symbol.

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However, interestingly enough, there's another way to view the generating function for $\mathrm{PDO}(n)$, and this approach has allowed Shishuo Fu and me to refine the curious identity above in a really beautiful way.

A Different View of the Generating Function

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$$A_k(q) = \sum_{0<m_1<m_2<\cdots<m_k} \frac{q^{m_1+m_2+\cdots+m_k}}{(1-q^{m_1})^2 \ldots (1-q^{m_k})^2},$$

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$$C_k(q) = \sum_{0<m_1<m_2<\cdots<m_k} \frac{q^{2m_1+2m_2+\cdots+2m_k-k}}{(1-q^{2m_1-1})^2 \ldots (1-q^{2m_k-1})^2}.$$

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These provide generalizations in the following sense.

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Fix a positive integer k. Define $a_{n,k} := \sum s_1 \cdots s_k$ where the sum is taken over all partitions of n of the form

$$n = s_1 m_1 + \cdots + s_k m_k$$

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When $k = 1$ this is simply $\sigma_1(n)$, the usual sum–of–divisors function.

One can then show that

$$A_k(q) = \sum_{n=1}^{\infty} a_{n,k} q^n.$$

A Different View of the Generating Function

Next, consider $c_{n,k} := \sum s_1 \cdots s_k$ where the sum is taken over all partitions of n of the form

$$n = s_1(2m_1 - 1) + \cdots + s_k(2m_k - 1)$$

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with $0 < m_1 < \cdots < m_k$.

Note that for $k = 1$, this is simply the sum over all divisors j of n such that n/j is an odd number.

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Note also that $c_{n,k} := \sum s_1 \cdots s_k$ is the sum of the products of the frequencies of all the parts in the odd–part partitions of n into exactly k parts.

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In analogous fashion, we have

$$C_k(q) = \sum_{n=1}^{\infty} c_{n,k} q^n.$$

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- Amdeberhan, Andrews, Tauraso, *arXiv:2409.20400*
- Amdeberhan, Ono, Singh, *Adv. Math.*
- Bachmann, *Res. Number Theory*
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First, note that for a given k, $C_k(q)$ provides a natural refinement of the generating function of $\mathrm{PDO}(n)$ in the sense that the coefficient of q^n in $C_k(q)$ provides the contribution to $\mathrm{PDO}(n)$ where the partitions in question have exactly k different part sizes.

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Said from a generating function perspective,

$$\sum_{n\geq 0} \mathrm{PDO}(n) q^n = \sum_{k\geq 0} C_k(q).$$

A Different View of the Generating Function

Secondly, we note that, in their 2013 *J. Reine Angew. Math.* paper, Andrews and Rose introduced the function

$$G(x,q) := 1 + 2\sum_{n\geq 1} T_{2n}(x/2)q^{n^2}$$

where, for $n \geq 0$, $T_n(x)$ is the n^{th} Chebyshev polynomial (of the first kind) defined by $T_n(\cos\theta) = \cos(n\theta)$.

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Note that the sum-product identity

$$\cos(A+B) + \cos(A-B) = 2\cos(A)\cos(B)$$

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Proposition: For $0 \leq m \leq n$, we have

$$T_{n+m}(x) + T_{n-m}(x) = 2T_m(x)T_n(x).$$

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As an aside, note that

$$\frac{f_2}{f_1^2} = \sum_{n\geq 0} \overline{p}(n)q^n$$

where $\overline{p}(n)$ counts the number of overpartitions of n.

Our Refinements

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The above comments imply that we can now define a "refined" version of $\mathrm{PDO}(n)$ which simultaneously takes into account the weight n of the partitions in question as well as the number of different part sizes in the partitions.

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Let $\mathcal{PDO}(n)$ denote the set of all PDO-partitions of n, so that $\mathrm{PDO}(n) = |\mathcal{PDO}(n)|$.

Moreover, let $\mathcal{PDO} := \bigcup_{n\geq 0} \mathcal{PDO}(n)$ with $\mathcal{PDO}(0)$ containing only the empty partition $\varnothing$.

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Lastly, let $\ell_d(\lambda)$ denote the number of different part sizes in the partition λ.

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Theorem: For every $n \geq 0$, we have

$$\sum_{\lambda \in \mathcal{PDO}(2n)} x^{\ell_d(\lambda)} =$$

$$\sum_{k=0}^{n} \left(\sum_{\alpha \in \mathcal{PDO}(k)} x^{\ell_d(\alpha)} \right) \left(\sum_{\beta \in \mathcal{PDO}(n-k)} x^{\ell_d(\beta)} \right).$$

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Equivalently, writing

$$\mathrm{PDO}(x,q) := \sum_{\lambda \in \mathcal{PDO}} x^{\ell_d(\lambda)} q^{|\lambda|},$$

we have for all $n \geq 0$,

$$[q^{2n}]\mathrm{PDO}(x,q) = [q^n] \left(\mathrm{PDO}(x,q)\right)^2.$$

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- there are 8 which contain exactly one part size (all of which are constructed from the ordinary partition $1+1+1+1+1+1+1+1$), and

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Example: Let $n = 4$. Of the 22 partitions into odd parts with designated summands which are counted by $\mathrm{PDO}(8)$, we see that

- there are 8 which contain exactly one part size (all of which are constructed from the ordinary partition $1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$), and
- there are 14 which contain exactly two different part sizes (these are constructed from designating the parts in $7 + 1$, $5 + 3$, $5 + 1 + 1 + 1$, $3 + 3 + 1 + 1$, and $3 + 1 + 1 + 1 + 1 + 1$).

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If we denote by $(\alpha|\beta)$ the pairs of PDO partitions where α has weight k and β has weight $4-k$, then we see that we can naturally break these into two subsets.

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- the 8 pairs of PDO partitions with only one designated part are the following:

$$(\varnothing|1'+1+1+1), \quad (\varnothing|1+1'+1+1),$$
$$(\varnothing|1+1+1'+1), \quad (\varnothing|1+1+1+1'),$$
$$(1'+1+1+1|\varnothing), \quad (1+1'+1+1|\varnothing),$$
$$(1+1+1'+1|\varnothing), \quad (1+1+1+1'|\varnothing)$$

Our Refinements

- the 14 pairs of PDO partitions with exactly two designated parts present are the following:

$$(\varnothing|3'+1'),\quad (3'+1'|\varnothing),\quad (1'|3'),\quad (3'|1'),$$
$$(1'+1|1'+1),\quad (1'+1|1+1'),$$
$$(1+1'|1'+1),\quad (1+1'|1+1'),$$
$$(1'|1'+1+1),\quad (1'|1+1'+1),\quad (1'|1+1+1'),$$
$$(1'+1+1|1'),\quad (1+1'+1|1'),\quad (1+1+1'|1')$$

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The 2–dissection of $G(x, q)$ is easy since $G(x, q)$ is naturally written as a sum.

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The 2–dissection of the generating function for $\overline{p}(n)$ appears in the 2005 paper of Hirschhorn and Sellers on overpartitions.

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Combining all the facts above reveals that the refined generating function identity is equivalent to the following identity:

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$$[q^{2n}]\left(\frac{f_2}{f_1^2}G(x,q)\right) = [q^n]\left(\frac{f_2}{f_1^2}G(x,q)\right)^2.$$

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After incorporating the 2–dissections for $G(x, q)$ and the generating function for $\overline{p}(n)$, along with performing some simplifications, we see that we need to prove the following:

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$$\frac{f_4^5}{f_2^2 f_8^2}\left(1+2\sum_{n\geq 1} T_{4n} q^{2n^2}\right) + 4q\frac{f_8^2}{f_4}\left(\sum_{n\geq 1} T_{4n-2} q^{2n^2-2n}\right) = \left(1+2\sum_{n\geq 1} T_{2n} q^{n^2}\right)^2,$$

where we have suppressed the argument $x/2$ in all Chebyshev polynomials $T_n(x/2)$.

Our Refinements

As an interesting aside, note that

$$\frac{f_4^5}{f_2^2 f_8^2} \quad \text{and} \quad \frac{f_8^2}{f_4}$$

are clearly related to Ramanujan's theta functions

$$\varphi(q) := 1 + 2\sum_{n\geq 1} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2},$$

$$\psi(q) := \sum_{n\geq 1} q^{\binom{n}{2}} = \frac{f_2^2}{f_1}.$$

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$$\psi(q) := \sum_{n\geq 1} q^{\binom{n}{2}} = \frac{f_2^2}{f_1}.$$

After a bit more simplification, we reduce the above equation to an even simpler equality, and we close out the proof then via a pair of elementary bijections.

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For a PDO partition $\lambda \in \mathrm{PDO}$, let $\ell_d^o(\lambda)$ be the number of different parts that occur an odd number of times in λ, and write $P_1(x, y, q) := \sum_{\lambda \in \mathcal{PDO}} x^{\ell_d(\lambda)} y^{\ell_d^o(\lambda)} q^{|\lambda|}$.

Our Refinements

In the time that remains, let me provide a two–parameter refinement of the original curious identity (which is a one–parameter refinement of our first refinement).

For a PDO partition $\lambda \in \mathrm{PDO}$, let $\ell_d^o(\lambda)$ be the number of different parts that occur an odd number of times in λ, and write $P_1(x, y, q) := \sum_{\lambda \in \mathcal{PDO}} x^{\ell_d(\lambda)} y^{\ell_d^o(\lambda)} q^{|\lambda|}$.

For example, $\ell_d^o(7' + 3 + 3' + 1 + 1 + 1 + 1' + 1) = 2$.

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For a PDO partition pair $(\alpha|\beta)$, let $\ell_r(\alpha, \beta)$ be the number of different part sizes that occur (simultaneously) in both α and β.

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Now let

$$P_2(x,y,q) := \sum_{(\alpha|\beta)\in\mathcal{PDO}\times\mathcal{PDO}} x^{\ell_d(\alpha)+\ell_d(\beta)} y^{2\ell_r(\alpha,\beta)} q^{|\alpha|+|\beta|}.$$

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We then have the following:

Our Refinements

$$P_1(x,y,q) = \prod_{m\geq 1}\left(1+2x\frac{q^{4m-2}}{(1-q^{4m-2})^2}+xy\frac{q^{2m-1}(1+q^{4m-2})}{(1-q^{4m-2})^2}\right)$$
$$= \prod_{m\geq 1}\left(1+xy\frac{q^{2m-1}}{(1-q^{2m-1})^2}+2x(1-y)\frac{q^{4m-2}}{(1-q^{4m-2})^2}\right),$$

and

$$P_2(x,y,q) = \prod_{m\geq 1}\left(1+2x\frac{q^{2m-1}}{(1-q^{2m-1})^2}+x^2y^2\left(\frac{q^{2m-1}}{(1-q^{2m-1})^2}\right)^2\right)$$
$$= \prod_{m\geq 1}\left(\left(1+xy\frac{q^{2m-1}}{(1-q^{2m-1})^2}\right)^2+2x(1-y)\frac{q^{2m-1}}{(1-q^{2m-1})^2}\right).$$

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Moreover, for all $n \geq 0$ we have

$$[q^{2n}]P_1(x, y, q) = [q^n]P_2(x, y, q).$$

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Let me demonstrate the above result with a brief example.

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Our proof involves tools similar to those mentioned above, along with the extension of some of the results of Andrews and Rose.

Let me demonstrate the above result with a brief example.

Example: Let $n = 4$ and $k = j = 2$.

Our Refinements

There are **ten** PDO partitions in $\mathcal{PDO}(8)$ which contain exactly two different part sizes, and each of them occur an odd number of times (these are derived from designating the parts in $7+1$, $5+3$, $5+1+1+1$, and $3+1+1+1+1+1$).

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On the other hand, the following **ten** PDO partition pairs are all of those having combined weight 4, with 2 designated parts and $j/2 = 1$ simultaneously shared part size.

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On the other hand, the following **ten** PDO partition pairs are all of those having combined weight 4, with 2 designated parts and $j/2 = 1$ simultaneously shared part size.

$$(1'+1|1'+1), \quad (1'+1|1+1'),$$
$$(1+1'|1'+1), \quad (1+1'|1+1'),$$
$$(1'|1'+1+1), \quad (1'|1+1'+1), \quad (1'|1+1+1'),$$
$$(1'+1+1|1'), \quad (1+1'+1|1'), \quad (1+1+1'|1')$$

Closing Thoughts

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- As a complete sidenote, if we return to the function $\mathrm{PD}(n)$ of Andrews, Lewis, and Lovejoy, where there are no parity restrictions on the parts in question, then we see that
$$\sum_{n\geq 0}\mathrm{PD}(n)q^n = \sum_{k\geq 0}A_k(q).$$

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$$\sum_{n\geq 0} \mathrm{PD}(n)q^n = \sum_{k\geq 0} A_k(q).$$
It is not clear whether this can be utilized in any beneficial way; even so, it's worth highlighting as this yields (in analogous fashion) a refinement of the function $\mathrm{PD}(n)$ by the number of parts present in each such partition.

Closing Thoughts

I'll close with a few comments:

- In essence, our proofs reduce to proving a few trigonometric identities (which are, at their core, statements about Chebyshev polynomials). For the most part, once we have reduced the work to these identities, their proofs are relatively straightforward.

Closing Thoughts

I'll close with a few comments:

- When Shishuo and I began our discussions at the Andrews–Berndt conference in June 2024, our primary goal was to find a purely bijective / combinatorial proof of the original identity that I shared at the beginning of this talk. Unfortunately, such a proof has proven elusive. Nevertheless, we are very happy to have proven these refinements of the original identity.

Closing Thoughts

And with that I will close.

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Thanks very much for attending today.

James Sellers
University of Minnesota Duluth

A New View of Odd-Part Partitions with Designated Summands

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