Elementary Proofs of Congruences Modulo 5 for Overpartitions with Restricted Odd Differences

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Finishing The Proof of the Congruences

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Opening Comments

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For example, the partitions of $n = 4$ are given by

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1,$$

while the overpartitions of $n = 4$ are

$$4, \quad \overline{4}, \quad 3+1, \quad \overline{3}+1, \quad 3+\overline{1}, \quad \overline{3}+\overline{1}, \quad 2+2, \quad 2+\overline{2},$$

$$2+1+1, \quad \overline{2}+1+1, \quad 2+1+\overline{1}, \quad \overline{2}+1+\overline{1}, \quad 1+1+1+1, \quad 1+1+1+\overline{1}.$$

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In 2015, Bringmann, Dousse, Lovejoy, and Mahlburg defined the function $\overline{t}(n)$ to be the number of overpartitions of weight n where (i) the difference between two successive parts may be odd only if the larger part is overlined and (ii) if the smallest part is odd then it is overlined.

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For example, $\overline{t}(4) = 8$ where the overpartitions in question are

$$4, \quad \overline{4}, \quad 3+\overline{1}, \quad \overline{3}+\overline{1}, \quad 2+2, \quad 2+\overline{2}, \quad \overline{2}+1+\overline{1}, \quad 1+1+1+\overline{1}.$$

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In their 2015 paper, BDLM prove that the generating function for $\overline{t}(n)$ is given by

$$\sum_{n\geq 0} \overline{t}(n)q^n = \frac{f_3}{f_1 f_2},$$

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where

$$f_k = f_k(q) := (q^k; q^k)_\infty \qquad \text{with} \qquad (a;q)_\infty := \prod_{i \geq 0} (1 - aq^i).$$

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They also proved that $\bar{t}(n)$ satisfies a nice characterization modulo 3.

Theorem 1

For all $n \geq 1$,

$$\bar{t}(n) \equiv \begin{cases} (-1)^{k+1} \pmod 3 & \text{if } n = k^2 \text{ for some integer } k, \\ 0 \pmod 3 & \text{otherwise.} \end{cases}$$

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In particular, in 2023, Hanson and Smith utilized the theory of modular forms to prove the following two congruences modulo 5:

For all $n \geq 0$,

$$\overline{t}(80n + 40) \equiv \overline{t}(80n + 60) \equiv 0 \pmod{5}.$$

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Instead, we utilize well–known dissection properties which already appear in the literature, along with a generalization of Chern and Tang's work in 2021, to initially determine the behavior of the generating function for $\overline{t}(5n)$ modulo 5.

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Instead, we utilize well–known dissection properties which already appear in the literature, along with a generalization of Chern and Tang's work in 2021, to initially determine the behavior of the generating function for $\overline{t}(5n)$ modulo 5.

This is, by far, the most involved aspect of our proof.

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Our goal in this work is straightforward – to provide an alternative proof of the above pair of congruences which does not rely on modular forms.

Instead, we utilize well–known dissection properties which already appear in the literature, along with a generalization of Chern and Tang's work in 2021, to initially determine the behavior of the generating function for $\overline{t}(5n)$ modulo 5.

This is, by far, the most involved aspect of our proof.

Once this is complete, we simply need to 2–dissect the generating function for $\overline{t}(5n)$ further until we reach a congruence modulo 5 satisfied by the generating function for $\overline{t}(20n)$.

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Once this is done, the proof of the above pair of congruences modulo 5 is readily finished.

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First, we require the following 2-dissection results, all of which already appear in the literature.

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Lemma 2

$$
\begin{aligned}
f_1^2 &= \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\,\frac{f_2 f_{16}^2}{f_8},\\
\frac{f_1^3}{f_3} &= \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2},\\
\frac{f_3^3}{f_1} &= \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\,\frac{f_{12}^3}{f_4},\\
\frac{f_1}{f_3} &= \frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\,\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}},\\
\frac{1}{f_1 f_3} &= \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\,\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}},\\
\frac{f_1^2}{f_3^2} &= \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}.
\end{aligned}
$$

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Next, we require the following identity which we easily proved using the results already mentioned above.

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Lemma 3

We have

$$\frac{f_1^6}{f_3^2} = \frac{f_2 f_4 f_6^5}{f_{12}^3} - 6q\frac{f_2^2 f_4^2 f_{12}^2}{f_6^2} + 10q^2\frac{f_2^4 f_{12}^6}{f_4^2 f_6^4}.$$

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Note the coefficient 10 above; remember, we will be working modulo 5 for the remainder of this work...

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Next, we introduce the Rogers–Ramanujan continued fraction

$$R := R(q) = \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty},$$

which first appeared in the work of Rogers and later in Ramanujan's first letter to Hardy.

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The following 5–dissections involving R are well–known.

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Lemma 4

We have

$$f_1 = f_{25}\left(\frac{1}{R(q^5)} - q - q^2 R(q^5)\right),$$

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\,\Phi\big(R(q^5),\, q\big),$$

where

$$\Phi(r,Q) = \frac{1}{r^4} + \frac{Q}{r^3} + \frac{2Q^2}{r^2} + \frac{3Q^3}{r} + 5Q^4 - 3Q^5 r + 2Q^6 r^2 - Q^7 r^3 + Q^8 r^4.$$

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As immediate consequences of the previous lemma, we obtain

$$f_3 = f_{75}\left(\frac{1}{R(q^{15})} - q^3 - q^6\,R(q^{15})\right),$$

$$\frac{1}{f_k} = \frac{f_{25k}^5}{f_{5k}^6}\,\Phi\big(R(q^{5k}),\,q^k\big).$$

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To simplify the notation in our work below, we set

$$K = q^{-1}\frac{f_2 f_5^5}{f_1 f_{10}^5}, \qquad S = q^{-2}\frac{f_1^3 f_3^3}{f_5^3 f_{15}^3}, \qquad T = q^{-2}\frac{f_3 f_5^5}{f_1 f_{15}^5},$$

$$U = q^{-1}\frac{f_2^3 f_3^3}{f_1^2 f_6^2 f_{10} f_{15}}, \qquad V = \frac{f_1^2 f_6 f_{30}}{f_3 f_{10}^2 f_{15}}.$$

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For $\alpha \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathbb{Z}$, Chern and Tang (2021) introduced the two families

$$\frac{1}{q^{\alpha} R(q)^{\alpha+2\beta} R(q^2)^{2\alpha-\beta}} + (-1)^{\alpha+\beta} q^{\alpha} R(q)^{\alpha+2\beta} R(q^2)^{2\alpha-\beta},$$
$$\frac{1}{q^{\alpha} R(q)^{2\alpha+3\beta} R(q^3)^{\alpha-\beta}} + (-1)^{\alpha} q^{\alpha} R(q)^{2\alpha+3\beta} R(q^3)^{\alpha-\beta},$$

and represented them in terms of eta quotients involving K, S, T through recurrence relations.

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and represented them in terms of eta quotients involving K, S, T through recurrence relations.

In our first manuscript mentioned in the introduction, we extended the results of Chern and Tang to the following three-parameter family, which is extremely beneficial for representing the generating functions in question.

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For a triple $(m, n, p) \in \mathbb{Z}^3$, we set

$$w(m,n,p) := q^{-p} R(q)^{-2m-n} R(q^2)^{m-n-p} R(q^3)^{n-p},$$

and define

$$G(m,n,p) := w(m,n,p) + \frac{(-1)^{m+n}}{w(m,n,p)}.$$

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Then, we immediately have $G(0, 0, 0) = 2$ and the reflection property

$$G(-m, -n, -p) = (-1)^{m+n} G(m, n, p).$$

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Then, we immediately have $G(0, 0, 0) = 2$ and the reflection property

$$G(-m, -n, -p) = (-1)^{m+n} G(m, n, p).$$

We can then prove the following recurrences for all $(m, n, p) \in \mathbb{Z}^3$.

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$$G(m+1,n,p) = \frac{4}{K}\,G(m,n,p) + G(m-1,n,p),$$
$$G(m,n+1,p) = (1-V)\,G(m,n,p) + G(m,n-1,p),$$
$$G(m,n,p+1) = (U-2)\,G(m,n,p) - G(m,n,p-1).$$

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$$G(m+1,n,p) = \frac{4}{K}\,G(m,n,p) + G(m-1,n,p),$$
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$$G(m,n,p+1) = (U-2)\,G(m,n,p) - G(m,n,p-1).$$

Moreover, we have the initial values

$$G(0,0,0) = 2, \quad G(1,0,0) = \frac{4}{K}, \quad G(0,1,0) = 1-V, \quad G(0,0,1) = U-2,$$
$$G(1,1,1) = K + 2 + \frac{4}{K}, \quad G(1,1,0) = 2 + \frac{9}{T}, \quad G(0,1,1) = K,$$
$$G(1,0,1) = \frac{1}{4}\left(T - S + \frac{9}{T} + 6\right).$$

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We also require the parameterization of f–quotients due to Alaca, Alaca, and Williams (2006).

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$$\varphi(q) := 1 + 2\sum_{n\geq 1} q^{n^2}$$

denote the classical theta function of Ramanujan, and set

$$s := \frac{\varphi^3(q^3)}{\varphi(q)}, \qquad t := \frac{\varphi^2(q) - \varphi^2(q^3)}{4q\,\varphi^2(q^3)}.$$

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Lemma 5

We have

$$f_1 = s^{1/2}\, t^{1/24} (1-2qt)^{1/2} (1+qt)^{1/8} (1+2qt)^{1/6} (1+4qt)^{1/8},$$
$$f_2 = s^{1/2}\, t^{1/12} (1-2qt)^{1/4} (1+qt)^{1/4} (1+2qt)^{1/12} (1+4qt)^{1/4},$$
$$f_3 = s^{1/2}\, t^{1/8} (1-2qt)^{1/6} (1+qt)^{1/24} (1+2qt)^{1/2} (1+4qt)^{1/24},$$
$$f_6 = s^{1/2}\, t^{1/4} (1-2qt)^{1/12} (1+qt)^{1/12} (1+2qt)^{1/4} (1+4qt)^{1/12}.$$

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Lastly, as we close this section, we note the following lemma which follows from the Binomial Theorem.

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Lemma 6

For a prime p and positive integers k and l,

$$f_l^{p^k} \equiv f_{lp}^{p^{k-1}} \pmod{p^k}.$$

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Lastly, as we close this section, we note the following lemma which follows from the Binomial Theorem.

Lemma 6

For a prime p and positive integers k and l,

$$f_l^{p^k} \equiv f_{lp}^{p^{k-1}} \pmod{p^k}.$$

With all the above tools in hand, we are now in a position to prove our main theorem.

Proof of Our Main Theorem

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We begin by establishing the following generating function result.

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Theorem 7

We have

$$\sum_{n\geq 0} \bar{t}(5n)q^n \equiv \frac{f_1 f_2 f_3^3}{f_6^2} \pmod 5.$$

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Editorial comment: This is, by far, the most difficult result that we must prove...

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$$\sum_{n\geq 0} \overline{t}(5n)q^n \equiv \frac{f_1 f_2 f_3^3}{f_6^2} \pmod 5.$$

Editorial comment: This is, by far, the most difficult result that we must prove... Everything else that is needed follows extremely easily...

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Proof of Our Main Theorem

We begin by establishing the following generating function result.

Theorem 7

We have

$$\sum_{n\geq 0} \overline{t}(5n)q^n \equiv \frac{f_1 f_2 f_3^3}{f_6^2} \pmod 5.$$

Editorial comment: This is, by far, the most difficult result that we must prove... Everything else that is needed follows extremely easily...

Proof: Using the 5–dissections that involve R mentioned above, we have

$$\sum_{n\geq 0} \overline{t}(n)q^n = \frac{f_{25}^5 f_{50}^5 f_{75}}{f_5^6 f_{10}^6}\, \Phi\big(R(q^5), q\big)\, \Phi\big(R(q^{10}), q^2\big) \left(\frac{1}{R(q^{15})} - q^3 - q^6\, R(q^{15})\right).$$

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Proof of Our Main Theorem

Extracting the terms in which the exponents of q are congruent to 0 modulo 5 and then replacing q^5 by q gives

$$\sum_{n\geq 0} \bar{t}(5n)q^n = \frac{f_5^5 f_{10}^5 f_{15}}{f_1^6 f_2^6}\,\Lambda,$$

where Λ denotes the sum of the terms of the product

$$\Phi\big(R(q^5),q\big)\,\Phi\big(R(q^{10}),q^2\big)\big(R(q^{15})^{-1}-q^3-q^6R(q^{15})\big)$$

whose q-exponent is divisible by 5, with q^5 then replaced by q.

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whose q-exponent is divisible by 5, with q^5 then replaced by q.

Thus Λ can be expressed as a Laurent polynomial in $R(q)$, $R(q^2)$, and $R(q^3)$ with coefficients in $\mathbb{Z}[q]$.

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Proof of Our Main Theorem

Carrying out this finite extraction and reducing every coefficient modulo 5 simplifies Λ to

$$\Lambda \equiv \frac{L_0}{R(q^3)} + L_1 + R(q^3)\, L_2 \pmod 5,$$

where L_0, L_1, L_2 are the following Laurent polynomials in $R(q)$ and $R(q^2)$:

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where L_0, L_1, L_2 are the following Laurent polynomials in $R(q)$ and $R(q^2)$:

$$\begin{aligned} L_0 = {} & R(q)^{-4}R(q^2)^{-4} + q\big(2R(q)R(q^2)^{-4} + 3R(q)^{-1}R(q^2)^{-3} + 2R(q)^{-3}R(q^2)^{-2}\big) \\ & + q^2\big(R(q)^4R(q^2)^{-3} + 4R(q)^2R(q^2)^{-2} + 2R(q)^{-4}R(q^2)\big) \\ & + q^3\big(4R(q)R(q^2) + R(q)^{-1}R(q^2)^2 + 4R(q)^{-3}R(q^2)^3\big) \\ & + q^4\big(2R(q)^4R(q^2)^2 + 3R(q)^2R(q^2)^3\big)\,, \end{aligned}$$

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$$
\begin{aligned}
L_1 = {} & q\left(3R(q)^{-2}R(q^2)^{-4} + 4R(q)^{-4}R(q^2)^{-3}\right) \\
& + q^2\left(R(q)^3R(q^2)^{-4} + 3R(q)R(q^2)^{-3} + 4R(q)^{-1}R(q^2)^{-2} + 2R(q)^{-3}R(q^2)^{-1}\right) \\
& + q^3\left(3R(q)^4R(q^2)^{-2} + 4R(q)^2R(q^2)^{-1} + R(q)^{-2}R(q^2) + 3R(q)^{-4}R(q^2)^2\right) \\
& + q^4\left(2R(q)^3R(q^2) + R(q)R(q^2)^2 + 3R(q)^{-1}R(q^2)^3 + 4R(q)^{-3}R(q^2)^4\right) \\
& + q^5\left(R(q)^4R(q^2)^3 + 3R(q)^2R(q^2)^4\right),
\end{aligned}
$$

Proof of Our Main Theorem

$$\begin{aligned} L_2 = {} & q^2\left(3R(q)^{-2}R(q^2)^{-3} + 3R(q)^{-4}R(q^2)^{-2}\right) \\ & + q^3\left(R(q)^3R(q^2)^{-3} + R(q)R(q^2)^{-2} + R(q)^{-1}R(q^2)^{-1}\right) \\ & + q^4\left(2R(q)^4R(q^2)^{-1} + R(q)^{-2}R(q^2)^2 + R(q)^{-4}R(q^2)^3\right) \\ & + q^5\left(2R(q)^3R(q^2)^2 + 2R(q)R(q^2)^3 + 2R(q)^{-1}R(q^2)^4\right) \\ & + q^6\left(4R(q)^4R(q^2)^4\right). \end{aligned}$$

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$$\begin{aligned} L_2 = q^2\big(&3R(q)^{-2}R(q^2)^{-3} + 3R(q)^{-4}R(q^2)^{-2}\big) \\ &+ q^3\big(R(q)^3R(q^2)^{-3} + R(q)R(q^2)^{-2} + R(q)^{-1}R(q^2)^{-1}\big) \\ &+ q^4\big(2R(q)^4R(q^2)^{-1} + R(q)^{-2}R(q^2)^2 + R(q)^{-4}R(q^2)^3\big) \\ &+ q^5\big(2R(q)^3R(q^2)^2 + 2R(q)R(q^2)^3 + 2R(q)^{-1}R(q^2)^4\big) \\ &+ q^6\big(4R(q)^4R(q^2)^4\big)\,. \end{aligned}$$

So we have

$$\sum_{n\geq 0}\bar{t}(5n)q^n \equiv \frac{f_5^5 f_{10}^5 f_{15}}{f_1^6 f_2^6}\left(\frac{L_0}{R(q^3)} + L_1 + R(q^3)L_2\right) \pmod 5.$$

Proof of Our Main Theorem

Now a key step!

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We pair the twelve terms of $L_0/R(q^3)$ with the twelve terms of $R(q^3)L_2$, while the sixteen terms of L_1 (with coefficient $R(q^3)^0$) form eight internal pairs.

Proof of Our Main Theorem

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This pairing is done in a natural way with an eye towards rewriting our results in terms of $G(m,n,p)$ for specific values of m, n, and p.

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Indeed, this is EXACTLY why we needed to construct $G(m,n,p)$ and understand its properties!

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Indeed, this is EXACTLY why we needed to construct $G(m,n,p)$ and understand its properties!

Collecting these twenty blocks modulo 5, we obtain:

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Proof of Our Main Theorem

$$\begin{aligned}\frac{L_0}{R(q^3)} + L_1 + R(q^3)L_2 \equiv & \left(\frac{1}{R(q)^4R(q^2)^4R(q^3)} - q^6\, R(q)^4R(q^2)^4R(q^3)\right) \\ & + 2q\left(\frac{R(q)}{R(q^2)^4R(q^3)} + q^4\,\frac{R(q^2)^4R(q^3)}{R(q)}\right) \\ & \vdots \\ & + 3q^3\left(\frac{R(q)^4}{R(q^2)^2} + \frac{R(q^2)^2}{R(q)^4}\right) \\ & + 4q^3\left(\frac{R(q)^2}{R(q^2)} - \frac{R(q^2)}{R(q)^2}\right) \pmod 5\end{aligned}$$

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Proof of Our Main Theorem

$$
\begin{aligned}
&= q^3 \big[G(1,2,3) + 2\,G(-1,1,2) + 3\,G(0,1,2) + 2\,G(1,1,2) \\
&\quad + G(-2,0,1) + 4\,G(-1,0,1) + 2\,G(2,0,1) + 4\,G(0,-1,0) \\
&\quad + G(1,-1,0) + 4\,G(2,-1,0) + 3\,G(1,2,1) + 3\,G(0,2,1) \\
&\quad + 3\,G(0,2,2) + 4\,G(1,2,2) + G(-2,1,1) + 3\,G(-1,1,1) \\
&\quad + 4\,G(0,1,1) + 2\,G(1,1,1) + 3\,G(-2,0,0) + 4\,G(-1,0,0)\big],
\end{aligned}
$$

Proof of Our Main Theorem

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\begin{aligned}
&= q^3 \big[G(1,2,3) + 2\,G(-1,1,2) + 3\,G(0,1,2) + 2\,G(1,1,2) \\
&\quad + G(-2,0,1) + 4\,G(-1,0,1) + 2\,G(2,0,1) + 4\,G(0,-1,0) \\
&\quad + G(1,-1,0) + 4\,G(2,-1,0) + 3\,G(1,2,1) + 3\,G(0,2,1) \\
&\quad + 3\,G(0,2,2) + 4\,G(1,2,2) + G(-2,1,1) + 3\,G(-1,1,1) \\
&\quad + 4\,G(0,1,1) + 2\,G(1,1,1) + 3\,G(-2,0,0) + 4\,G(-1,0,0)\big],
\end{aligned}
$$

where this last equality follows from recognizing that each such pairing above can be rewritten in terms of $G(m,n,p)$.

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&\quad + G(1,-1,0) + 4\,G(2,-1,0) + 3\,G(1,2,1) + 3\,G(0,2,1) \\
&\quad + 3\,G(0,2,2) + 4\,G(1,2,2) + G(-2,1,1) + 3\,G(-1,1,1) \\
&\quad + 4\,G(0,1,1) + 2\,G(1,1,1) + 3\,G(-2,0,0) + 4\,G(-1,0,0)\big],
\end{aligned}
$$

where this last equality follows from recognizing that each such pairing above can be rewritten in terms of $G(m,n,p)$.

We next wish to simplify modulo 5.

Proof of Our Main Theorem

By finitely many applications of the recurrences and initial values I mentioned above, along with the reflection property, we obtain the following equalities.

Proof of Our Main Theorem

By finitely many applications of the recurrences and initial values I mentioned above, along with the reflection property, we obtain the following equalities.

$$G(1,2,3) = (U-1)(U-3)\left[\frac{1}{4}\left(T - S + \frac{9}{T}\right) + \frac{3}{2} + (1-V)\left(K + 2 + \frac{4}{K}\right)\right] - (U-2)\left[(1-V)\left(2 + \frac{9}{T}\right) + \frac{4}{K}\right],$$

$$G(-1,1,2) = (U-2)\left(K - 2 + \frac{4}{K}\right) + \frac{4(1-V)}{K} - 2 - \frac{9}{T},$$

$$G(0,1,2) = K(U-2) + V - 1,$$

$$G(1,1,2) = (U-2)\left(K + 2 + \frac{4}{K}\right) - 2 - \frac{9}{T},$$

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$$G(-2,0,1) = (U-2)\left(1+\frac{16}{K^2}\right) - \frac{1}{K}\left(T - S + \frac{9}{T}\right) - \frac{6}{K},$$

$$G(-1,0,1) = \frac{1}{4}\left(T - S + \frac{9}{T}\right) + \frac{3}{2} - \frac{4(U-2)}{K},$$

$$G(2,0,1) = (U-2) + \frac{1}{K}\left(T - S + \frac{9}{T}\right) + \frac{6}{K},$$

$$G(0,-1,0) = V - 1,$$

$$G(1,-1,0) = 2 + \frac{9}{T} - \frac{4(1-V)}{K},$$

$$G(2,-1,0) = -(1-V)\left(1+\frac{16}{K^2}\right) + \frac{8}{K} + \frac{36}{KT},$$

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Proof of Our Main Theorem

$$G(1,2,1) = \frac{1}{4}\left(T - S + \frac{9}{T}\right) + \frac{3}{2} + (1-V)\left(K + 2 + \frac{4}{K}\right),$$

$$G(0,2,1) = (1-V)K + U - 2,$$

$$G(0,2,2) = (1-V)K(U-2) - (1-V)^2 + (U-2)^2 - 2,$$

$$G(1,2,2) = (U-2)\left[\frac{1}{4}\left(T - S + \frac{9}{T}\right) + \frac{3}{2}\right] + (1-V)\left[(U-2)\left(K + 2 + \frac{4}{K}\right) - 2 - \frac{9}{T}\right] - \frac{4}{K},$$

$$G(-2,1,1) = K - 4 + \frac{8}{K} - \frac{16}{K^2},$$

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Proof of Our Main Theorem

$$G(-1,1,1) = K - 2 + \frac{4}{K},$$
$$G(0,1,1) = K,$$
$$G(1,1,1) = K + 2 + \frac{4}{K},$$
$$G(-2,0,0) = 2 + \frac{16}{K^2},$$
$$G(-1,0,0) = -\frac{4}{K}.$$

Proof of Our Main Theorem

$$
\begin{aligned}
G(-1,1,1) &= K - 2 + \frac{4}{K}, \\
G(0,1,1) &= K, \\
G(1,1,1) &= K + 2 + \frac{4}{K}, \\
G(-2,0,0) &= 2 + \frac{16}{K^2}, \\
G(-1,0,0) &= -\frac{4}{K}.
\end{aligned}
$$

We now reduce the twenty evaluations in the above list modulo 5.

Proof of Our Main Theorem

Setting $A := S + T^{-1} + 4T$ and completing a SIGNIFICANT number of reductions modulo 5, we obtain

$$\frac{L_0}{R(q^3)} + L_1 + R(q^3)L_2 \equiv q^3K^{-2}T^{-1}P(K,T,A,U,V) \pmod 5$$

where

Proof of Our Main Theorem

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where

$$\begin{aligned}
&P(K,T,A,U,V)\\
&= KT\big[KA + (K^2+2K+4)(1-V) + 2K\big](U-2)^2\\
&\quad + \big[4K^2TA + K(2K^2T + KT + K + T)(1-V)\\
&\qquad\qquad + 2K^3T + 2K^2T + KT + T\big](U-2)\\
&\quad + KT(K+4)A + 2K^2T(1-V)^2 + (4K^2 + 2KT + T)(1-V)\\
&\quad + 2K^2T + 3K^2 + 4KT + 4K + 2T.
\end{aligned}$$

Proof of Our Main Theorem

From the work we completed above, this means

$$\sum_{n\geq 0} \overline{t}(5n)q^n \equiv \frac{f_5^5 f_{10}^5 f_{15}}{f_1^6 f_2^6} q^3 K^{-2} T^{-1} P(K,T,A,U,V) \pmod 5.$$

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Here and below we write

$$g := K - 3 - \frac{4}{K} = q^{-1}\frac{f_1^2 f_2^2}{f_5^2 f_{10}^2},$$

where the last equality follows from one of our lemmas above.

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Here and below we write

$$g := K - 3 - \frac{4}{K} = q^{-1}\frac{f_1^2 f_2^2}{f_5^2 f_{10}^2},$$

where the last equality follows from one of our lemmas above.

Thanks to the Alaca–Alaca–Williams parameterization, we have the following:

Proof of Our Main Theorem

Lemma 8

$$K \equiv \frac{(1-2qt)^6(1+2qt)^2}{qt\,(1+qt)^3(1+4qt)^3} \pmod 5,$$

$$T \equiv \frac{(1-2qt)^8(1+qt)^2(1+4qt)^2}{q^2t^2\,(1+2qt)^8} \pmod 5,$$

$$g \equiv \frac{1}{s^8\,qt\,(1-2qt)^6(1+qt)^3(1+2qt)^2(1+4qt)^3} \pmod 5,$$

$$U \equiv \frac{1}{s^4\,qt\,(1-2qt)^2(1+qt)(1+2qt)^2(1+4qt)} \pmod 5,$$

$$V \equiv \frac{1}{s^4\,(1-2qt)^2(1+qt)^2(1+2qt)^2(1+4qt)^2} \pmod 5,$$

$$S \equiv \frac{1}{s^{12}\,q^2t^2\,(1-2qt)^8(1+qt)^2(1+2qt)^8(1+4qt)^2} \pmod 5.$$

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Proof of Our Main Theorem

Using facts about the s and t from the Alaca–Alaca–Williams parameterization, we then prove the following "collapse" lemma.

Proof of Our Main Theorem

Using facts about the s and t from the Alaca–Alaca–Williams parameterization, we then prove the following "collapse" lemma.

Lemma 9

We have

$$K^{-2}T^{-1}P(K,T,A,U,V) \equiv U\,g^2 \pmod 5.$$

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Proof of Our Main Theorem

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Lemma 9

We have

$$K^{-2}T^{-1}P(K,T,A,U,V) \equiv U\,g^2 \pmod 5.$$

Applying the above lemma, we obtain

$$\sum_{n\geq 0} \overline{t}(5n)q^n \equiv \frac{f_5^5 f_{10}^5 f_{15}}{f_1^6 f_2^6} q^3 U g^2 \pmod 5.$$

Proof of Our Main Theorem

Substituting the f-quotient results that we have for U and g, we get

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(5n)q^n &\equiv \frac{f_5^5 f_{10}^5 f_{15}}{f_1^6 f_2^6}q^3\left(q^{-1}\frac{f_2^3 f_3^3}{f_1^2 f_6^2 f_{10} f_{15}}\right)\left(q^{-1}\frac{f_1^2 f_2^2}{f_5^2 f_{10}^2}\right)^2 \pmod 5\\ &\equiv \frac{f_2 f_3^3 f_5}{f_1^4 f_6^2} \pmod 5\\ &= \frac{f_1 f_2 f_3^3}{f_6^2}\cdot\frac{f_5}{f_1^5}\\ &\equiv \frac{f_1 f_2 f_3^3}{f_6^2} \pmod 5.\end{aligned}$$

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Proof of Our Main Theorem

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$$\begin{aligned}\sum_{n\geq 0}\overline{t}(5n)q^n &\equiv \frac{f_5^5 f_{10}^5 f_{15}}{f_1^6 f_2^6}q^3\left(q^{-1}\frac{f_2^3 f_3^3}{f_1^2 f_6^2 f_{10} f_{15}}\right)\left(q^{-1}\frac{f_1^2 f_2^2}{f_5^2 f_{10}^2}\right)^2 \pmod 5\\ &\equiv \frac{f_2 f_3^3 f_5}{f_1^4 f_6^2} \pmod 5\\ &= \frac{f_1 f_2 f_3^3}{f_6^2}\cdot\frac{f_5}{f_1^5}\\ &\equiv \frac{f_1 f_2 f_3^3}{f_6^2} \pmod 5.\end{aligned}$$

This completes the proof of our main theorem.

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Now that we have a generating function result for $\overline{t}(5n)$, we can move forward to generating functions for $\overline{t}(40n)$ and $\overline{t}(40n+20)$ modulo 5.

Finishing The Proof of the Congruences

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Theorem 10

We have

$$\sum_{n\geq 0}\overline{t}(40n)q^n \equiv \frac{f_1^6}{f_3^2}\cdot\frac{f_8f_{12}^2}{f_2^2f_4f_{24}} - 4q\,\frac{1}{f_1f_3}\cdot\frac{f_2^2f_4^2f_6^2f_{24}}{f_8f_{12}} \pmod 5,$$

$$\sum_{n\geq 0}\overline{t}(40n+20)q^n \equiv 4\frac{1}{f_1f_3}\cdot\frac{f_2^3f_6f_8f_{12}^2}{f_4f_{24}} - \frac{f_1^6}{f_3^2}\cdot\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}} \pmod 5.$$

Finishing The Proof of the Congruences

Proof: From our main theorem, we have

$$\sum_{n\geq 0}\overline{t}(5n)q^n \equiv \frac{f_2}{f_6^2}\cdot\frac{f_3^3}{f_1}\cdot f_1^2 \pmod 5$$

$$= \frac{f_2}{f_6^2}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\,\frac{f_{12}^3}{f_4}\right)\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\,\frac{f_2 f_{16}^2}{f_8}\right).$$

Finishing The Proof of the Congruences

Proof: From our main theorem, we have

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$$= \frac{f_2}{f_6^2}\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\,\frac{f_{12}^3}{f_4}\right)\left(\frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\,\frac{f_2 f_{16}^2}{f_8}\right).$$

Extracting the terms with even exponents of q and then replacing q^2 by q gives

$$\sum_{n\geq 0}\overline{t}(10n)q^n \equiv \frac{f_2 f_4^5}{f_6 f_8^2} - 2q\frac{f_1^2 f_6^3 f_8^2}{f_2 f_3^2 f_4} \pmod 5$$
$$= \frac{f_2 f_4^5}{f_6 f_8^2} - 2q\frac{f_6^3 f_8^2}{f_2 f_4}\left(\frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}\right).$$

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Isolating the terms in which the exponents of q are even and then replacing q^2 by q, we obtain

$$\sum_{n\geq 0} \bar{t}(20n)q^n \equiv \frac{f_1}{f_3}\left(\frac{f_2^5}{f_4^2} + 4q\frac{f_4^3 f_6 f_{12}}{f_2^2}\right) \pmod 5$$
$$= \left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}\right)\left(\frac{f_2^5}{f_4^2} + 4q\frac{f_4^3 f_6 f_{12}}{f_2^2}\right).$$

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Finishing The Proof of the Congruences

Isolating the terms in which the exponents of q are even and then replacing q^2 by q, we obtain

$$\sum_{n\geq 0} \overline{t}(20n)q^n \equiv \frac{f_1}{f_3}\left(\frac{f_2^5}{f_4^2} + 4q\frac{f_4^3 f_6 f_{12}}{f_2^2}\right) \pmod 5$$
$$= \left(\frac{f_2 f_{16} f_{24}^2}{f_6^2 f_8 f_{48}} - q\frac{f_2 f_8^2 f_{12} f_{48}}{f_4 f_6^2 f_{16} f_{24}}\right)\left(\frac{f_2^5}{f_4^2} + 4q\frac{f_4^3 f_6 f_{12}}{f_2^2}\right).$$

Now, extracting the terms with even exponents of q and then replacing q^2 by q yields the result for $\overline{t}(40n)$. Similarly, isolating the terms with odd exponents of q, dividing both sides by q, and then replacing q^2 by q gives the result for $\overline{t}(40n+20)$.

Finishing The Proof of the Congruences

Proof of Our Congruences

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$$\sum_{n\geq 0}\bar{t}(40n)q^n \equiv \frac{f_1^6}{f_3^2}\cdot\frac{f_8f_{12}^2}{f_2^2f_4f_{24}} - 4q\,\frac{1}{f_1f_3}\cdot\frac{f_2^2f_4^2f_6^2f_{24}}{f_8f_{12}} \pmod 5$$

$$\equiv \left(\frac{f_2f_4f_6^5}{f_{12}^3} + 4q\frac{f_2^2f_4^2f_{12}^2}{f_6^2}\right)\frac{f_8f_{12}^2}{f_2^2f_4f_{24}}$$

$$- 4q\left(\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right)\frac{f_2^2f_4^2f_6^2f_{24}}{f_8f_{12}} \pmod 5$$

$$= \frac{f_6^5f_8}{f_2f_{12}f_{24}} - 4q^2\,\frac{f_4^7f_{24}^3}{f_2^2f_8^3f_{12}^2}$$

$$\equiv Y_0(q^2) \pmod 5,$$

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$$\sum_{n\geq 0}\overline{t}(40n)q^n \equiv \frac{f_1^6}{f_3^2}\cdot\frac{f_8f_{12}^2}{f_2^2f_4f_{24}} - 4q\,\frac{1}{f_1f_3}\cdot\frac{f_2^2f_4^2f_6^2f_{24}}{f_8f_{12}} \pmod 5$$

$$\equiv \left(\frac{f_2f_4f_6^5}{f_{12}^3} + 4q\frac{f_2^2f_4^2f_{12}^2}{f_6^2}\right)\frac{f_8f_{12}^2}{f_2^2f_4f_{24}}$$

$$- 4q\left(\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right)\frac{f_2^2f_4^2f_6^2f_{24}}{f_8f_{12}} \pmod 5$$

$$= \frac{f_6^5f_8}{f_2f_{12}f_{24}} - 4q^2\,\frac{f_4^7f_{24}^3}{f_2^2f_8^3f_{12}^2}$$

$$\equiv Y_0(q^2) \pmod 5,$$

so that $\overline{t}(40(2n+1)) = \overline{t}(80n+40) \equiv 0 \pmod 5$.

Finishing The Proof of the Congruences

and ...

Finishing The Proof of the Congruences

and ...

$$\begin{aligned}\sum_{n\geq 0}\bar{t}(40n+20)q^n &\equiv 4\frac{1}{f_1f_3}\cdot\frac{f_2^3f_6f_8f_{12}^2}{f_4f_{24}}-\frac{f_1^6}{f_3^2}\cdot\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}} \pmod 5\\ &\equiv 4\left(\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right)\frac{f_2^3f_6f_8f_{12}^2}{f_4f_{24}}\\ &\quad-\left(\frac{f_2f_4f_6^5}{f_{12}^3}+4q\frac{f_2^2f_4^2f_{12}^2}{f_6^2}\right)\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}} \pmod 5\\ &\equiv 4\frac{f_2f_8^3f_{12}^7}{f_4^2f_6^3f_{24}^3}-\frac{f_4^3f_6^6f_{24}}{f_2^2f_8f_{12}^4}\\ &\equiv Y_1(q^2) \pmod 5,\end{aligned}$$

Finishing The Proof of the Congruences

and ...

$$\begin{aligned}\sum_{n\geq 0}\overline{t}(40n+20)q^n &\equiv 4\frac{1}{f_1f_3}\cdot\frac{f_2^3f_6f_8f_{12}^2}{f_4f_{24}}-\frac{f_1^6}{f_3^2}\cdot\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}} \pmod 5\\ &\equiv 4\left(\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}+q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right)\frac{f_2^3f_6f_8f_{12}^2}{f_4f_{24}}\\ &\quad-\left(\frac{f_2f_4f_6^5}{f_{12}^3}+4q\frac{f_2^2f_4^2f_{12}^2}{f_6^2}\right)\frac{f_4^2f_6f_{24}}{f_2^3f_8f_{12}} \pmod 5\\ &\equiv 4\frac{f_2f_8^3f_{12}^7}{f_4^2f_6^3f_{24}^3}-\frac{f_4^3f_6^6f_{24}}{f_2^2f_8f_{12}^4}\\ &\equiv Y_1(q^2) \pmod 5,\end{aligned}$$

so that $\overline{t}(40(2n+1)+20)=\overline{t}(80n+60)\equiv 0 \pmod 5$.

Closing Thoughts

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Our work above can also be used to provide a new elementary proof of the following theorem, first proven by Chern and Hao via modular forms and later proved in elementary fashion by Lin, Liu, Wang, and Xiao.

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Theorem 11

For all $n \geq 0$, $\overline{t}(45n + 30) \equiv 0 \pmod{5}$.

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Our work above can also be used to provide a new elementary proof of the following theorem, first proven by Chern and Hao via modular forms and later proved in elementary fashion by Lin, Liu, Wang, and Xiao.

Theorem 11

For all $n \geq 0$, $\overline{t}(45n + 30) \equiv 0 \pmod{5}$.

In order to prove the above theorem, we require two additional 3-dissection results.

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Lemma 12

We have

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2},$$
$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q \frac{f_3 f_{18}^2}{f_6 f_9}.$$

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Lemma 12

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$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q \frac{f_3 f_{18}^2}{f_6 f_9}.$$

Using our main theorem as the starting point, we have

$$\sum_{n \geq 0} \bar{t}(5n) q^n \equiv \frac{f_1 f_2 f_3^3}{f_6^2} \pmod 5$$
$$= \frac{f_3^3}{f_6^2}(f_1 f_2).$$

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From the above lemma, we then know

$$\sum_{n\geq 0} \overline{t}(15n)q^{3n} \equiv \frac{f_3^3}{f_6^2}\left(\frac{f_6 f_9^4}{f_3 f_{18}^2}\right) \pmod 5$$

or

$$\begin{aligned}\sum_{n\geq 0} \overline{t}(15n)q^{n} &\equiv \frac{f_1^2 f_3^4}{f_2 f_6^2} \pmod 5 \\ &= \frac{f_3^4}{f_6^2}\left(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}\right)\end{aligned}$$

again from the 3–dissections above.

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From the above lemma, we then know

$$\sum_{n\geq 0} \overline{t}(15n)q^{3n} \equiv \frac{f_3^3}{f_6^2}\left(\frac{f_6 f_9^4}{f_3 f_{18}^2}\right) \pmod 5$$

or

$$\begin{aligned}\sum_{n\geq 0} \overline{t}(15n)q^{n} &\equiv \frac{f_1^2 f_3^4}{f_2 f_6^2} \pmod 5\\ &= \frac{f_3^4}{f_6^2}\left(\frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}\right)\end{aligned}$$

again from the 3–dissections above. Note that, when expanded as a power series in q, the last expression above will contain no terms of the form q^{3n+2}.

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Therefore, for all $n \geq 0$,

$$\overline{t}(15(3n+2)) = \overline{t}(45n+30) \equiv 0 \pmod{5}.$$

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And with that I will close.

Closing Thoughts

Therefore, for all $n \geq 0$,

$$\overline{t}(15(3n+2)) = \overline{t}(45n+30) \equiv 0 \pmod{5}.$$

And with that I will close.

Thanks very much for attending my talk!

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