Elementary Proofs of Infinite Families of Congruences for Merca's Cubic Partitions

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Acknowledgements

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The results that I will share with you today have been posted on arXiv:

https://arxiv.org/abs/2208.11249

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- Share some thoughts regarding cubic partitions
- Discuss Merca's recent work on cubic partitions, with special focus on his recent congruences and the proofs he provided
- Describe work that Robson and I have since completed to prove Merca's results as well as (newly identified) infinite families of congruences for these objects using truly elementary methods

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Cubic partitions were introduced by Hei-Chi Chan in 2010 in connection with Ramanujan's cubic continued fraction.

Introductory Thoughts

In a recent publication in the *Ramanujan Journal*, Mircea Merca extensively studied a function which he denoted as $A(n)$ and is defined to be the difference between the number of cubic partitions of n into an even numbers of parts and the number of cubic partitions of n into an odd numbers of parts.

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In a recent publication in the *Ramanujan Journal*, Mircea Merca extensively studied a function which he denoted as $A(n)$ and is defined to be the difference between the number of cubic partitions of n into an even numbers of parts and the number of cubic partitions of n into an odd numbers of parts.

Merca noted that the generating function for $A(n)$ is given by

$$\sum_{n=0}^{\infty} A(n)q^n = (q;q^2)_\infty(q^2;q^4)_\infty = \frac{f_1}{f_4}$$

where $(a;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)\dots$ and

$$f_a^b = (q^a;q^a)_\infty^b.$$

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The relevant generating functions found via Smoot's package are the following:

Introductory Thoughts

$$\sum_{n=0}^{\infty} A(9n+5)\,q^n$$
$$= -\frac{3q(q^2;q^2)_\infty^3(q^3;q^3)_\infty(q^{12};q^{12})_\infty^6}{(q;q)_\infty^2(q^4;q^4)_\infty^3(q^6;q^6)_\infty^5} + \frac{9q^2(q^2;q^2)_\infty^5(q^3;q^3)_\infty(q^{12};q^{12})_\infty^{10}}{(q;q)_\infty^2(q^4;q^4)_\infty^7(q^6;q^6)_\infty^7}$$
$$+ \frac{3q^3(q^2;q^2)_\infty^7(q^3;q^3)_\infty(q^{12};q^{12})_\infty^{14}}{(q;q)_\infty^2(q^4;q^4)_\infty^{11}(q^6;q^6)_\infty^9} - \frac{9q^4(q^2;q^2)_\infty^9(q^3;q^3)_\infty(q^{12};q^{12})_\infty^{18}}{(q;q)_\infty^2(q^4;q^4)_\infty^{15}(q^6;q^6)_\infty^{11}}.$$

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$$
\begin{aligned}
\sum_{n=0}^{\infty} A(27n+26)\, q^n
&= -\frac{9(q^2;q^2)_\infty^2 (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^4}{(q;q)_\infty^3 (q^4;q^4)_\infty^{-1} (q^6;q^6)_\infty^6} \\
&\quad + \frac{3q\,(q^2;q^2)_\infty^4 (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^8}{(q;q)_\infty^3 (q^4;q^4)_\infty^3 (q^6;q^6)_\infty^8} \\
&\quad + \frac{342q^2 (q^2;q^2)_\infty^6 (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{12}}{(q;q)_\infty^3 (q^4;q^4)_\infty^7 (q^6;q^6)_\infty^{10}} \\
&\quad - \frac{462q^3 (q^2;q^2)_\infty^8 (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{16}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{11} (q^6;q^6)_\infty^{12}} \\
&\quad - \frac{3087q^4 (q^2;q^2)_\infty^{10} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{20}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{15} (q^6;q^6)_\infty^{14}} \\
&\quad + \frac{4833q^5 (q^2;q^2)_\infty^{12} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{24}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{19} (q^6;q^6)_\infty^{16}} \\
&\quad + \frac{9315q^6 (q^2;q^2)_\infty^{14} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{28}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{23} (q^6;q^6)_\infty^{18}} \\
&\quad - \frac{14580q^7 (q^2;q^2)_\infty^{16} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{32}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{27} (q^6;q^6)_\infty^{20}} \\
&\quad - \frac{10935q^8 (q^2;q^2)_\infty^{18} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{36}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{31} (q^6;q^6)_\infty^{22}} \\
&\quad + \frac{16767q^9 (q^2;q^2)_\infty^{20} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{40}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{35} (q^6;q^6)_\infty^{24}} \\
&\quad + \frac{4374q^{10} (q^2;q^2)_\infty^{22} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{44}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{39} (q^6;q^6)_\infty^{26}} \\
&\quad - \frac{6561q^{11} (q^2;q^2)_\infty^{24} (q^3;q^3)_\infty^2 (q^{12};q^{12})_\infty^{48}}{(q;q)_\infty^3 (q^4;q^4)_\infty^{43} (q^6;q^6)_\infty^{28}}.
\end{aligned}
$$

Elementary Proofs of Merca's Congruences

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These functions satisfy many interesting properties, including the following:

Elementary Proofs of Merca's Congruences

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Next, we will need some well–known 3–dissection results in order to construct our proofs.

Elementary Proofs of Merca's Congruences

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$$\frac{1}{\phi(-q)} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}$$

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$$\frac{1}{\psi(q)} = \frac{f_3^2 f_9^3}{f_6^6} - q\frac{f_3^3 f_{18}^3}{f_6^7} + q^2\frac{f_3^4 f_{18}^6}{f_6^8 f_9^3}$$

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$$\sum_{n=0}^{\infty} A(3n+2)q^{3n+2} = q^2 \frac{f_6^4 f_9^5 f_{36}^3}{f_3^4 f_{12}^4 f_{18}^4} - 2q^2 \frac{f_6 f_{18}^5}{f_3^2 f_9 f_{12}^3}.$$

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Dividing by q^2 and replacing q^3 by q yields

$$\begin{aligned}\sum_{n=0}^{\infty} A(3n+2)q^{n} &= \frac{f_2^4 f_3^5 f_{12}^3}{f_1^4 f_4^4 f_6^4} - 2\frac{f_2 f_6^5}{f_1^2 f_3 f_4^3} \\ &\equiv \left(\frac{f_2}{f_1 f_4}\right) \frac{f_3^4 f_{12}^2}{f_6^3} - 2\left(\frac{f_2}{f_1^2}\right) \frac{f_6^5}{f_3 f_{12}} \pmod{3}.\end{aligned}$$

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This proves Merca's first congruence. □

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$$\sum_{n=0}^{\infty} A(3n+2)q^n \equiv \left(\frac{f_2}{f_1 f_4}\right) \frac{f_3^4 f_{12}^2}{f_6^3} - 2 \left(\frac{f_2}{f_1^2}\right) \frac{f_6^5}{f_3 f_{12}} \pmod{3}.$$

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$$\sum_{n=0}^{\infty} A(9n+8)q^{3n+2} \equiv q^2 \frac{f_6 f_9^3 f_{36}^3}{f_{12}^2 f_{18}^3} - 8q^2 \frac{f_6^7 f_{18}^3}{f_3^7 f_{12}} \pmod 3.$$

Elementary Proofs of Merca's Congruences

Dividing by q^2 and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} A(9n+8)q^n \equiv \frac{f_2 f_3^3 f_{12}^3}{f_4^2 f_6^3} - 8\frac{f_2^7 f_6^3}{f_1^7 f_4} \pmod 3$$
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Now we employ the second and fourth lemmas to extract the terms of the form q^{3n+2} from the congruence above.

The resulting congruence after division by q^2 and replacing q^3 by q is given by the following:

Elementary Proofs of Merca's Congruences

$$\sum_{n=0}^{\infty} A(27n+26)q^n$$

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$$\sum_{n=0}^{\infty} A(27n+26)q^n \equiv -\frac{f_1^3 f_{12}^3}{f_4^4} - 8\frac{f_2^9 f_3^3 f_{12}^3}{f_1^6 f_4^4 f_6^3} \pmod{3}$$

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And that proves Merca's second congruence. □

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In essence, internal congruences of this type sometimes exist and can provide the induction step of the proof of an infinite family of congruences where the modulus does not grow as the arithmetic progressions change.

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So we need to perform a few more dissections like the ones we completed above.

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Next, we revisit the congruence expression we obtained earlier for $A(9n+8)$ (remember, this was needed on our way to $A(27n+26)$) and use the second and fourth dissection lemmas to extract the terms of the form q^{3n} to obtain a generating function expression for $A(27n+8)$:

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Note that

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And with that I will close. Thanks for your attention!

Elementary Proofs of Infinite Families of Congruences for Merca's Cubic Partitions

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September 2022