Infinite Families of Congruences Modulo Powers of 2 for Partitions into Odd Parts with Designated Summands

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Opening Comments

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- Thanks to my co–author Shane Chern (Dalhousie University) for our very fruitful collaboration; the latter portion of my talk today will cover the material on which Shane and I collaborated.

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- Along the way, I will provide some historical background regarding this function.

- The results in the first part of the talk are proved via straightforward generating function manipulations.

- The later results with Shane rely heavily on tools from modular forms as well as an inductive argument.

Background

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These are built by taking unrestricted integer partitions and designating exactly one of each occurrence of a part.

For example, there are 10 partitions with designated summands of weight 4:

$$4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1, \quad 2'+1+1'$$

$$1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'$$

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In the same paper, Andrews, Lewis, and Lovejoy also considered the restricted partitions with designated summands wherein all parts must be odd, and they denoted the corresponding enumeration function by $PDO(n)$.

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Beginning with Andrews, Lewis, and Lovejoy, a wide variety of Ramanujan–like congruences have been proven for $PD(n)$ and $PDO(n)$.

Here are some examples of such work:

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Herden et al. proved a number of arithmetic properties satisfied by several functions, including $PDO(n)$.

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Conjecture: For $n \geq 0$, we have

$$\begin{aligned} PDO(16n+12) &\equiv 0 \pmod{4}, \\ PDO(24n+20) &\equiv 0 \pmod{4}, \\ PDO(25n+5) &\equiv 0 \pmod{4}, \\ PDO(32n+24) &\equiv 0 \pmod{4}, \\ PDO(48n+26) &\equiv 0 \pmod{4} \end{aligned}$$

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This is a hint of something much larger; indeed, these two congruences are true, and they belong to an easily–described infinite family of congruences modulo 4:

Theorem: For all $\alpha \geq 0$ and all $n \geq 0$,

$$PDO(2^{\alpha}(4n+3)) \equiv 0 \pmod{4}.$$

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Our first goal in this talk is to outline proofs of the two theorems above.

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Our first goal in this talk is to outline proofs of the two theorems above.

All of the proof techniques used to prove these two families are elementary, relying on classical q–series identites and generating function manipulations.

Proofs of These Two Infinite Families

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As noted by Andrews, Lewis, and Lovejoy, the generating function for $PDO(n)$ is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}$$

where $f_r = (1-q^r)(1-q^{2r})(1-q^{3r})(1-q^{4r})\dots$ is the usual q–Pochhammer symbol.

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In order to prove our results, we require several elementary generating function dissection tools, most of which are well–known 2–dissection results that allow us to manipulate the generating function for $PDO(n)$ to our advantage.

Proofs of These Two Infinite Families

Lemma:

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}.$$

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$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}.$$

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Lemma:

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q \frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}},$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q \frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}.$$

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Using the generating function for $PDO(n)$ given by Andrews, Lewis, and Lovejoy, as well as the lemmas above, it is a straightforward exercise to prove the following:

$$\sum_{n=0}^{\infty} PDO(2n)q^n = \frac{f_4^2 f_6^4}{f_1^2 f_3^2 f_{12}^2}, \quad \text{and}$$

$$\sum_{n=0}^{\infty} PDO(2n+1)q^n = \frac{f_2^6 f_{12}^2}{f_1^4 f_4^2 f_6^2}.$$

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(The above dissections appeared in Andrews, Lewis, and Lovejoy's original paper.)

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\sum_{n=0}^{\infty} PDO(4n)q^n &= \frac{f_4^4 f_6^8}{f_1^4 f_3^4 f_{12}^4} + q\frac{f_2^{12} f_{12}^4}{f_1^8 f_4^4 f_6^4}, \\
\sum_{n=0}^{\infty} PDO(4n+1)q^n &= \frac{f_2^{12} f_6^2}{f_1^8 f_3^2 f_4^4}, \\
\sum_{n=0}^{\infty} PDO(4n+2)q^n &= 2\frac{f_2^6 f_6^2}{f_1^6 f_3^2}, \quad \text{and} \\
\sum_{n=0}^{\infty} PDO(4n+3)q^n &= 4\frac{f_4^4 f_6^2}{f_1^4 f_3^2}.
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\end{aligned}$$

With these tools in hand, we can now proceed to proving the mod 4 and mod 8 families of congruences we mentioned earlier.

Proofs of These Two Infinite Families

For now, we focus on the mod 4 family of congruences:

Theorem: For all $\alpha \geq 0$ and all $n \geq 0$,

$$PDO(2^{\alpha}(4n+3)) \equiv 0 \pmod 4.$$

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$$\sum_{n=0}^{\infty} PDO(2n)q^n \equiv \frac{f_4^2}{f_1^2 f_3^2} \pmod{4}, \quad \text{and}$$

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These are easily seen thanks to the following:

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Corollary: For all $n \geq 0$,

$$\begin{aligned} PDO(4n+3) &\equiv 0 \pmod{4}, \\ PDO(6n+3) &\equiv 0 \pmod{4}, \quad \text{and} \\ PDO(6n+5) &\equiv 0 \pmod{4}. \end{aligned}$$

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Theorem:

$$\sum_{n=0}^{\infty} PDO(4n)q^n \equiv \left(\frac{f_2^3}{f_6}\right)^2 + qf_{12}^2 \pmod 4 \quad \text{and}$$

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Because of the "structure" of the results above, several corollaries follow immediately.

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The following theorem provides an "internal congruence" modulo 4 which is satisfied by $PDO(n)$, and this serves as the "engine" for the induction step of our proof.

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This follows immediately from our generating function result for $PDO(4n)$ (mod 4) (mentioned above) and the fact that

$$\sum_{n=0}^{\infty} PDO(n)q^n \equiv \left(\frac{f_2^3}{f_6}\right)^2 + qf_{12}^2 \pmod 4$$

which was proven by Herden et. al.

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The proof of our infinite family of congruences modulo 4 now follows very quickly.

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Thanks to our internal congruence modulo 4, we have the following:

Proofs of These Two Infinite Families

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This gives our result when the values of α are even.

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Similarly,

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This gives our result when the values of α are odd.

And that completes our proof of this infinite family of mod 4 congruences.

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Before we move on to our infinite family of congruences modulo 8, we note in passing that we also now have the following infinite families of congruences modulo 4:

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Theorem: For all $\alpha \geq 0$ and all $n \geq 0$,

$$\begin{aligned} PDO(4^{\alpha}(6n+3)) &\equiv 0 \pmod 4, \\ PDO(4^{\alpha}(6n+5)) &\equiv 0 \pmod 4. \end{aligned}$$

Proofs of These Two Infinite Families

We now transition to a sketch of the proof of the infinite family of congruences modulo 8:

Theorem: For all $\alpha \geq 0$ and all $n \geq 0$,

$$PDO(2^{\alpha}(8n+7)) \equiv 0 \pmod{8}.$$

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The proof idea here is identical to the proof for the family of congruences modulo 4.

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Theorem: For all $\alpha \geq 0$ and all $n \geq 0$,

$$PDO(2^{\alpha}(8n+7)) \equiv 0 \pmod{8}.$$

The proof idea here is identical to the proof for the family of congruences modulo 4.

We prove the first few cases individually, and then prove an internal congruence that takes care of the proof by induction.

Proofs of These Two Infinite Families

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Proof: From our earlier work, we know

$$\sum_{n=0}^{\infty} PDO(4n+3)q^n = 4\frac{f_4^4 f_6^2}{f_1^4 f_3^2}$$

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Because the function $\frac{f_4^4 f_6^2}{f_2^2 f_6} = \frac{f_4^4 f_6}{f_2^2}$ is an even function of q, we immediately conclude that, for all $n \geq 0$,

$$PDO(4(2n+1)+3) = PDO(8n+7) \equiv 0 \pmod{8}.$$

$\square$

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Theorem: For all $n \geq 0$, $PDO(16n+14) \equiv 0 \pmod{8}$.

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$$\sum_{n=0}^{\infty} PDO(4n+2)q^n = 2\frac{f_2^6 f_6^2}{f_1^6 f_3^2}$$

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Next, we prove the $\alpha = 1$ case of this infinite family of congruences.

Theorem: For all $n \geq 0$, $PDO(16n+14) \equiv 0 \pmod{8}$.

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$$\begin{aligned}\sum_{n=0}^{\infty} PDO(4n+2)q^n &= 2\frac{f_2^6 f_6^2}{f_1^6 f_3^2} \\ &= 2f_2^6 f_6^2 \left(\frac{1}{f_1^4}\right)\left(\frac{1}{f_1 f_3}\right)^2.\end{aligned}$$

Proofs of These Two Infinite Families

Using our dissection lemmas and collecting those terms wherein the powers of q are odd, we can conclude

$$\begin{aligned}
&\sum_{n=0}^{\infty} PDO(8n+6)q^{2n+1} \\
\equiv\ & 2\left(\frac{f_4^{14}f_6^2}{f_2^8f_8^4}\right)\left(2q\frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2}\cdot\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}\right) \pmod{8} \\
\equiv\ & 4q\frac{f_4^{18}f_{12}^4}{f_2^{14}f_6^4f_8^4} \pmod{8}.
\end{aligned}$$

Proofs of These Two Infinite Families

This implies

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(8n+6)q^n &\equiv 4\frac{f_2^{18}f_6^4}{f_1^{14}f_3^4f_4^4} \pmod 8\\ &\equiv 4\frac{f_2^{11}f_6^2}{f_4^4} \pmod 8.\end{aligned}$$

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Since the last expression above is an even function of q, we immediately know that, for all $n \geq 0$,

$$PDO(8(2n+1)+6) = PDO(16n+14) \equiv 0 \pmod 8.$$

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And then we prove the following internal congruence:

Theorem: For all $n \geq 0$, $PDO(16n) \equiv PDO(4n) \pmod{8}$.

Proof: Using the same kind of dissection techniques on $PDO(8n)$, we can show that

Proofs of These Two Infinite Families

$$\begin{aligned}\sum_{n=0}^{\infty} PDO(16n)q^{2n} &\equiv \frac{f_8^4}{f_2^4 f_6^4} + q^2 \frac{f_2^{24} f_{24}^4}{f_2^8 f_{12}^4 f_4^8} \pmod 8 \\ &\equiv \frac{f_8^4}{f_2^4 f_6^4} + q^2 \frac{f_2^{24} f_{24}^4}{f_2^8 f_{12}^4 f_2^{16}} \pmod 8 \\ &\equiv \frac{f_8^4}{f_2^4 f_6^4} + q^2 \frac{f_{24}^4}{f_{12}^4} \pmod 8.\end{aligned}$$

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This means we know

Proofs of These Two Infinite Families

$$\sum_{n=0}^{\infty} PDO(16n)q^n \equiv \frac{f_4^4}{f_1^4 f_3^4} + q\frac{f_{12}^4}{f_6^4} \pmod 8$$

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thanks to the generating function for $PDO(4n)$ which we proved earlier. $\square$

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This now gives us all the tools we need to prove this infinite family of congruences modulo 8.

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$$\begin{aligned} PDO(8n+7) &\equiv 0 \pmod{8}, \\ PDO(16n+14) &\equiv 0 \pmod{8}, \\ PDO(32n+28) &\equiv 0 \pmod{8}, \\ PDO(64n+56) &\equiv 0 \pmod{8}. \end{aligned}$$

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This gives our result when the values of α are odd. □

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This manuscript is currently under review at *INTEGERS* (basically as a follow–up to the Herden et al. paper which appeared in *INTEGERS*).

Extending The Internal Congruences

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The second of these two internal congruences is part of an infinite family of internal congruences!

Extending The Internal Congruences

Theorem: For all $k \geq 0$ and all $n \geq 0$,

$$PDO(2^{2k+3}n) \equiv PDO(2^{2k+1}n) \pmod{2^{2k+3}}.$$

Extending The Internal Congruences

Theorem: For all $k \geq 0$ and all $n \geq 0$,

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Note that, when n is replaced by $2n$, we have

$$PDO(2^{2k+4}n) \equiv PDO(2^{2k+2}n) \pmod{2^{2k+3}}.$$

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The $k = 0$ case of this result is

$$PDO(2^4 n) \equiv PDO(2^2 n) \pmod{2^3}$$

which is the second of our results above.

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James Sellers
University of Minnesota Duluth

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Extending The Internal Congruences

We utilize several classical tools to prove this family of internal congruences, including:

- generating function dissections via the U operator,
- various modular relations and recurrences involving a Hauptmodul on the classical modular curve $X_0(6)$,
- and an induction argument which provides the final step in proving the necessary divisibilities.

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$$\delta = \delta(q) := \frac{f_4 f_6^2}{f_1 f_3 f_{12}},$$

$$\gamma = \gamma(q) := \frac{f_1^5 f_2^5 f_6^5}{f_3^{15}},$$

$$\xi = \xi(q) := \frac{f_2^5 f_6}{f_1 f_3^5},$$

$$\kappa = \kappa(q) := \frac{\gamma(q^2)^2}{\gamma(q)}.$$

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Note that

$$\delta(q) = \sum_{n=0}^{\infty} PDO(n) q^n.$$

Extending The Internal Congruences

We further define for $k \geq 2$,

$$\Lambda_k = \Lambda_k(q) := \gamma(q)^{2^{k-2}} \sum_{n=0}^{\infty} PDO(2^k n) q^n.$$

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We let U be the *unitizing operator of degree two*, given by

$$U\left(\sum_n a_n q^n\right) := \sum_n a_{2n} q^n.$$

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We now summarize the approach to completing the proof of our result.

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Modular relation for γ and ξ:

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$$\gamma^6 = 59049\xi^{10} - 262440\xi^{11} + 466560\xi^{12} - 414720\xi^{13} + 184320\xi^{14} - 32768\xi^{15}.$$

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Modular relation for γ and ξ:

$$\gamma^6 = 59049\xi^{10} - 262440\xi^{11} + 466560\xi^{12} - 414720\xi^{13} + 184320\xi^{14} - 32768\xi^{15}.$$

The discovery, and proof, of this result relies on an analysis of the order of γ^6 and ξ at each of their cusps.

Extending The Internal Congruences

Modular relations for κ and ξ:

Initial cases:

James Sellers
University of Minnesota Duluth

Extending The Internal Congruences

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$$\begin{aligned}
U\left(\kappa\right) &= 5\xi^3 - 20\xi^4 + 16\xi^5, \\
U\left(\xi\right) &= 5\xi - 4\xi^2, \\
U\left(\kappa^2\right) &= -\xi^5 + 50\xi^6 - 400\xi^7 + 1120\xi^8 - 1280\xi^9 + 512\xi^{10}, \\
U\left(\kappa\xi\right) &= 3\xi^3 - 18\xi^4 + 16\xi^5, \\
U\left(\xi^2\right) &= -9\xi + 58\xi^2 - 80\xi^3 + 32\xi^4.
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Each of the above can be shown using cusp analysis. We opted to "automate" the proof using a combination of Smoot's Mathematica implementation of Radu's algorithm as well as Garvan's Maple package `ETA`.

Extending The Internal Congruences

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Theorem: For any $i \geq 2$ and $j \geq 0$,

$$\zeta_{i,j} = \left(10\xi^3 - 40\xi^4 + 32\xi^5\right) \cdot \zeta_{i-1,j} - \left(\xi^5\right) \cdot \zeta_{i-2,j}.$$

Also, for any $i \geq 0$ and $j \geq 2$,

$$\zeta_{i,j} = \left(10\xi - 8\xi^2\right) \cdot \zeta_{i,j-1} - \left(9\xi - 8\xi^2\right) \cdot \zeta_{i,j-2}.$$

Extending The Internal Congruences

We can combine the two recurrences above and derive that for $i, j \geq 2$,

$$\begin{aligned}\zeta_{i,j} = &\left(10\xi - 8\xi^2\right)\left(10\xi^3 - 40\xi^4 + 32\xi^5\right) \cdot \zeta_{i-1,j-1}\\ &- \left(9\xi - 8\xi^2\right)\left(10\xi^3 - 40\xi^4 + 32\xi^5\right) \cdot \zeta_{i-1,j-2}\\ &- \left(10\xi - 8\xi^2\right)\left(\xi^5\right) \cdot \zeta_{i-2,j-1}\\ &+ \left(9\xi - 8\xi^2\right)\left(\xi^5\right) \cdot \zeta_{i-2,j-2}.\end{aligned}$$

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With the initial cases that we demonstrated, along with this recurrence, we can prove in straightforward fashion that $U\left(\kappa^i \xi^j\right) \in \mathbb{Z}[\xi]$ for any $i, j \geq 0$.

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Modular relations for Λ_k and ξ:

Theorem: For any $k \geq 2$,

$$\Lambda_k \in \mathbb{Z}[\xi].$$

Extending The Internal Congruences

More precisely, if we write

$$\Lambda_k := \sum_m c_k(m)\xi^m,$$

then

$$\Lambda_2 = 3\xi^2 - 2\xi^3,$$

and for $k \geq 3$, we recursively have

$$\Lambda_k = \sum_\ell c_{k-1}(\ell) \cdot \zeta_{2^{k-3},\ell},$$

where $\zeta_{i,j}$ is given above.

Extending The Internal Congruences

In order to handle the "internal" nature of the results in question, we then define a new family of auxiliary functions for $k \geq 3$:

$$\Phi_k(q) := \gamma(q)^{2^k} \left(\sum_{n=0}^{\infty} PDO(2^{k+2}n)q^n - \sum_{n=0}^{\infty} PDO(2^k n)q^n \right)$$

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In light of earlier work, we have

$$\begin{aligned} \Phi_k &= \Lambda_{k+2} - \gamma^{3 \cdot 2^{k-2}} \Lambda_k \\ &= \Lambda_{k+2} - \left(\gamma^6\right)^{2^{k-3}} \Lambda_k. \end{aligned}$$

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We can then show the following:

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$$\begin{aligned}\Phi_3 = {} & 34012224\xi^{14} - 396809280\xi^{15} + 2061728640\xi^{16} \\ & - 6195823488\xi^{17} + 11887534080\xi^{18} - 15250636800\xi^{19} \\ & + 13309968384\xi^{20} - 7840727040\xi^{21} + 2994733056\xi^{22} \\ & - 671088640\xi^{23} + 67108864\xi^{24}, \quad and \quad \dots\end{aligned}$$

Extending The Internal Congruences

... for $k \geq 4$, we recursively have

$$\Phi_k = \sum_{\ell} F_{k-1}(\ell) \cdot \zeta_{2^{k-1},\ell},$$

where $\zeta_{i,j}$ is given above.

Extending The Internal Congruences

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... and our infinite family of internal congruences modulo 2^{2k+3} follows.

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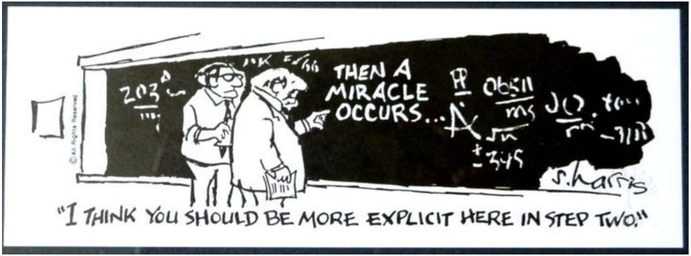

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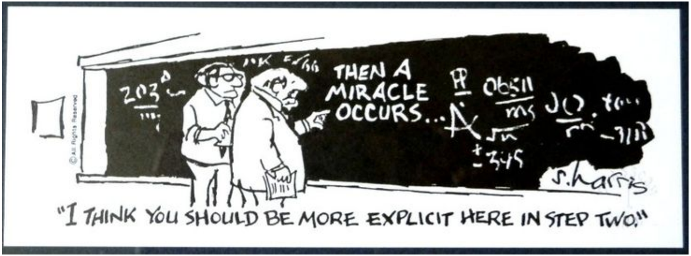

... I encourage you to see the details in our manuscript:

https://arxiv.org/abs/2308.04348

Closing Thoughts

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Theorem: For all $\alpha \geq 0$ and all $n \geq 0$,

$$PDO\big(2^{\alpha}(4n+3)\big) \equiv 0 \pmod{4},$$
$$PDO\big(2^{\alpha}(8n+7)\big) \equiv 0 \pmod{8}.$$

Closing Thoughts

It would be gratifying to see other cases of our family of internal congruences used to assist in proving divisibility properties satisfied by $PDO(n)$ for higher powers of 2 (similar to the results above).

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This would mean we would need to find the appropriate "basis step" Ramanujan–like congruences that are satisfied by PDO modulo some higher power of 2.

I'd be happy to chat with anyone interested in such an endeavor!

Closing Thoughts

Second, in the course of proving congruences modulo arbitrary powers for the coefficients of an eta-product

$$\mathrm{H}(q) = \sum_{n=0}^{\infty} h(n)q^n,$$

the usual strategy is to find a suitable basis $\{\xi_1, \xi_2, \ldots, \xi_L\}$ of the corresponding modular space such that each dissection slice, accompanied by a certain multiplier (usually an eta-product),

$$\lambda_m \sum_{n=0}^{\infty} h(p^m n + t_m)q^n,$$

can be represented as a polynomial in $\mathbb{Z}[\xi_1, \xi_2, \ldots, \xi_L]$.

Closing Thoughts

For example, when proving the congruences modulo powers of 5 for the partition function (e.g., see Watson, 1938), two specific multipliers take turns showing up, i.e., $\lambda_{2M-1} = \lambda$ and $\lambda_{2M} = \lambda'$ for two certain series λ and λ'.

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However, in this work, the multipliers γ, γ^2, γ^4, γ^8, $\ldots$ never overlap.

Meanwhile, an important outcome of cycling the multipliers in the previous studies is that it is typically sufficient to represent each degree p unitization $U_p\left(\kappa^i \xi^j\right)$ as a polynomial in ξ for a certain series κ, with the exponent $i \in \{0, 1\}$.

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In contrast, when there are endless possibilities for the multipliers, we have to extend the consideration of i to infinity, thereby substantially increasing the amount of required p-adic analysis.

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And with that I will close.

Thanks very much for attending today.

Infinite Families of Congruences Modulo Powers of 2 for Partitions into Odd Parts with Designated Summands

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