Cranks Witnessing an Infinite Family of Congruences for a Sum of Partition Functions

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Dissertation Background and History

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$$p(4) = 5$$

Definition 1

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \ldots, \lambda_m$ such that $\sum_{i=1}^{m} \lambda_i = n$. The λ_i are called the parts of the partition and we write $p(n)$ to denote the number of partitions of n.

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Definition 2

Let λ be a partition of n into parts from the set $[m]$. We write λ in "multiplicity notation," so that $\lambda = (1^{e_1}, 2^{e_2}, \ldots, m^{e_m})$ is the partition with exactly e_i parts of size i for each $i \in [m]$.

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Partition Numbers and Ramanujan Congruences

Here is the sequence of partition numbers.

$$\{p(n)\}_{n=0}^{\infty} = 1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, \ldots$$

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In 1919, Ramanujan proved the following patterns in this sequence.[6]

Theorem 3

For all nonnegative integers k,

$$p(5k+4) \equiv 0 \pmod{5} \tag{1}$$
$$p(7k+5) \equiv 0 \pmod{7} \tag{2}$$
$$p(11k+6) \equiv 0 \pmod{11}. \tag{3}$$

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$$c(\lambda) = \begin{cases} l(\lambda) & if \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & if \omega(\lambda) > 0. \end{cases}$$

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Let ℓ be a prime and consider $\{p(n,3) \pmod{\ell}\}_{n=0}^{\infty}$

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$$225, 234, 243, 252, 261, 271, 280, 290, 300, 310, 320, 331, 341, 352, 363, 374, 385, 397, 408, 420,$$
$$432, 444, 456, 469, 481, 494, 507, 520, 533, 547, 560, 574, 588, 602, 616, 631, 645, 660, 675, 690,$$
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$\ell = 3$: We see regular intervals of partitions congruent to 0 modulo 3.

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$$1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19, 21, 24, 27, 30, 33, 37, 40, 44, 48, 52, 56, 61, 65, 70, 75, 80, 85, 91, 96, 102, 108, 114, 120, 127, 133, 140, 147, 154, 161, 169, 176, 184, 192, 200, 208, 217, 225, 234, 243, 252, 261, 271, 280, 290, 300, 310, 320, 331, 341, 352, 363, 374, 385, 397, 408, 420, 432, 444, 456, 469, 481, 494, 507, 520, 533, 547, 560, 574, 588, 602, 616, 631, 645, 660, 675, 690, 705, 721, 736, 752, 768, 784, 800, 817, 833, 850, 867, 884, 901, 919, 936, 954, 972, 990, 1008, 1027, \ldots$$

Theorem 7 (The Interval Theorem. Kronholm (2007))

For any prime ℓ, any nonnegative integer k, and any $2 \leq m \leq \ell + 1$, we have

$$p(\ell \cdot \mathrm{lcm}(m)k - t, m) \equiv 0 \pmod{\ell} \tag{7}$$

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$$\{p(n,3) \pmod 3\}_{n\geq 0} \implies p(18k-t,3) \equiv 0 \pmod 3 \text{ for } 0 < t < 6 \tag{8}$$

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$p(0, 4) =$
$p(1, 4) =$
$p(2, 4) =$
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$p(5, 4) =$
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$p(0, 4) = 1$
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$p(2, 4) = 2$
$p(3, 4) = 3$
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$p(22,4) = 136$	$+$	$p(4,4) = 5$	$= 141 \equiv 0 \pmod 3$
$p(21,4) = 120$	$+$	$p(5,4) = 6$	$= 126 \equiv 0 \pmod 3$
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Sum and Difference Theorem of Partition Numbers

Theorem 10 (G., Kronholm)

Let ℓ be an odd prime. Set $n' = \ell lcm(m)(k+1) - r - \binom{\frac{m^2+m}{2}}{}$ and $n = \ell lcm(m)k + r$. Then for

$$-\left(\frac{m^2+m}{2}\right) + 1 \leq r \leq \ell\, lcm(m) - \left(\frac{m^2+m}{2}\right) \tag{10}$$

and $k \geq 0$, we have

$$p(n', m) + (-1)^m p(n, m) \equiv 0 \pmod{\ell}. \tag{11}$$

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p(36k+18,4) & + & p(36k+8,4) & \equiv 0 \pmod 3 \\
p(17,4)=72 & + & p(9,4)=18 & =90 \equiv 0 \pmod 3 \\
p(16,4)=64 & + & p(10,4)=23 & =87 \equiv 0 \pmod 3 \\
p(15,4)=54 & + & p(11,4)=27 & =81 \equiv 0 \pmod 3 \\
p(14,4)=47 & + & p(12,4)=34 & =81 \equiv 0 \pmod 3 \\
p(13,4)=39 & & & =39 \equiv 0 \pmod 3
\end{array}
$$

A Continuation of $p(n,4) \pmod 3$

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&p(15,4) = 54 &+\quad &p(11,4) = 27 &&= 81 \equiv 0 \pmod 3\\
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Definition 11

A polynomial of degree d, $P(q) = a_0 + a_1 q + \cdots + a_d q^d$, is said to be reciprocal if

$$a_i = a_{d-i} \tag{12}$$

. Equivalently, if

$$q^d P\left(q^{-1}\right) = P(q). \tag{13}$$

Proof by Example of Theorem 10: Anti/Reciprocal Polynomials

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Proof by Example of Theorem 10: Anti/Reciprocal Polynomials

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Proof by Example of the Sum/Difference Congruence

We will show that

$$P(q) = \frac{1 - q^{3lcm(4)}}{(q;q)_4} \pmod{3} \tag{20}$$

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Proof by Example of the Sum/Difference Theorem

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Thus P(q) is anti-reciprocal polynomial and

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Cranks Witnessing the Interval Theorem

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Multiplicity Based Statistics: $p(8,4) = 15$, $\tau = (0,1,1,1)$

Definition 14

Given a partition $\lambda = (1^{e_1}, 2^{e_2}, \ldots, m^{e_m})$, we define a *multiplicity-based statistic* (*MB statistic*) $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \mathbb{Z}^m$ to be

$$\tau(\lambda) = \sum_{i=1}^{m} \tau_i e_i.$$

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$(1^2, 2^1, 3^0, 4^1)$	$(1^4, 2^0, 3^0, 4^1)$	$(1^0, 2^1, 3^2, 4^0)$
$(1^2, 2^0, 3^2, 4^0)$	$(1^1, 2^2, 3^1, 4^0)$	$(1^3, 2^1, 3^1, 4^0)$
$(1^5, 2^0, 3^1, 4^0)$	$(1^0, 2^4, 3^0, 4^0)$	$(1^2, 2^3, 3^0, 4^0)$
$(1^4, 2^2, 3^0, 4^0)$	$(1^6, 2^1, 3^0, 4^0)$	$(1^8, 2^0, 3^0, 4^0)$

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$$0 \cdot 1 +$$

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$$0\cdot 1+1\cdot 2+$$

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Given a partition $\lambda = (1^{e_1}, 2^{e_2}, \ldots, m^{e_m})$, we define a multiplicity-based statistic (MB statistic) $\tau = (\tau_1, \tau_2, \ldots, \tau_m) \in \mathbb{Z}^m$ to be

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What Constitutes a Crank? $\mathcal{M}_\tau(r, \ell, n, m)$

Definition 15

- For a given partition statistic τ and a positive integer ℓ, we allow τ to classify the partitions of n into ℓ subclasses by letting $\mathcal{M}_\tau(r, \ell, n, m)$ be the set of partitions λ of n into parts from $[m]$ such that $\tau(\lambda) \equiv r \pmod{\ell}$.
- Also, we define $M_\tau(r, \ell, n, m) = |\mathcal{M}_\tau(r, \ell, n, m)|$. [4]

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Let $\mathfrak{p}(n)$ denote the set of partitions of n. For a given n, if the statistic $\tau : \mathfrak{p}(n, m) \to \mathbb{Z}$ is equally distributed over every residue class modulo ℓ, we say that τ is a crank modulo ℓ, witnessing the ℓ-divisibility of $p(n, m)$.
That is, if

$$M_\tau(i, \ell, n, m) = \frac{p(n, m)}{\ell} \tag{29}$$

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A MB statistic τ is a crank for the congruences of The Interval Theorem (Theorem 7) if the components of the tuple $\widehat{\left(\frac{\tau_i}{i}\right)}_{i=1}^{m}$ are distinct modulo ℓ, and $\tau_\ell \not\equiv 0 \pmod{\ell}$ in the tuple $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$.

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For $2 \leq m \leq \ell + 1$, let

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A function $f(k)$ is a quasipolynomial if there exist d polynomials $f_0(k), f_1(k), \ldots, f_{d-1}(k)$ such that:

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What the Quasipolynomial Program Does: Formulas for $p(n, 4)$

$$\sum_{n=0}^{\infty} p(n,4)q^n = \frac{1}{(q;q)_4}$$

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$$= \frac{1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8\cdots+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38}}{(1-q^{12})^4}$$

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Now we multiply and collect like terms to establish twelve formulas describing $p(n,4)$ for all n.

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$$= \frac{(1+q+q^2+\cdots+q^{11})(1+q^2+q^4+q^6+q^8+q^{10})(1+q^3+q^6+q^9)(1+q^4+q^8)}{(q;q)_4(1+q+q^2+\cdots+q^{11})(1+q^2+q^4+q^6+q^8+q^{10})(1+q^3+q^6+q^9)(1+q^4+q^8)}$$

$$= \frac{1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8\cdots+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38}}{(1-q^{12})^4}$$

With $\frac{1}{(1-q^{12})^4} = \sum_{k\geq 0}\binom{k+3}{3}q^{12k}$, we now rewrite it as

$$(1+q+2q^2+3q^3+5q^4+6q^5+9q^6+\cdots+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38})\times\sum_{k\geq 0}\binom{k+3}{3}q^{12k}.$$

Now we multiply and collect like terms to establish twelve formulas describing $p(n,4)$ for all n.

What the Quasipolynomial Program Does: Formulas for $p(n,4)$

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,4)q^n &= (1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10}\\
&\quad +27q^{11}+30q^{12}+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19}\\
&\quad +48q^{20}+48q^{21}+44q^{22}+42q^{23}+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28}\\
&\quad +18q^{29}+15q^{30}+11q^{31}+9q^{32}+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38})\\
&\quad \times \sum_{k\geq 0} \binom{k+3}{3} q^{12k}\end{aligned}$$

What the Quasipolynomial Program Does: Formulas for $p(n,4)$

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,4)q^n \quad = \quad & (1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10} \\ & +27q^{11}+30q^{12}+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19} \\ & +48q^{20}+48q^{21}+44q^{22}+42q^{23}+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28} \\ & +18q^{29}+15q^{30}+11q^{31}+9q^{32}+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38}) \\ & \times \sum_{k\geq 0} \binom{k+3}{3} q^{12k}\end{aligned}$$

Multiply and collect like terms:

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$$\begin{aligned}\sum_{n=0}^{\infty} p(n,4)q^n \quad = \quad & (1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10} \\ & +27q^{11}+30q^{12}+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19} \\ & +48q^{20}+48q^{21}+44q^{22}+42q^{23}+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28} \\ & +18q^{29}+15q^{30}+11q^{31}+9q^{32}+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38}) \\ & \times \sum_{k\geq 0} \binom{k+3}{3} q^{12k}\end{aligned}$$

Multiply and collect like terms: For example, how many ways are there to get an exponent of $12k+5$?

What the Quasipolynomial Program Does: Formulas for $p(n,4)$

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,4)q^n \quad = \quad & (1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10} \\ & +27q^{11}+30q^{12}+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19} \\ & +48q^{20}+48q^{21}+44q^{22}+42q^{23}+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28} \\ & +18q^{29}+15q^{30}+11q^{31}+9q^{32}+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38}) \\ & \times \sum_{k\geq 0} \binom{k+3}{3} q^{12k}\end{aligned}$$

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$$\sum_{k=0}^{\infty} p(12k+5,4)q^{12k+5}$$

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Multiply and collect like terms: For example, how many ways are there to get an exponent of $12k+5$?

$$\sum_{k=0}^{\infty} p(12k+5,4)q^{12k+5} = \Big(6q^5 + 48q^{17}$$

What the Quasipolynomial Program Does: Formulas for $p(n,4)$

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Multiply and collect like terms: For example, how many ways are there to get an exponent of $12k+5$?

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$$\sum_{n=0}^{\infty} p(n,4)q^n = (1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10}$$
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What the Quasipolynomial Program Does: Formulas for $p(n,4)$

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$$= \sum_{k\geq 0} \left(6\binom{k+3}{3}+48\binom{k+2}{3}+18\binom{k+1}{3}\right) q^{12k+5}. \quad \text{Hence,}$$

$$p(12k+5,4) = 6\binom{k+3}{3}+48\binom{k+2}{3}+18\binom{k+1}{3}$$

What the Quasipolynomial Program Does: Formulas for $p(n,4)$

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,4)q^n &= (1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10}\\ &\quad +27q^{11}+30q^{12}+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19}\\ &\quad +48q^{20}+48q^{21}+44q^{22}+42q^{23}+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28}\\ &\quad +18q^{29}+15q^{30}+11q^{31}+9q^{32}+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38})\\ &\quad \times \sum_{k\geq 0}\binom{k+3}{3}q^{12k}\end{aligned}$$

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$$= \sum_{k\geq 0}\left(6\binom{k+3}{3}+48\binom{k+2}{3}+18\binom{k+1}{3}\right)q^{12k+5}.$$ Hence,

$$p(12k+5,4) = 6\binom{k+3}{3}+48\binom{k+2}{3}+18\binom{k+1}{3} = 12k^3+30k^2+24k+6.$$

What the Quasipolynomial Program Does: Formulas for $p(n,4)$

$$\begin{aligned}\sum_{n=0}^{\infty} p(n,4)q^n \quad = \quad & (1+q+2q^2+3q^3+5q^4+6q^5+9q^6+11q^7+15q^8+18q^9+23q^{10} \\ & +27q^{11}+30q^{12}+35q^{13}+39q^{14}+42q^{15}+44q^{16}+48q^{17}+48q^{18}+50q^{19} \\ & +48q^{20}+48q^{21}+44q^{22}+42q^{23}+39q^{24}+35q^{25}+30q^{26}+27q^{27}+23q^{28} \\ & +18q^{29}+15q^{30}+11q^{31}+9q^{32}+6q^{33}+5q^{34}+3q^{35}+2q^{36}+q^{37}+q^{38}) \\ & \times \sum_{k\geq 0} \binom{k+3}{3} q^{12k}\end{aligned}$$

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$$= \sum_{k\geq 0} \left(6\binom{k+3}{3}+48\binom{k+2}{3}+18\binom{k+1}{3}\right) q^{12k+5}. \quad \text{Hence,}$$

$$p(12k+5,4) = 6\binom{k+3}{3}+48\binom{k+2}{3}+18\binom{k+1}{3} = 12k^3+30k^2+24k+6.$$

Twelve Formulas for $p(n, 4)$

This process allows us to describe $p(n, 4)$ with twelve formulas:

Twelve Formulas for $p(n, 4)$

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$$\begin{aligned}
p(12k, 4) &= 1\binom{k+3}{3} + 30\binom{k+2}{3} + 39\binom{k+1}{3} + 2\binom{k}{3} &= 12k^3 + 15k^2 + 6k + 1\\
p(12k+1, 4) &= 1\binom{k+3}{3} + 35\binom{k+2}{3} + 35\binom{k+1}{3} + 1\binom{k}{3} &= 12k^3 + 18k^2 + 8k + 1\\
p(12k+2, 4) &= 2\binom{k+3}{3} + 39\binom{k+2}{3} + 30\binom{k+1}{3} + 1\binom{k}{3} &= 12k^3 + 21k^2 + 12k + 2\\
p(12k+3, 4) &= 3\binom{k+3}{3} + 42\binom{k+2}{3} + 27\binom{k+1}{3} &= 12k^3 + 24k^2 + 15k + 3\\
p(12k+4, 4) &= 5\binom{k+3}{3} + 44\binom{k+2}{3} + 23\binom{k+1}{3} &= 12k^3 + 27k^2 + 20k + 5\\
p(12k+5, 4) &= 6\binom{k+3}{3} + 48\binom{k+2}{3} + 18\binom{k+1}{3} &= 12k^3 + 30k^2 + 24k + 6\\
p(12k+6, 4) &= 9\binom{k+3}{3} + 48\binom{k+2}{3} + 15\binom{k+1}{3} &= 12k^3 + 33k^2 + 30k + 9\\
p(12k+7, 4) &= 11\binom{k+3}{3} + 50\binom{k+2}{3} + 11\binom{k+1}{3} &= 12k^3 + 36k^2 + 35k + 11\\
p(12k+8, 4) &= 15\binom{k+3}{3} + 48\binom{k+2}{3} + 9\binom{k+1}{3} &= 12k^3 + 39k^2 + 42k + 15\\
p(12k+9, 4) &= 18\binom{k+3}{3} + 48\binom{k+2}{3} + 6\binom{k+1}{3} &= 12k^3 + 42k^2 + 48k + 18\\
p(12k+10, 4) &= 23\binom{k+3}{3} + 44\binom{k+2}{3} + 5\binom{k+1}{3} &= 12k^3 + 45k^2 + 56k + 23\\
p(12k+11, 4) &= 27\binom{k+3}{3} + 42\binom{k+2}{3} + 3\binom{k+1}{3} &= 12k^3 + 48k^2 + 63k + 27
\end{aligned}$$

Twelve Formulas for $p(n,4)$

This process allows us to describe $p(n,4)$ with twelve formulas:

$$
\begin{aligned}
p(12k,4) &= 1\binom{k+3}{3} + 30\binom{k+2}{3} + 39\binom{k+1}{3} + 2\binom{k}{3} &&= 12k^3+15k^2+6k+1\\
p(12k+1,4) &= 1\binom{k+3}{3} + 35\binom{k+2}{3} + 35\binom{k+1}{3} + 1\binom{k}{3} &&= 12k^3+18k^2+8k+1\\
p(12k+2,4) &= 2\binom{k+3}{3} + 39\binom{k+2}{3} + 30\binom{k+1}{3} + 1\binom{k}{3} &&= 12k^3+21k^2+12k+2\\
p(12k+3,4) &= 3\binom{k+3}{3} + 42\binom{k+2}{3} + 27\binom{k+1}{3} &&= 12k^3+24k^2+15k+3\\
p(12k+4,4) &= 5\binom{k+3}{3} + 44\binom{k+2}{3} + 23\binom{k+1}{3} &&= 12k^3+27k^2+20k+5\\
p(12k+5,4) &= 6\binom{k+3}{3} + 48\binom{k+2}{3} + 18\binom{k+1}{3} &&= 12k^3+30k^2+24k+6\\
p(12k+6,4) &= 9\binom{k+3}{3} + 48\binom{k+2}{3} + 15\binom{k+1}{3} &&= 12k^3+33k^2+30k+9\\
p(12k+7,4) &= 11\binom{k+3}{3} + 50\binom{k+2}{3} + 11\binom{k+1}{3} &&= 12k^3+36k^2+35k+11\\
p(12k+8,4) &= 15\binom{k+3}{3} + 48\binom{k+2}{3} + 9\binom{k+1}{3} &&= 12k^3+39k^2+42k+15\\
p(12k+9,4) &= 18\binom{k+3}{3} + 48\binom{k+2}{3} + 6\binom{k+1}{3} &&= 12k^3+42k^2+48k+18\\
p(12k+10,4) &= 23\binom{k+3}{3} + 44\binom{k+2}{3} + 5\binom{k+1}{3} &&= 12k^3+45k^2+56k+23\\
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\end{aligned}
$$

Quasipolynomial Python Program: $\ell = 3$, $m = 4$, $\beta = (0,1,1,1)$

$$f(z,q) = \sum_{n=0}^{\infty} \sum_{r=-\infty}^{\infty} M_{\beta}(r,n,4) z^r q^n = \frac{1}{(1-q)(1-zq^2)(1-zq^3)(1-zq^4)} \tag{30}$$

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$$= \frac{\sum_{i=0}^{35} q^i \times \sum_{i=0}^{17} (zq^2)^i \times \sum_{i=0}^{11} (zq^3)^i \times \sum_{i=0}^{8} (zq^4)^i}{(1-q^{36})(1-z^{15}q^{36})(1-z^{12}q^{36})(1-z^9q^{36})}$$

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$$= \frac{\sum_{i=0}^{35} q^i \times \sum_{i=0}^{17} (zq^2)^i \times \sum_{i=0}^{11} (zq^3)^i \times \sum_{i=0}^{8} (zq^4)^i}{(1-q^{36})(1-z^{15}q^{36})(1-z^{12}q^{36})(1-z^9q^{36})} = \frac{A(z^3,q) + zB(z^3,q) + z^2C(z^3,q)}{(1-q^{36})(1-z^{15}q^{36})(1-z^{12}q^{36})(1-z^9q^{36})} \tag{31}$$

Quasipolynomial Python Program: $\ell = 3$, $m = 4$, $\beta = (0,1,1,1)$

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$$f(\zeta,q) = \sum_{n=0}^{\infty} \sum_{r=0}^{2} M_{\beta}(r,3,n,4) \zeta^r q^n = \frac{A(1,q) + \zeta B(1,q) + \zeta^2 C(1,q)}{(1-q^{36})^4} = (A(1,q) + \zeta B(1,q) + \zeta^2 C(1,q)) \times \sum_{k=0}^{\infty} \binom{k+3}{3} q^{36k}$$

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Constituents of Quasipolynomials: $\ell = 3$, $m = 4$, $\beta = (0, 1, 1, 1)$

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$$n' = \ell lcm(m)(k+1) - r - \left(\frac{m^2+m}{2}\right) \tag{32}$$

$$n = \ell lcm(m)k + r. \tag{33}$$

Constituents of Quasipolynomials: $\ell = 3$, $m = 4$, $\beta = (0,1,1,1)$

$M_\beta(0,3,36k,4) = 108k^3 + 45k^2 + 6k + 1$	$M_\beta(0,3,36k+26,4) = 108k^3 + 279k^2 + 240k + 68$
$M_\beta(1,3,36k,4) = 108k^3 + 45k^2 + 6k + 0$	$M_\beta(1,3,36k+26,4) = 108k^3 + 279k^2 + 240k + 69$
$M_\beta(2,3,36k,4) = 108k^3 + 45k^2 + 6k + 0$	$M_\beta(2,3,36k+26,4) = 108k^3 + 279k^2 + 240k + 69$
$M_\beta(0,3,36k+1,4) = 108k^3 + 54k^2 + 8k + 1$	$M_\beta(0,3,36k+25,4) = 108k^3 + 270k^2 + 224k + 61$
$M_\beta(1,3,36k+1,4) = 108k^3 + 54k^2 + 8k + 0$	$M_\beta(1,3,36k+25,4) = 108k^3 + 270k^2 + 224k + 62$
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$M_\beta(0,3,36k+2,4) = 108k^3 + 63k^2 + 12k + 1$	$M_\beta(0,3,36k+24,4) = 108k^3 + 261k^2 + 210k + 56$
$M_\beta(1,3,36k+2,4) = 108k^3 + 63k^2 + 12k + 1$	$M_\beta(1,3,36k+24,4) = 108k^3 + 261k^2 + 210k + 57$
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$M_\beta(0,3,36k+3,4) = 108k^3 + 72k^2 + 15k + 1$	$M_\beta(0,3,36k+23,4) = 108k^3 + 252k^2 + 195k + 50$
$M_\beta(1,3,36k+3,4) = 108k^3 + 72k^2 + 15k + 2$	$M_\beta(1,3,36k+23,4) = 108k^3 + 252k^2 + 195k + 51$
$M_\beta(2,3,36k+3,4) = 108k^3 + 72k^2 + 15k + 0$	$M_\beta(2,3,36k+23,4) = 108k^3 + 252k^2 + 195k + 49$
$M_\beta(0,3,36k+4,4) = 108k^3 + 81k^2 + 20k + 1$	$M_\beta(0,3,36k+22,4) = 108k^3 + 243k^2 + 182k + 46$
$M_\beta(1,3,36k+4,4) = 108k^3 + 81k^2 + 20k + 3$	$M_\beta(1,3,36k+22,4) = 108k^3 + 243k^2 + 182k + 46$
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$M_\beta(1,3,36k+5,4) = 108k^3+90k^2+24k+3$	$M_\beta(1,3,36k+21,4) = 108k^3+234k^2+168k+40$
$M_\beta(2,3,36k+5,4) = 108k^3+90k^2+24k+2$	$M_\beta(2,3,36k+21,4) = 108k^3+234k^2+168k+39$
$M_\beta(0,3,36k+6,4) = 108k^3+99k^2+30k+2$	$M_\beta(0,3,36k+20,4) = 108k^3+225k^2+156k+37$
$M_\beta(1,3,36k+6,4) = 108k^3+99k^2+30k+3$	$M_\beta(1,3,36k+20,4) = 108k^3+225k^2+156k+35$
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$M_\beta(0,3,36k+7,4) = 108k^3+108k^2+35k+3$	$M_\beta(0,3,36k+19,4) = 108k^3+216k^2+143k+32$
$M_\beta(1,3,36k+7,4) = 108k^3+108k^2+35k+3$	$M_\beta(1,3,36k+19,4) = 108k^3+216k^2+143k+30$
$M_\beta(2,3,36k+7,4) = 108k^3+108k^2+35k+5$	$M_\beta(2,3,36k+19,4) = 108k^3+216k^2+143k+32$
$M_\beta(0,3,36k+8,4) = 108k^3+117k^2+42k+5$	$M_\beta(0,3,36k+18,4) = 108k^3+207k^2+132k+28$
$M_\beta(1,3,36k+8,4) = 108k^3+117k^2+42k+4$	$M_\beta(1,3,36k+18,4) = 108k^3+207k^2+132k+27$
$M_\beta(2,3,36k+8,4) = 108k^3+117k^2+42k+6$	$M_\beta(2,3,36k+18,4) = 108k^3+207k^2+132k+29$
$M_\beta(0,3,36k+9,4) = 108k^3+126k^2+48k+7$	$M_\beta(0,3,36k+17,4) = 108k^3+198k^2+120k+23$
$M_\beta(1,3,36k+9,4) = 108k^3+126k^2+48k+5$	$M_\beta(1,3,36k+17,4) = 108k^3+198k^2+120k+24$
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$M_\beta(1,3,36k+5,4) = 108k^3 + 90k^2 + 24k + 3$	$M_\beta(1,3,36k+21,4) = 108k^3 + 234k^2 + 168k + 40$
$M_\beta(2,3,36k+5,4) = 108k^3 + 90k^2 + 24k + 2$	$M_\beta(2,3,36k+21,4) = 108k^3 + 234k^2 + 168k + 39$
$M_\beta(0,3,36k+6,4) = 108k^3 + 99k^2 + 30k + 2$	$M_\beta(0,3,36k+20,4) = 108k^3 + 225k^2 + 156k + 37$
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$M_\beta(0,3,36k+9,4) = 108k^3 + 126k^2 + 48k + 7$	$M_\beta(0,3,36k+17,4) = 108k^3 + 198k^2 + 120k + 23$
$M_\beta(1,3,36k+9,4) = 108k^3 + 126k^2 + 48k + 5$	$M_\beta(1,3,36k+17,4) = 108k^3 + 198k^2 + 120k + 24$
$M_\beta(2,3,36k+9,4) = 108k^3 + 126k^2 + 48k + 6$	$M_\beta(2,3,36k+17,4) = 108k^3 + 198k^2 + 120k + 25$

Constituents of Quasipolynomials: $\ell = 3$, $m = 4$, $\beta = (0,1,1,1)$

$M_\beta(0,3,36k+10,4) = 108k^3 + 135k^2 + 56k + 9$	$M_\beta(0,3,36k+16,4) = 108k^3 + 189k^2 + 110k + 20$
$M_\beta(1,3,36k+10,4) = 108k^3 + 135k^2 + 56k + 7$	$M_\beta(1,3,36k+16,4) = 108k^3 + 189k^2 + 110k + 22$
$M_\beta(2,3,36k+10,4) = 108k^3 + 135k^2 + 56k + 7$	$M_\beta(2,3,36k+16,4) = 108k^3 + 189k^2 + 110k + 22$
$M_\beta(0,3,36k+11,4) = 108k^3 + 144k^2 + 63k + 10$	$M_\beta(0,3,36k+15,4) = 108k^3 + 180k^2 + 99k + 17$
$M_\beta(1,3,36k+11,4) = 108k^3 + 144k^2 + 63k + 9$	$M_\beta(1,3,36k+15,4) = 108k^3 + 180k^2 + 99k + 19$
$M_\beta(2,3,36k+11,4) = 108k^3 + 144k^2 + 63k + 8$	$M_\beta(2,3,36k+15,4) = 108k^3 + 180k^2 + 99k + 18$
$M_\beta(0,3,36k+12,4) = 108k^3 + 153k^2 + 72k + 12$	$M_\beta(0,3,36k+14,4) = 108k^3 + 171k^2 + 90k + 15$
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$M_\beta(0,3,36k+13,4) = 108k^3 + 162k^2 + 80k + 13$	$M_\beta(0,3,36k+13,4) = 108k^3 + 162k^2 + 80k + 13$
$M_\beta(1,3,36k+13,4) = 108k^3 + 162k^2 + 80k + 14$	$M_\beta(1,3,36k+13,4) = 108k^3 + 162k^2 + 80k + 14$
$M_\beta(2,3,36k+13,4) = 108k^3 + 162k^2 + 80k + 12$	$M_\beta(2,3,36k+13,4) = 108k^3 + 162k^2 + 80k + 12$

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$M_\beta(0,3,36k+31,4) = 108k^3 + 324k^2 + 323k + 107$	$M_\beta(0,3,36k+31,4) = 108k^3 + 324k^2 + 323k + 107$
$M_\beta(1,3,36k+31,4) = 108k^3 + 324k^2 + 323k + 107$	$M_\beta(1,3,36k+31,4) = 108k^3 + 324k^2 + 323k + 107$
$M_\beta(2,3,36k+31,4) = 108k^3 + 324k^2 + 323k + 107$	$M_\beta(2,3,36k+31,4) = 108k^3 + 324k^2 + 323k + 107$
$M_\beta(0,3,36k+32,4) = 108k^3 + 333k^2 + 342k + 117$	$M_\beta(0,3,36k+30,4) = 108k^3 + 315k^2 + 306k + 99$
$M_\beta(1,3,36k+32,4) = 108k^3 + 333k^2 + 342k + 117$	$M_\beta(1,3,36k+30,4) = 108k^3 + 315k^2 + 306k + 99$
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$M_\beta(0,3,36k+33,4) = 108k^3 + 342k^2 + 360k + 126$	$M_\beta(0,3,36k+29,4) = 108k^3 + 306k^2 + 288k + 90$
$M_\beta(1,3,36k+33,4) = 108k^3 + 342k^2 + 360k + 126$	$M_\beta(1,3,36k+29,4) = 108k^3 + 306k^2 + 288k + 90$
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$M_\beta(0,3,36k+34,4) = 108k^3 + 351k^2 + 380k + 137$	$M_\beta(0,3,36k+28,4) = 108k^3 + 297k^2 + 272k + 83$
$M_\beta(1,3,36k+34,4) = 108k^3 + 351k^2 + 380k + 137$	$M_\beta(1,3,36k+28,4) = 108k^3 + 297k^2 + 272k + 83$
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$M_\beta(0,3,36k+35,4) = 108k^3 + 360k^2 + 399k + 147$	$M_\beta(0,3,36k+27,4) = 108k^3 + 288k^2 + 255k + 75$
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Crank Class Behavior

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Recall: A Continuation of $p(n,4) \pmod 3$

$$
\begin{array}{llll}
p(26,4) = 206 & + & p(0,4) = 1 & = 207 \equiv 0 \pmod 3 \\
p(25,4) = 185 & + & p(1,4) = 1 & = 186 \equiv 0 \pmod 3 \\
p(24,4) = 169 & + & p(2,4) = 2 & = 171 \equiv 0 \pmod 3 \\
p(23,4) = 150 & + & p(3,4) = 3 & = 153 \equiv 0 \pmod 3 \\
p(22,4) = 136 & + & p(4,4) = 5 & = 141 \equiv 0 \pmod 3 \\
p(21,4) = 120 & + & p(5,4) = 6 & = 126 \equiv 0 \pmod 3 \\
p(20,4) = 108 & + & p(6,4) = 9 & = 117 \equiv 0 \pmod 3 \\
p(19,4) = 94 & + & p(7,4) = 11 & = 105 \equiv 0 \pmod 3 \\
p(18,4) = 84 & + & p(8,4) = 15 & = 99 \equiv 0 \pmod 3 \\
p(17,4) = 72 & + & p(9,4) = 18 & = 90 \equiv 0 \pmod 3 \\
p(16,4) = 64 & + & p(10,4) = 23 & = 87 \equiv 0 \pmod 3 \\
p(15,4) = 54 & + & p(11,4) = 27 & = 81 \equiv 0 \pmod 3 \\
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“Fixed” and “Flipped” Crank Classes

$p(18, 4) = 84$ and $p(8, 4) = 15$ organize into 3 unbalanced crank classes under $\beta = (0, 1, 1, 1)$.

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$$M_\beta(0,3,18,4) = 28 \qquad + \qquad M_\beta(0,3,8,4) = 5$$

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The "fixing" and "flipping" of the crank classes is consistent over the entire period.

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Dissertation Conjecture

This brings us to our dissertation conjecture.

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Conjecture 22 (G., Kronholm)

Let ℓ and m be given, and suppose τ is any MB crank witnessing the congruences of the Interval Theorem.

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Let $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$ be an MB statistic with each component $0 \leq \tau_a \leq \ell - 1$. We define a function

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Let's look at our tables of constituents again, this time paying attention to how they fix and flip.

Constituents of Quasipolynomials: $\ell = 3$, $m = 4$, $\beta = (0,1,1,1)$

$M_\beta(0,3,36k,4) = 108k^3 + 45k^2 + 6k + 1$	$M_\beta(0,3,36k+26,4) = 108k^3 + 279k^2 + 240k + 68$
$M_\beta(1,3,36k,4) = 108k^3 + 45k^2 + 6k + 0$	$M_\beta(1,3,36k+26,4) = 108k^3 + 279k^2 + 240k + 69$
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$M_\beta(0,3,36k+1,4) = 108k^3 + 54k^2 + 8k + 1$	$M_\beta(0,3,36k+25,4) = 108k^3 + 270k^2 + 224k + 61$
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$M_\beta(1,3,36k+2,4) = 108k^3 + 63k^2 + 12k + 1$	$M_\beta(1,3,36k+24,4) = 108k^3 + 261k^2 + 210k + 57$
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$M_\beta(0,3,36k+3,4) = 108k^3 + 72k^2 + 15k + 1$	$M_\beta(0,3,36k+23,4) = 108k^3 + 252k^2 + 195k + 50$
$M_\beta(1,3,36k+3,4) = 108k^3 + 72k^2 + 15k + 2$	$M_\beta(1,3,36k+23,4) = 108k^3 + 252k^2 + 195k + 51$
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$M_\beta(1,3,36k+5,4) = 108k^3 + 90k^2 + 24k + 3$	$M_\beta(1,3,36k+21,4) = 108k^3 + 234k^2 + 168k + 40$
$M_\beta(2,3,36k+5,4) = 108k^3 + 90k^2 + 24k + 2$	$M_\beta(2,3,36k+21,4) = 108k^3 + 234k^2 + 168k + 39$
$M_\beta(0,3,36k+6,4) = 108k^3 + 99k^2 + 30k + 2$	$M_\beta(0,3,36k+20,4) = 108k^3 + 225k^2 + 156k + 37$
$M_\beta(1,3,36k+6,4) = 108k^3 + 99k^2 + 30k + 3$	$M_\beta(1,3,36k+20,4) = 108k^3 + 225k^2 + 156k + 35$
$M_\beta(2,3,36k+6,4) = 108k^3 + 99k^2 + 30k + 4$	$M_\beta(2,3,36k+20,4) = 108k^3 + 225k^2 + 156k + 36$
$M_\beta(0,3,36k+7,4) = 108k^3 + 108k^2 + 35k + 3$	$M_\beta(0,3,36k+19,4) = 108k^3 + 216k^2 + 143k + 32$
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$M_\beta(0,3,36k+8,4) = 108k^3 + 117k^2 + 42k + 5$	$M_\beta(0,3,36k+18,4) = 108k^3 + 207k^2 + 132k + 28$
$M_\beta(1,3,36k+8,4) = 108k^3 + 117k^2 + 42k + 4$	$M_\beta(1,3,36k+18,4) = 108k^3 + 207k^2 + 132k + 27$
$M_\beta(2,3,36k+8,4) = 108k^3 + 117k^2 + 42k + 6$	$M_\beta(2,3,36k+18,4) = 108k^3 + 207k^2 + 132k + 29$
$M_\beta(0,3,36k+9,4) = 108k^3 + 126k^2 + 48k + 7$	$M_\beta(0,3,36k+17,4) = 108k^3 + 198k^2 + 120k + 23$
$M_\beta(1,3,36k+9,4) = 108k^3 + 126k^2 + 48k + 5$	$M_\beta(1,3,36k+17,4) = 108k^3 + 198k^2 + 120k + 24$
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$M_\beta(0,3,36k+10,4) = 108k^3 + 135k^2 + 56k + 9$	$M_\beta(0,3,36k+16,4) = 108k^3 + 189k^2 + 110k + 20$
$M_\beta(1,3,36k+10,4) = 108k^3 + 135k^2 + 56k + 7$	$M_\beta(1,3,36k+16,4) = 108k^3 + 189k^2 + 110k + 22$
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$M_\beta(0,3,36k+11,4) = 108k^3 + 144k^2 + 63k + 10$	$M_\beta(0,3,36k+15,4) = 108k^3 + 180k^2 + 99k + 17$
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$M_\beta(0,3,36k+12,4) = 108k^3 + 153k^2 + 72k + 12$	$M_\beta(0,3,36k+14,4) = 108k^3 + 171k^2 + 90k + 15$
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$M_\beta(0,3,36k+32,4) = 108k^3 + 333k^2 + 342k + 117$	$M_\beta(0,3,36k+30,4) = 108k^3 + 315k^2 + 306k + 99$
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Recasting the Generating Function for $p(n, m)$

Recall it is possible to recast the generating function for $p(n, m)$ so that it is in terms of binomial coefficients. For example

$$\sum_{n=0}^{\infty} p(n,4)q^n = \frac{1}{(q;q)_4} = (1 + q + 2q^2 + 3q^3 + 5q^4 + 6q^5 + 9q^6 + \cdots$$
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Definition 25

Given ℓ an odd prime, ζ a primitive ℓ^{th} root of unity, $m, j \geq 1$, and MB statistic $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$, we call the polynomial below the "Ehrhart MB statistic numerator":

$$E_\tau(\zeta, q) = \prod_{j=1}^{m} \sum_{i=0}^{\frac{\mathrm{lcm}(m)-j}{j}} \left(\zeta^{\tau_i} q^j\right)^i = \prod_{j=1}^{m} \frac{1 - \zeta^{\tau_j\left(\frac{\mathrm{lcm}(m)-j}{j}+1\right)} q^{\left(\frac{\mathrm{lcm}(m)-j}{j}+1\right)j}}{1 - \zeta^{\tau_j} q^j} \tag{37}$$

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Conjugate Reciprocal Polynomials

Definition 12

A polynomial of degree d, $P(q) = a_0 + a_1 q + \cdots + a_d q^d$, is said to be reciprocal if

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A polynomial of degree d, with complex coefficients $\{a_i\}$, $P(q) = \sum_{i=0}^{d} a_i q^i = a_0 + a_1 q + \cdots + a_d q^d$, is said to be conjugate reciprocal if

$$a_i = \overline{a_{d-i}}. \tag{42}$$

Equivalently, if

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Definition of a "Flipped" Reciprocal Polynomial

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Let ℓ be an odd prime and ζ be a primitive ℓ^{th} root of unity, and y be a complex number. Given a polynomial

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Example: "Flipped" Reciprocal Polynomial

Given $\ell = 3, m = 4, \zeta = e^{\frac{2\pi i}{3}}, \tau = (0, 1, 1, 1)$, here is the Ehrhart numerator we get by recasting the crank generating function.

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$$+\cdots+$$

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$$+10q^{123}+9e^{-\frac{2i\pi}{3}}q^{123}+8e^{\frac{2i\pi}{3}}q^{123}+9q^{124}+7e^{-\frac{2i\pi}{3}}q^{124}+7e^{\frac{2i\pi}{3}}q^{124}+{\color{red}7}q^{125}+{\color{teal}5}e^{-\frac{2i\pi}{3}}q^{125}+{\color{blue}6}e^{\frac{2i\pi}{3}}q^{125}$$

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$$7q^9 + 6e^{-\frac{2i\pi}{3}} q^9 + 5e^{\frac{2i\pi}{3}} q^9$$

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Remark (Future Work)

How might these generalizations, namely flipped reciprocal polynomials, apply to a proof that there are cranks witnessing the Sum and Difference Theorem?

Future Work

Conjecture 22 (G., Kronholm)

Let ℓ and m be given, and suppose $\boldsymbol{\tau}$ is any MB crank witnessing the congruences of the Interval Theorem. Then $\boldsymbol{\tau}$ also witnesses the congruences of the Sum and Difference Theorem in the following way: For $i \in \{0, 1, \ldots, \ell - 1\}$,

$$M_{\boldsymbol{\tau}}(i, \ell, n', m) \pm M_{\boldsymbol{\tau}}(-i - \sigma(\boldsymbol{\tau}), \ell, n, m) = \frac{p(n', m) + p(n, m)}{\ell} \tag{45}$$

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Theorem 16 (Eichhorn, Kronholm, and Larsen (2022))

A MB statistic τ is a crank for the congruences of The Interval Theorem (Theorem 7) if the components of the tuple $\widehat{\left(\frac{\tau_i}{i}\right)}_{i=1}^{m}$ are distinct modulo ℓ, and $\tau_\ell \not\equiv 0 \pmod{\ell}$ in the tuple $\tau = (\tau_1, \tau_2, \ldots, \tau_m)$.

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Thank you for your time today

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