Generalizations of Bender's variant of the q-Vandermonde convolution

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The Chu–Vandermonde formula

The Chu–Vandermonde formula is

$$\binom{x}{N} = \sum_{n=0}^{N} \binom{a}{n}\binom{x-a}{N-n}.$$

Although this identity appears in work of Vandermonde (1772), its earliest appearance is in a book by Chu Shih-Chieh (1303). Needham and Ling (1959) point out that Chu had identities equivalent to

$$\sum_{n=1}^{N} \binom{n+x-1}{x} = \binom{N+x}{x+1} \quad \text{and} \quad \sum_{n=1}^{N} \binom{n+x-1}{x}\binom{N+a-n}{a} = \sum_{n=1}^{N} \binom{n+a+x-1}{a+x}.$$

Proofs

- By equating coefficients of z on either side of $(1+z)^a(1+z)^b = (1+z)^{a+b}$
- By induction on N (Vandermonde)
- As the terminating case of Gauss's summation formula for a ${}_2F_1$ series with unit argument
- The obvious combinatorial argument

Notation

For $0 < |q| < 1$, we define

$$(x)_\infty = \prod_{n=0}^{\infty}(1-xq^n), \qquad (x)_a = \frac{(x)_\infty}{(xq^a)_\infty}, \qquad (x_1, x_2, \dots, x_k)_a = (x_1)_a(x_2)_a \dots (x_k)_a.$$

Thus $(x)_0 = 1$ and, when n is a positive integer,

$$(x)_n = (1-x)(1-qx)(1-q^2x)\dots(1-q^{n-1}x).$$

If n is a negative integer then

$$1/(x)_n = (1-xq^{-1})(1-xq^{-2})\dots(1-xq^n).$$

The q-binomial coefficient $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_q$ is defined by

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q)_a}{(q)_b(q)_{a-b}}.$$

The q-Vandermonde formula, valid for every integer $N \geq 0$, is

$$\begin{bmatrix} x \\ N \end{bmatrix}_q = \sum_{n=0}^{N} q^{(N-n)(a-n)} \begin{bmatrix} a \\ n \end{bmatrix}_q \begin{bmatrix} x-a \\ N-n \end{bmatrix}_q .$$

In the limit $q \to 1$ it reduces to the Chu–Vandermonde formula.

Proofs

- By equating coefficients of z on either side of $\frac{(az)_\infty}{(z)_\infty} \cdot \frac{(abz)_\infty}{(az)_\infty} = \frac{(abz)_\infty}{(z)_\infty}$
- By induction on N
- As the terminating case of Heine's q-analogue of Gauss's ${}_2F_1$ summation
- Combinatorially, by counting partitions

Sequences of Bender's type

Definition

A sequence $(u_n)_{n\geq 0}$ will be said to be of Bender's type if it satisfies the following conditions

(i) for each $n \geq 0$, either $u_{n+1} = u_n$ or $u_{n+1} = u_n + 1$;

(ii) $u_n \to \infty$ as $n \to \infty$.

Bender's generalized q-Vandermonde formula

Theorem (Bender, 1971)

Let (u_n) be an integer sequence of Bender's type. Then, for any integer $N \geq 0$,

$$\begin{bmatrix} x \\ N \end{bmatrix}_q = \sum_{n=0}^{N} q^{(N-n)(u_{n+1}-n)} \begin{bmatrix} u_n \\ n \end{bmatrix}_q \begin{bmatrix} x - u_{n+1} \\ N-n \end{bmatrix}_q$$

When the sequence (u_n) is constant (for $n \leq N$), this is the q-Vandermonde formula.

Proofs

- By counting partitions into distinct parts (Bender)
- By counting subspaces of a finite-dimensional vector space over $\mathbb{F}_q$ (Bender; see also Andrews, 1974)
- By rearranging the q-Vandermonde formula (Evans, 1978)
- By lattice paths (Sulanke, 1981; Gessel, 1984)

Integer partitions

Let n be any positive integer. A *partition* of n is an ordered tuple of positive integers $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ such that

$$\pi_1 \geq \pi_2 \geq \ldots \geq \pi_k \qquad \text{and} \qquad \pi_1 + \pi_2 + \ldots + \pi_k = n.$$

Example. There are seven partitions of 5:

$$\begin{array}{llll} (5) & (2,2,1) & (4,1) & (2,1,1,1) \\ (3,2) & (1,1,1,1,1) & (3,1,1) & \end{array}$$

By convention, there is one partition of 0, known as the empty partition. We write $\mathcal{P}$ for the set of all partitions and $\mathcal{D}$ for the set of all partitions into distinct parts.

For a partition $\pi \in \mathcal{P}$, we define the following functions:

$$\begin{aligned} \sigma(\pi) &= \text{ the sum of the parts of } \pi, \\ \nu(\pi) &= \text{ the number of parts of } \pi, \\ \lambda(\pi) &= \text{ the largest part of } \pi. \end{aligned}$$

The generating function for partitions into distinct parts of size at most n is

$$(-qx)_n = \sum_{\substack{\pi\in\mathcal{D}\\ \lambda(\pi)\leq n}} x^{\nu(\pi)} q^{\sigma(\pi)}.$$

The q-binomial coefficient can also be interpreted as a generating function for partitions:

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \sum_{\substack{\pi\in\mathcal{P}\\ \nu(\pi)\leq n\\ \lambda(\pi)\leq m-n}} q^{\sigma(\pi)}, \qquad q^{\frac{n(n+1)}{2}} \begin{bmatrix} m \\ n \end{bmatrix}_q = \sum_{\substack{\pi\in\mathcal{D}\\ \nu(\pi)=n\\ \lambda(\pi)\leq m}} q^{\sigma(\pi)}.$$

A combinatorial interpretation of Bender's formula

The following proof is given in Bender's paper: First, multiply both sides by $q^{N(N+1)/2}$. The identity becomes

$$q^{\frac{N(N+1)}{2}}\begin{bmatrix}M\\N\end{bmatrix}_q=\sum_{n=0}^{N}q^{\frac{n(n+1)}{2}}\begin{bmatrix}u_n\\n\end{bmatrix}_q\cdot q^{\frac{(N-n)(N-n+1)}{2}+(N-n)u_{n+1}}\begin{bmatrix}M-u_{n+1}\\N-n\end{bmatrix}_q.$$

This may be interpreted as

$$\sum_{\substack{\pi\in\mathcal{D}\\ \nu(\pi)=N\\ \lambda(\pi)\le M}}q^{\sigma(\pi)}=\sum_{n=0}^{N}\underbrace{\Bigg(\sum_{\substack{\pi\in\mathcal{D}\\ \nu(\pi)=n\\ \lambda(\pi)\le u_n}}q^{\sigma(\pi)}\Bigg)\Bigg(\sum_{\substack{\pi\in\mathcal{D}\\ \nu(\pi)=N-n\\ \text{each }\pi_j\in[u_{n+1}+1,M]}}q^{\sigma(\pi)}\Bigg)}_{\substack{\text{This counts partitions }\pi\in\mathcal{D}\text{ with }\pi_n\le u_n\\ \text{and }\pi_{n+1}>u_{n+1}.\text{ (Writing }\pi_1\le\pi_2\le\ldots)}}.$$

The non-terminating extension of Bender's formula

Theorem (B.T., 2023)

Let (u_n) be an integer sequence of Bender's type. Then, for $|x| < 1$,

$$\sum_{n=0}^{\infty} \frac{(ax)_{u_{n+1}-n-1}(a)_{u_n}(b)_n}{(a)_{u_n-n}(q)_n(abx)_{u_{n+1}-1}} x^n = \frac{(ax, bx)_\infty}{(x, abx)_\infty}.$$

The special case in which $u_n = n$ for every n is the q-Gauss identity,

$$\sum_{n=0}^{\infty} \frac{(a, b)_n}{(q, abx)_n} x^n = \frac{(ax, bx)_\infty}{(x, abx)_\infty}.$$

The series terminates when $b = q^{-N}$ for some integer $N \geq 0$. The identity is then

$$\sum_{n=0}^{N} \frac{(ax)_{u_{n+1}-n-1}(a)_{u_n}(q^{-N})_n}{(a)_{u_n-n}(q)_n(axq^{-N})_{u_{n+1}-1}} x^n = \frac{(xq^{-N})_N}{(axq^{-N})_N}.$$

After making the substitution $(a, x) \mapsto (q^{a+1}, q^{N-x})$, this reduces to Bender's formula, with $a + u_n$ in place of u_n.

A further extension of Bender's formula

The q-Gauss identity is a special case of the ${}_6\phi_5$ summation, which may be written in the form

$$\sum_{n=0}^{\infty} \frac{(a,b,c)_n (abcx)_{n-1}}{(q,abx,acx,bcx)_n} (1 - abcxq^{2n-1}) x^n = \frac{(ax,bx,cx,abcx)_\infty}{(x,abx,acx,bcx)_\infty}.$$

A natural question to ask is whether this formula admits a generalization of Bender's type.

Theorem (B.T., 2025)

Let (u_n) be an integer sequence of Bender's type. Then, for $|x| < 1$,

$$\sum_{n=0}^{\infty} \frac{(ax)_{u_{n+1}-n-1} (a)_{u_n} (b,c)_n (abcx)_{u_n-1}}{(a)_{u_n-n} (q,bcx)_n (abx,acx)_{u_{n+1}-1}} \left(1 - abcx(u_{n+1}-u_n) q^{n+u_n-1}\right) x^n = \frac{(ax,bx,cx,abcx)_\infty}{(x,abx,acx,bcx)_\infty}.$$

The terminating case

Corollary (B.T., 2025)

Let (u_n) be a sequence of Bender's type. Then

$$\sum_{n=0}^{N} \frac{q^{(N-n)(u_{n+1}-n)}}{\begin{bmatrix} b+N-x+u_{n+1} \\ N \end{bmatrix}_q} \begin{bmatrix} u_n \\ n \end{bmatrix}_q \begin{bmatrix} x-u_{n+1} \\ N-n \end{bmatrix}_q \begin{bmatrix} b+n \\ x \end{bmatrix}_q \frac{1-(u_{n+1}-u_n)q^{b-x+n+u_n+1}}{1-(u_{n+1}-u_n)q^{b-x+u_n+1}}$$

$$= \begin{bmatrix} b \\ x-N \end{bmatrix}_q$$

- The special case of this formula when u_n is constant for $n \leq N$ is the q-Saalschütz summation.
- In the limit $b \to \infty$, the formula reduces to Bender's summation.

The proof in outline

- Suppose we take the differences $u_{n+1} - u_n$ to equal 1 for all $n \geq N$. Then $u_n = n - N + u_N$ for $n \geq N$ and the formula to be proved becomes

$$\sum_{n=0}^{N-1} \frac{(ax)_{u_{n+1}-n-1}(a)_{u_n}(b,c)_n(abcx)_{u_n-1}}{(a)_{u_n-n}(q,bcx)_n(abx,acx)_{u_{n+1}-1}}\left(1 - abcx(u_{n+1}-u_n)q^{n+u_n-1}\right)x^n$$
$$+\frac{(ax)_{u_N-N}}{(a)_{u_N-N}} \sum_{n=N}^{\infty} \frac{(a)_{n+u_N-N}(b,c)_n(abcx)_{n+u_N-N-1}}{(q,bcx)_n(abx,acx)_{n+u_N-N}}(1 - abcxq^{2n+u_N-N-1})x^n = \frac{(ax,bx,cx,abcx)_\infty}{(x,abx,acx,bcx)_\infty}.$$

- We can prove this by induction on N. The case $N = 0$ is just the ${}_6\phi_5$ summation, slightly rewritten. The induction step requires that $u_{n+1} - u_n$ equals 0 or 1 for each n.
- The general case follows by letting $N \to \infty$. Tannery's theorem may be used to show that the second term on the left-hand side is $O(|x|^{u_N})$ in this limit. Since $|x| < 1$ and $u_N \to \infty$, the sum must tend to zero.

A special case related to the q-binomial theorem

When $b = c = 0$, the formula becomes

$$\sum_{n=0}^{\infty} \frac{(ax)_{u_{n+1}-n-1}(a)_{u_n}}{(a)_{u_n-n}(q)_n} x^n = \frac{(ax)_\infty}{(x)_\infty}.$$

This identity contains the q-binomial theorem as its special case $u_n = n$. When $a = q$ it may be written in the form

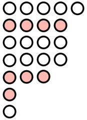
$$\sum_{n=0}^{\infty} (-1)^n (-qx)_{u_{n+1}-n} \begin{bmatrix} u_n \\ n \end{bmatrix}_q (qx)^n = 1.$$

There is really no loss of generality in specializing the value of a since rescaling a just amounts to shifting the terms of (u_n). When the u_n are positive integers, this identity can be proved combinatorially.

Overpartitions

The set of overpartitions is given formally by $\mathcal{P}^{(1)} = \mathcal{P} \times \mathcal{D}$. Informally, an overpartition is a partition in which some parts are 'marked' and all of the marked parts are distinct from one another.

Example. Let $\pi = \big((5,4,4,1),(4,3,1)\big)$. We represent this with a Ferrers diagram:



It is conventional to list the marked and unmarked parts together—so here we may write

$$\pi = (5, \overline{4}, 4, 4, \overline{3}, 1, \overline{1}).$$

By identifying a partition $\pi \in \mathcal{P}$ with the overpartition $(\pi, \varnothing) \in \mathcal{P}^{(1)}$, we may regard $\mathcal{P}$ as a subset of $\mathcal{P}^{(1)}$. It matters which parts are marked. For example,

$$(\overline{5}, 4, 4, 4, \overline{3}, 1, 1)$$

is different from the overpartition π shown above.

Generating functions for overpartitions

Let us write

$$\begin{aligned}\sigma(\pi) &= \text{ the sum of all of the parts of } \pi,\\ \nu_{\mathrm{u}}(\pi) &= \text{ the number of unmarked parts of } \pi,\\ \nu_{\mathrm{m}}(\pi) &= \text{ the number of marked parts of } \pi.\end{aligned}$$

It is well known that

$$\sum_{\pi\in\mathcal{P}^{(1)}} a^{\nu_{\mathrm{m}}(\pi)} x^{\nu_{\mathrm{u}}(\pi)} q^{\sigma(\pi)} = \frac{(-aq)_\infty}{(qx)_\infty}$$

and we can use this to give a combinatorial interpretation of the q-binomial theorem. If we also introduce

$$\begin{aligned}\lambda_{\mathrm{u}}(\pi) &= \text{ the largest unmarked part of } \pi,\\ \lambda_{\mathrm{m}}(\pi) &= \text{ the largest marked part of } \pi,\end{aligned}$$

then

$$\sum_{\substack{\pi\in\mathcal{P}^{(1)}\\ \nu_{\mathrm{u}}(\pi)=n\\ \lambda_{\mathrm{u}}(\pi)\le\ell,\,\lambda_{\mathrm{m}}(\pi)\le m}} x^{\nu_{\mathrm{m}}(\pi)} q^{\sigma(\pi)} = \Bigg(\sum_{\substack{\pi\in\mathcal{P}\\ \nu(\pi)=n\\ \lambda(\pi)\le\ell}} q^{\sigma(\pi)}\Bigg)\Bigg(\sum_{\substack{\pi\in\mathcal{D}\\ \lambda(\pi)\le m}} x^{\nu(\pi)} q^{\sigma(\pi)}\Bigg) = q^n \begin{bmatrix}\ell+n-1\\ n\end{bmatrix}_q (-qx)_m.$$

A combinatorial interpretation

When the u_n are positive integers, the identity

$$\sum_{n=0}^{\infty}(-1)^n(-qx)_{u_{n+1}-n}\begin{bmatrix}u_n\\ n\end{bmatrix}_q(qx)^n=1$$

admits the following interpretation.

Theorem (B.T., 2025)

Let $(u_n)_{n\geq 0}$ be any sequence of positive integers of Bender's type. Let M and N be positive integers. Let $A(M,N)$ denote the number of overpartitions of N into exactly M parts such that $\nu_{\mathrm{u}}(\pi)$ is even and

$$\nu_{\mathrm{u}}(\pi)+\lambda_{\mathrm{u}}(\pi)\leq u_{\nu_{\mathrm{u}}(\pi)}+1, \qquad \nu_{\mathrm{u}}(\pi)+\lambda_{\mathrm{m}}(\pi)\leq u_{\nu_{\mathrm{u}}(\pi)+1}.$$

Let $B(M,N)$ denote the number of overpartitions of N into exactly M parts such that $\nu_{\mathrm{u}}(\pi)$ is odd and these same inequalities hold. Then $A(M,N)=B(M,N)$.

An example

Suppose we take $M = 3$ and $N = 8$, and the sequence (u_n) begins

$$u_0 = u_1 = 4, \qquad u_2 = u_3 = 5, \qquad u_4 = 6.$$

The overpartitions of 8 into 3 parts are shown in the following table:

Overpartitions satisfying both inequalities		Overpartitions satisfying the first inequality but not the second	Overpartitions satisfying the second inequality but not the first
$\nu_{\mathrm{u}}(\pi)$ even	$\nu_{\mathrm{u}}(\pi)$ odd		
$(\overline{4}, \overline{3}, \overline{1})$	$(4, \overline{3}, \overline{1})$	$(\overline{6}, \overline{1}, 1)$	$(6, \overline{1}, 1)$
$(4, \overline{3}, 1)$	$(\overline{4}, \overline{3}, 1)$	$(\overline{6}, 1, 1)$	$(6, 1, 1)$
$(4, 3, \overline{1})$	$(\overline{4}, 3, \overline{1})$	$(\overline{5}, \overline{2}, \overline{1})$	$(5, \overline{2}, \overline{1})$
$(4, \overline{2}, 2)$	$(\overline{4}, \overline{2}, 2)$	$(\overline{5}, \overline{2}, 1)$	$(5, \overline{2}, 1)$
$(\overline{3}, 3, 2)$	$(3, 3, 2)$	$(\overline{5}, 2, \overline{1})$	$(5, 2, \overline{1})$
$(3, 3, \overline{2})$	$(\overline{3}, 3, \overline{2})$	$(\overline{5}, 2, 1)$	$(5, 2, 1)$
		$(\overline{4}, 3, 1)$	$(4, 3, 1)$
		$(\overline{4}, 2, 2)$	$(4, 2, 2)$

Another example

Suppose now we take $M = 3$ and $N = 10$, the sequence (u_n) being the same as before. The overpartitions of 10 into 3 parts are shown in the following table:

Overpartitions satisfying both inequalities		Overpartitions satisfying the first inequality but not the second		Overpartitions satisfying the second inequality but not the first		Overpartitions satisfying neither inequality
$\nu_{\mathrm{u}}(\pi)$ even	$\nu_{\mathrm{u}}(\pi)$ odd					
$(4,4,\overline{2})$	$(\overline{4},4,\overline{2})$	$(\overline{8},1,1)$	$(\overline{8},\overline{1},1)$	$(8,1,1)$	$(8,\overline{1},1)$	$(5,\overline{4},1)$
$(4,\overline{3},3)$	$(\overline{4},\overline{3},3)$	$(\overline{7},\overline{2},\overline{1})$	$(\overline{7},\overline{2},1)$	$(7,\overline{2},\overline{1})$	$(7,\overline{2},1)$	
		$(\overline{7},2,1)$	$(\overline{7},2,\overline{1})$	$(7,2,1)$	$(7,2,\overline{1})$	
		$(\overline{6},\overline{3},\overline{1})$	$(\overline{6},3,1)$	$(6,\overline{3},\overline{1})$	$(6,3,1)$	
		$(\overline{6},3,\overline{1})$	$(\overline{6},\overline{3},1)$	$(6,3,\overline{1})$	$(6,\overline{3},1)$	
		$(\overline{6},2,2)$	$(\overline{6},\overline{2},2)$	$(6,2,2)$	$(6,\overline{2},2)$	
		$(\overline{5},\overline{4},\overline{1})$	$(\overline{5},\overline{4},1)$	$(5,\overline{4},\overline{1})$		
		$(\overline{5},4,\overline{1})$	$(\overline{5},4,1)$	$(5,4,\overline{1})$	$(5,4,1)$	
		$(\overline{5},\overline{3},\overline{2})$	$(\overline{5},\overline{3},2)$	$(5,\overline{3},\overline{2})$	$(5,\overline{3},2)$	
		$(\overline{5},3,\overline{2})$	$(\overline{5},3,2)$	$(5,3,\overline{2})$	$(5,3,2)$	
		$(\overline{4},4,2)$	$(\overline{4},3,3)$	$(4,4,2)$	$(4,3,3)$	

A bijective proof

Let S denote the set of overpartitions of N into M parts which satisfy the two inequalities

$$\nu_{\mathrm{u}}(\pi) + \lambda_{\mathrm{u}}(\pi) \leq u_{\nu_{\mathrm{u}}(\pi)} + 1 \qquad \text{and} \qquad \nu_{\mathrm{u}}(\pi) + \lambda_{\mathrm{m}}(\pi) \leq u_{\nu_{\mathrm{u}}(\pi)+1}.$$

On the set of all overpartitions, let $\iota : \pi \mapsto \tilde{\pi}$ be defined by the condition that π and $\tilde{\pi}$ are identical except that if largest part of π is marked then the largest part of $\tilde{\pi}$ is unmarked and vice versa. It suffices to show that $\tilde{\pi} \in S$ whenever $\pi \in S$. Suppose first that $\pi \in S$ and the largest part of π is marked. Then

$$\nu_{\mathrm{u}}(\tilde{\pi}) = \nu_{\mathrm{u}}(\pi) + 1, \qquad \lambda_{\mathrm{u}}(\tilde{\pi}) = \lambda_{\mathrm{m}}(\pi), \qquad \lambda_{\mathrm{m}}(\tilde{\pi}) \leq \lambda_{\mathrm{m}}(\pi) - 1,$$

so

$$\nu_{\mathrm{u}}(\tilde{\pi}) + \lambda_{\mathrm{u}}(\tilde{\pi}) = \nu_{\mathrm{u}}(\pi) + \lambda_{\mathrm{m}}(\pi) + 1 \leq u_{\nu_{\mathrm{u}}(\pi)+1} + 1 = u_{\nu_{\mathrm{u}}(\tilde{\pi})} + 1$$

and

$$\nu_{\mathrm{u}}(\tilde{\pi}) + \lambda_{\mathrm{m}}(\tilde{\pi}) \leq \nu_{\mathrm{u}}(\pi) + \lambda_{\mathrm{m}}(\pi) \leq u_{\nu_{\mathrm{u}}(\pi)+1} = u_{\nu_{\mathrm{u}}(\tilde{\pi})} \leq u_{\nu_{\mathrm{u}}(\tilde{\pi})+1}$$

so $\tilde{\pi} \in S$.

Suppose now that $\pi \in S$ and the largest part of π is unmarked. Then

$$\nu_{\mathrm{u}}(\tilde{\pi}) = \nu_{\mathrm{u}}(\pi) - 1, \qquad \lambda_{\mathrm{u}}(\tilde{\pi}) \le \lambda_{\mathrm{u}}(\pi), \qquad \lambda_{\mathrm{m}}(\tilde{\pi}) = \lambda_{\mathrm{u}}(\pi),$$

so

$$\nu_{\mathrm{u}}(\tilde{\pi}) + \lambda_{\mathrm{u}}(\tilde{\pi}) \le \nu_{\mathrm{u}}(\pi) + \lambda_{\mathrm{u}}(\pi) - 1 \le u_{\nu_{\mathrm{u}}(\pi)} = u_{\nu_{\mathrm{u}}(\tilde{\pi})+1} \le u_{\nu_{\mathrm{u}}(\tilde{\pi})} + 1$$

and

$$\nu_{\mathrm{u}}(\tilde{\pi}) + \lambda_{\mathrm{m}}(\tilde{\pi}) = \nu_{\mathrm{u}}(\pi) + \lambda_{\mathrm{u}}(\pi) - 1 \le u_{\nu_{\mathrm{u}}(\pi)} = u_{\nu_{\mathrm{u}}(\tilde{\pi})+1}$$

so again $\tilde{\pi} \in S$, as required.

WZ pairs

A *WZ pair* is a pair of functions $F, G : \mathbb{Z}^2_{\geq 0} \to \mathbb{C}$ which satisfy identically the relation
$$F(m+1,n) - F(m,n) = G(m,n+1) - G(m,n).$$
Let $\mathcal{U}$ denote the set of non-negative integer sequences $(u_n)_{n\geq 0}$ which are non-decreasing and for which $u_n \to \infty$ as $n \to \infty$. For any given sequence (u_n) in $\mathcal{U}$, let us write u_n^* for the number of terms which are $\leq n$.

Theorem (Amdeberhan and Zeilberger, 1997 (special case); B.T., 2023)

Let (F,G) be a WZ pair. Suppose moreover that F and G satisfy
$$\lim_{m\to\infty} F(m,n) = \lim_{m\to\infty} G(n,m) = 0$$
for every $n \geq 0$, and that there is a sequence of positive numbers $(c_n)_{n\geq 0}$ such that $|F(m,n)| \leq c_n$ and $|G(m,n)| \leq c_m$ for all integers $m, n \geq 0$ and the series $\sum_{n=0}^{\infty} c_n$ converges. Then
$$\sum_{n=0}^{\infty} F(0,n) = \sum_{n=0}^{\infty} F(u_n, n) + \sum_{n=0}^{\infty} G(n, u_n^*)$$
for any $u \in \mathcal{U}$.

A general theorem of Bender's type

If in this result we take (u_n) to be a sequence of Bender's type, then

$$\sum_{n=0}^{\infty} G(n, u_n^*) = \sum_{n=0}^{\infty} (u_{n+1} - u_n) G(u_n, n+1)$$

so, for a WZ pair (F, G) we have (under suitable conditions) that

$$\sum_{n=0}^{\infty} F(0, n) = \sum_{n=0}^{\infty} F(u_n, n) + \sum_{n=0}^{\infty} (u_{n+1} - u_n) G(u_n, n+1).$$

For example, consider the WZ pair

$$F(m, n) = \frac{(a)_{n-m+1}(ax)_{1-m} x^n}{(a)_{1-m}(q)_n}, \qquad G(m, n) = \frac{a q^{-m} (a)_{n-m}(ax)_{-m} x^n}{(a)_{1-m}(q)_{n-1}}.$$

For an integer sequence of Bender's type, this yields

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \sum_{n=0}^{\infty} \frac{(a)_{n+1-u_n}(ax)_{1-u_n}}{(a)_{1-u_n}(q)_n} x^n + a \sum_{n=0}^{\infty} \frac{(a)_{n+1-u_n}(ax)_{-u_n}}{(a)_{1-u_n}(q)_n} (u_{n+1} - u_n) q^{-u_n} x^{n+1}.$$

A general theorem of Bender's type

This can be simplified by combining the two series into one:

$$\frac{(ax)_\infty}{(x)_\infty} = \sum_{n=0}^{\infty} \frac{(a)_{n+1-u_n}(ax)_{-u_n}}{(a)_{1-u_n}(q)_n}\Big((1-axq^{-u_n}) + ax(u_{n+1}-u_n)q^{-u_n}\Big)x^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_{n+1-u_n}(ax)_{1-u_{n+1}}}{(a)_{1-u_n}(q)_n}x^n.$$

This is the same as the q-binomial-type summation obtained previously, but with $u_n \mapsto n+1-u_n$.

A procedure for generating more such identities

Suppose we begin with a q-series summation formula (finite or infinite) involving the variable a. We may write it in the form

$$\sum_{n=0}^{\infty} f_n(a) = g(a).$$

Let

$$F(m,n) = \frac{f_n(aq^m)}{g(aq^m)}.$$

We can follow the q-WZ method to produce a function G such that (F, G) is a WZ pair. We may then deduce from the foregoing that

$$\sum_{n=0}^{\infty} F(u_n, n) + \sum_{n=0}^{\infty} (u_{n+1} - u_n) G(u_n, n+1) = 1$$

for any sequence (u_n) of Bender's type.

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