Basic hypergeometric summation theorems with symmetry in four variables

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Overview.

- Jackson's problem on unilateral summation formulae with symmetry.
- A generalization of the ${}_6\phi_5$ summation with symmetry in four variables.
- Use of WZ pairs to prove summation formulae and derive new ones.
- Adaptation of classical methods from the theory of elliptic functions.
- Proofs of Ramanujan's ${}_1\psi_1$ and Bailey's ${}_6\psi_6$ identity using theta functions.
- Main result: an analogue of the ${}_6\psi_6$ identity with symmetry in four variables.
- Some partial fraction expansions.
- Open problems.

Notation.

The base, q, is fixed such that $0 < |q| < 1$.

The infinite q-Pochhammer symbol:

$$(a;q)_\infty = (a)_\infty := \prod_{m=0}^{\infty} (1 - aq^m).$$

The finite q-Pochhammer symbol:

$$(a;q)_n = (a)_n := \frac{(a)_\infty}{(aq^n)_\infty}.$$

Write $(a,b,c,\ldots)_n$ for $(a)_n(b)_n(c)_n \ldots$.

Theta functions:

$$\theta(x;q) = \theta(x) := (x, q/x, q)_\infty.$$

Definition

A *unilateral basic hypergeometric series* is a series $\sum_{n=0}^{\infty} u_n$ with the property that the ratio u_{n+1}/u_n of two successive terms is a rational function of q^n. A *bilateral basic hypergeometric series* is a series $\sum_{n=-\infty}^{\infty} u_n$ with the same property.

For non-negative integers r, s, the usual notation is

$$ {}_r\phi_s\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(q, b_1, b_2, \ldots, b_s)_n} \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{s-r+1} x^n, $$

$$ {}_r\psi_s\left(\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}; q, x\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_n}{(b_1, b_2, \ldots, b_s)_n} \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{s-r} x^n. $$

1. Cauchy's (1843) q-binomial theorem:

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{(ax)_\infty}{(x)_\infty}.$$

2. Heine's (1847) q-Gauss identity:

$$\sum_{n=0}^{\infty} \frac{(a,b)_n}{(q,abx)_n} x^n = \frac{(ax,bx)_\infty}{(x,abx)_\infty}.$$

3. The ${}_6\phi_5$ summation of Rogers (1895):

$$\sum_{n=0}^{\infty} \frac{(a,b,c)_n (abcx)_{n-1} (1-abcxq^{2n-1})}{(q,abx,bcx,acx)_n} x^n = \frac{(ax,bx,cx,abcx)_\infty}{(x,abx,bcx,acx)_\infty}.$$

Does the pattern continue?

Jackson considered this question in his paper *Summation of q-hypergeometric series.*

> The question arises, are similar summations by Gamma or q-Gamma functions possible for series symmetrical in 3, 4, 5 or higher number of elements? ... such summations are only possible for series symmetrical in two elements and for series symmetrical in three elements, ... such theorems do not exist for 4 or 5 or higher number of elements. — F.H. Jackson (1921)

In the appendix to his paper, he continued:

> The question is settled conclusively by an examination in the case of four variables of the following product, ... Of course this does not exclude the possibility of special cases of summation, where particular specified relations exist among the variables.

Jackson's problem (my interpretation).

Problem (Jackson)

Let k be any non-negative integer, let $\Pi_{\mathrm{e}}(k)$ denote the set of all products formed from an even number of the variables $a_1, a_2, \ldots, a_k$ and let $\Pi_{\mathrm{o}}(k)$ denote the set of all products formed from an odd number of these variables, with the understanding that $\Pi_{\mathrm{o}}(0) = \varnothing$ and $1 \in \Pi_{\mathrm{e}}(k)$ for every k. Is there a rational function R_k, symmetrical in its first k arguments, such that the equation

$$\sum_{n=0}^{\infty} x^n \prod_{m=0}^{n-1} R_k(a_1, a_2, \ldots, a_k; x, q^m) = \frac{\prod_{\alpha \in \Pi_{\mathrm{o}}(k)} (\alpha x)_\infty}{\prod_{\alpha \in \Pi_{\mathrm{e}}(k)} (\alpha x)_\infty}$$

holds identically for $|x| < 1$?

Jackson's problem (my interpretation).

For a given k, a solution R_k to this problem need not be unique. For example, in the case $k = 0$ one may take

$$R_0(x,y) = \frac{1}{1-qy} \quad \text{or} \quad R_0(x,y) = \frac{y^2}{(1-qy)(1-xy)},$$

corresponding to the two formulae

$$\sum_{n=0}^{\infty} \frac{x^n}{(q)_n} = \frac{1}{(x)_\infty}, \qquad \sum_{n=0}^{\infty} \frac{q^{n(n-1)}x^n}{(q,x)_n} = \frac{1}{(x)_\infty}.$$

Using WZ pairs, infinitely many solutions can be constructed for $k \leq 3$ using the q-binomial theorem, the q-Gauss identity and the ${}_6\phi_5$ summation.

Solution of Jackson's problem in the case $k = 4$.

Theorem

Let p_4 denote the polynomial

$$p_4(a,b,c,d;x,y)$$
$$= \frac{1}{1-\frac{abcdxy^3}{q}}\Bigg(\Big(1-\tfrac{abcxy^2}{q}\Big)\Big(1-\tfrac{abdxy^2}{q}\Big)\Big(1-\tfrac{acdxy^2}{q}\Big)\Big(1-\tfrac{bcdxy^2}{q}\Big)\Big(1-\tfrac{abcdxy^2}{q}\Big)\Big(1-\tfrac{abcdx^2y}{q^2}\Big)$$
$$+\tfrac{abcdx^2y}{q^2}(1-y)(1-ay)(1-by)(1-cy)(1-dy)\Big(1-\tfrac{(abcd)^2x^3y^4}{q^3}\Big)\Bigg).$$

Then, for $|x| < 1$,

$$\sum_{n=0}^{\infty}\frac{(a,b,c,d)_n(abcx,abdx,acdx,bcdx,abcdq^{n-1}x^2)_{n-1}}{(q,abx,acx,adx,bcx,bdx,cdx)_n(abcdx)_{2n}}p_4(a,b,c,d;x,q^n)x^n$$
$$=\frac{(ax,bx,cx,dx,abcx,abdx,acdx,bcdx)_\infty}{(x,abx,acx,adx,bcx,bdx,cdx,abcdx)_\infty}.$$

When $d = 0$ this reduces to the ${}_6\phi_5$ identity.
Moreover, the product on the right-hand side of the equation is of exactly the form which Jackson considered.

A limiting case of the identity.

Take $x \mapsto x/abcd$ and then let $a, b, c, d \to \infty$.

The resulting equation is

$$\sum_{n=0}^{\infty} \frac{q^{2n(n-1)}}{(q)_n (x)_{2n}} (1 - xq^{2n-1} + xq^{3n-1}) x^n = \frac{1}{(x)_\infty}.$$

The series has a combinatorial interpretation in terms of an $n \times 2n$ Durfee rectangle, with x counting the largest part of the partition.

In $1/(x)_\infty$, x counts the number of parts. The equation above therefore holds by conjugation.

The ordinary hypergeometric limit.

The q-Gauss identity and the ${}_6\phi_5$ summation are q-analogues of summation formulae for ordinary hypergeometric series.

The same is true of the summation formula in the previous theorem. It is a q-analogue of the following identity, valid for $\mathrm{Re}(x) > 0$.

$$\sum_{n=0}^{\infty} \frac{[a,b,c,d]_n [a+b+c+x, a+b+d+x, a+c+d+x, b+c+d+x, a+b+c+d+2x+n-1]_{n-1}}{n! [a+b+x, a+c+x, a+d+x, b+c+x, b+d+x, c+d+x]_n [a+b+c+d+x]_{2n}}$$
$$\times \frac{1}{a+b+c+d+x+3n-1} \Big((a+b+c+x+2n-1)(a+b+d+x+2n-1)(a+c+d+x+2n-1)$$
$$\times (b+c+d+x+2n-1)(a+b+c+d+x+2n-1)(a+b+c+d+2x+n-2)$$
$$+ n(a+n)(b+n)(c+n)(d+n)(2a+2b+2c+2d+3x+4n-3) \Big)$$
$$= \frac{\Gamma(x)\Gamma(a+b+x)\Gamma(a+c+x)\Gamma(a+d+x)\Gamma(b+c+x)\Gamma(b+d+x)\Gamma(c+d+x)\Gamma(a+b+c+d+x)}{\Gamma(a+x)\Gamma(b+x)\Gamma(c+x)\Gamma(d+x)\Gamma(a+b+c+x)\Gamma(a+b+d+x)\Gamma(a+c+d+x)\Gamma(b+c+d+x)}.$$

Here, $[a]_n = \Gamma(a+n)/\Gamma(a)$ is the ordinary Pochhammer symbol.

WZ pairs.

Definition

A *WZ pair* is a pair (F, G) of functions which satisfy the relation

$$F(m+1, n) - F(m, n) = G(m, n+1) - G(m, n).$$

If f is a function with the property that

$$f(x, y) - yf(qx, y)$$

is a symmetrical function of x and y, then the functions

$$F(m, n) = f(xq^m, yq^n)q^{mn}x^n y^m,$$
$$G(m, n) = f(yq^n, xq^m)q^{mn}x^n y^m$$

form a WZ pair.

General formulae for WZ pairs can then be used to find summation formulae involving the function f.

WZ pairs.

If f is continuous and $f(x,y) - yf(qx,y)$ is symmetrical in x and y, then the series $\sum_{n=0}^{\infty} f(x,yq^n)x^n$ is symmetrical in x and y and this symmetry is made explicit by the formula

$$\sum_{n=0}^{\infty} f(x,yq^n)x^n = \sum_{n=0}^{\infty} \Big(f(xq^n,yq^n) + xq^n f(yq^{n+1},xq^n) \Big) q^{n^2} x^n y^n.$$

If, in addition, $f(1,y) = 0$, then

$$\sum_{n=0}^{\infty} f(x,q^n)x^n = f(0,1).$$

The function

$$f(x,y) = \frac{(x, qy, abxy, acxy, adxy, bcxy, bdxy, cdxy, abcdxy^2, \frac{abcdx^2y}{q})_\infty}{(ax, bx, cx, dx, ay, by, cy, dy, \frac{abcxy}{q}, \frac{abdxy}{q}, \frac{acdxy}{q}, \frac{bcdxy}{q}, \frac{abcdx^2y^2}{q^2})_\infty} p_4(a,b,c,d;x,y)$$

has these properties.

A generalization of the Rogers–Fine identity.

Theorem

For $|x| < 1$,

$$\sum_{n=0}^{\infty} \frac{(ay,by,cy,dy)_n (abcxy,abdxy,acdxy,bcdxy)_{n-1} (abcdx^2y^2)_{2n-2}}{(y)_{n+1}(abxy,acxy,adxy,bcxy,bdxy,cdxy)_n (abcdxy^2)_{2n} (abcdx^2y)_{n-1}} p_4(a,b,c,d;x,yq^n) x^n$$
$$= \sum_{n=0}^{\infty} \frac{(ax,bx,cx,dx,ay,by,cy,dy)_n (abcxy,abdxy,acdxy,bcdxy)_{2n-1} (abcdx^2y^2)_{4n-2}}{(x,y)_{n+1}(abxy,acxy,adxy,bcxy,bdxy,cdxy)_{2n+1} (abcdx^2y)_{3n+1} (abcdxy^2)_{3n+1}}$$
$$\times \underbrace{P_4(a,b,c,d;xq^n,yq^n)}_{\text{a polynomial}} q^{n^2} x^n y^n.$$

The special case $b = c = d = 0$ is the Rogers–Fine identity.
The special case $c = d = 0$ is

$$\sum_{n=0}^{\infty} \frac{(ay,by)_n}{(y)_{n+1}(abxy)_n} x^n = \sum_{n=0}^{\infty} \frac{(ax,bx,ay,by)_n}{(x,y)_{n+1}(abxy)_{2n+1}} \left(1 - (a+b+ab)xyq^{2n} + abxy(x+y)q^{3n}\right) q^{n^2} x^n y^n.$$

This has a combinatorial interpretation involving a Durfee square.

Obtaining further results using WZ pairs.

Theorem

Let $\mathfrak{u} = \{u_n\}_{n=0}^{\infty}$ be any non-decreasing sequence of non-negative integers such that $u_n \to \infty$ as $n \to \infty$. Let $\{u_n^*\}_{n=0}^{\infty}$ be the sequence given by

$$u_n^* = \text{ the number of terms of the sequence } \mathfrak{u} \text{ which are } \leq n.$$

Then, for any WZ pair (F, G) (satisfying certain additional conditions which ensure convergence), the sum

$$\sum_{n=0}^{\infty} F(u_n, n) + \sum_{n=0}^{\infty} G(n, u_n^*)$$

is independent of the choice of the sequence $\mathfrak{u}$.

If $u_n = \lceil \frac{m}{\ell} n \rceil$ then $u_n^* = \lfloor \frac{\ell}{m} n \rfloor + 1$.

The WZ pair corresponding to the q-Gauss identity yields the following result: the series

$$\sum_{n=0}^{\infty} \frac{(ax,bx)_n(ay,by)_{u_n}}{(x)_{n+1}(y)_{u_n}(abxy)_{n+u_n}} q^{nu_n} x^{u_n} y^n + \sum_{n=0}^{\infty} \frac{(ax,bx)_{u_n^*}(ay,by)_n}{(x)_{u_n^*}(y)_{n+1}(abxy)_{n+u_n^*}} q^{nu_n^*} x^n y^{u_n^*}$$

is independent of the choice of the sequence $\mathfrak{u}$.

Multiplying by y and letting $y \to 1$ then leads to the identity

$$\sum_{n=0}^{\infty} \frac{(ax,bx)_n(a,b)_{u_n}}{(x)_{n+1}(q)_{u_n-1}(abx)_{n+u_n}} q^{nu_n} x^{u_n} + \sum_{n=0}^{\infty} \frac{(ax,bx)_{u_n^*}(a,b)_n}{(x)_{u_n^*}(q)_n(abx)_{n+u_n^*}} q^{nu_n^*} x^n = \frac{(ax,bx)_\infty}{(x,abx)_\infty},$$

which holds for arbitrary $\mathfrak{u}$.

The special case $u_n = 2n$, $u_n^* = \lfloor n/2 \rfloor + 1$ is a summation formula of the form

$$\sum_{n=0}^{\infty} \frac{q^{2n(n-1)}(a,b)_n(ax,bx)_{2n-2}}{(q)_n(x)_{2n}(abx)_{3n}} \left(\substack{\text{polynomial of} \\ \text{degree 10 in } q^n}\right) x^n = \frac{(ax,bx)_\infty}{(x,abx)_\infty}.$$

Spaces of theta functions.

Definition

For any positive integer n and any non-zero complex number C, let $T_n(C)$ denote the space of analytic functions f on $\mathbb{C}\backslash\{0\}$ which satisfy

$$f(qx) = \frac{(-1)^n}{Cx^n} f(x)$$

for every $x \in \mathbb{C}\backslash\{0\}$.

Lemma (Hermite, 1882)

$T_n(C)$ is a vector space of dimension n.

Recall $\theta(x) = (x, q/x, q)_\infty$.
$\theta \in T_1(1)$.

Spaces of theta functions.

Theorem (Classical)

Let $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be analytic. Then $f \in T_n(C)$ if and only if

$$f(x) = A\theta(\alpha_1 x)\theta(\alpha_2 x)\dots\theta(\alpha_n x)$$

for some complex number A and non-zero complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $\alpha_1\alpha_2\dots\alpha_n = C$.

Theorem (Appell, 1884)

Let $f, g \in T_n(C)$. Let x_1, x_2, $\dots$, x_n be non-zero complex numbers such that

(i) None of the ratios x_m/x_j is an integer power of q.

(ii) The product $Cx_1x_2\dots x_n$ is not an integer power of q.

If $f(x_m) = g(x_m)$ for $m = 1, 2, \dots, n$ then the functions f and g are identically equal.

The connection with the theory of elliptic functions.

Suppose $f, g \in T_n(C)$ are such that their zeros are distinct and let $q = \mathrm{e}^{\frac{2\pi\mathrm{i}\omega_2}{\omega_1}}$ where $\mathrm{Im}(\omega_2/\omega_1) > 0$. Then the function

$$F(u) = \frac{f\left(\mathrm{e}^{\frac{2\pi\mathrm{i}u}{\omega_1}}\right)}{g\left(\mathrm{e}^{\frac{2\pi\mathrm{i}u}{\omega_1}}\right)}$$

is an elliptic function with n zeros and n poles within each period parallelogram (counted with multiplicity).

It is meromorphic and satisfies

$$F(u + \omega_1) = F(u + \omega_2) = F(u).$$

The converse is also true: every elliptic function can be written in this form.

An example of the use of finite dimensionality.

Regarded as functions of x, the two products $\theta(ax)\theta(bcx)$ and $\theta(bx)\theta(acx)$ are clearly linearly independent.

They therefore form a basis for the two-dimensional space $T_2(abc)$.

$$\underbrace{\theta(cx)\theta(abx)}_{\in T_2(abc)} = A\theta(ax)\theta(bcx) + B\theta(bx)\theta(acx)$$

Since $\theta(1) = 0$, the values of A and B can be found by setting $x = 1/a$, $x = 1/b$:

$$A = \frac{\theta(c/b)\theta(a)}{\theta(a/b)\theta(c)}, \quad B = \frac{\theta(c/a)\theta(b)}{\theta(b/a)\theta(c)}.$$

The resulting equation is Weierstrass's three-term identity for theta functions. This proof is due to Gutzmer (1892).

Theta functions and bilateral basic hypergeometric series.

Given any convergent bilateral basic hypergeometric series, it is always possible to insert an extra parameter in such a way as to obtain a theta function.

Lemma

Let r, s and t be non-negative integers, not all equal to zero, such that $r+t \geq s$, and let p be a polynomial. Considered as a function of the variable y, with all other variables held fixed, the function

$$\left(\tfrac{q}{a_1 y}, \tfrac{q}{a_2 y}, \ldots, \tfrac{q}{a_r y}, b_1 y, b_2 y, \ldots, b_s y\right)_\infty \sum_{n=-\infty}^{\infty} \frac{(-1)^{tn} q^{\frac{tn(n-1)}{2}} (a_1 y, a_2 y, \ldots, a_r y)_n}{(b_1 y, b_2 y, \ldots, b_s y)_n} p(yq^n) x^n y^{tn}$$

belongs to the space $T_{r+t}(a_1 a_2 \ldots a_r x)$.

Example: the ${}_1\psi_1$ summation in terms of theta functions.

Let x be such that $|q/a| < |x| < 1$ and let

$$f(y) = \left(\tfrac{q}{ay}, qy\right)_\infty \sum_{n=-\infty}^{\infty} \frac{(ay)_n}{(qy)_n} x^n.$$

Then $f \in T_1(ax)$ and

$$f(1) = \left(\tfrac{q}{a}, q\right)_\infty \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} x^n = \frac{(\frac{q}{a}, q, ax)_\infty}{(x)_\infty}.$$

The space $T_1(ax)$ is one-dimensional; each of its elements is a multiple of $\theta(axy)$. Hence

$$f(y) = \frac{f(1)\theta(axy)}{\theta(ax)} = \frac{(\frac{q}{a})_\infty \, \theta(axy)}{(x, \frac{q}{ax})_\infty}.$$

This is Ramanujan's ${}_1\psi_1$ summation.

A theta-function proof of Bailey's ${}_6\psi_6$ summation formula.

Theorem (Bailey, 1936)

For $0 < |x| < 1$,

$$\sum_{n=-\infty}^{\infty} \frac{\left(ay, by, cy, \frac{abcxy}{q}\right)_n}{(qy, abxy, acxy, bcxy)_n}\left(1-abcxy^2q^{2n-1}\right)x^n = \frac{\left(\frac{q}{a}, \frac{q}{b}, \frac{q}{c}, ax, bx, cx\right)_\infty \theta\left(\frac{abcxy^2}{q}\right)}{\left(x, qy, \frac{q}{ay}, \frac{q}{by}, \frac{q}{cy}, abxy, acxy, bcxy, \frac{q^2}{abcxy}\right)_\infty}.$$

Proof.

Let

$$\Theta_3(a,b,c;x,y) = \frac{\left(x, qy, \frac{q}{ay}, \frac{q}{by}, \frac{q}{cy}, abxy, acxy, bcxy, \frac{q^2}{abcxy}\right)_\infty}{\left(\frac{q}{a}, \frac{q}{b}, \frac{q}{c}, ax, bx, cx\right)_\infty} \sum_{n=-\infty}^{\infty} \frac{\left(ay, by, cy, \frac{abcxy}{q}\right)_n}{(qy, abxy, acxy, bcxy)_n} \times\left(1-abcxy^2q^{2n-1}\right)x^n,$$

then

$$\Theta_3(a,b,c,x,qy) = \frac{q}{(abc)^2x^2y^4}\Theta_3(a,b,c,x,y) \qquad (n \mapsto n+1)$$

$$\Theta_3\left(a, \frac{q}{bx}, \frac{q}{cx}, x, \frac{1}{ay}\right) = -\frac{q}{abcxy^2}\Theta_3(a,b,c,x,y). \qquad (n \mapsto -n)$$

A theta function proof of Bailey's ${}_6\psi_6$ summation formula.

Proof (continued).

$$\begin{aligned}
\Theta_3(a,b,c,x,1) &= \theta\Big(\frac{abcx}{q}\Big), & \Theta_3\Big(a,b,c,x,\frac{q}{abx}\Big) &= \theta\Big(\frac{cq}{abx}\Big),\\
\Theta_3\Big(a,b,c,x,\frac{1}{a}\Big) &= \theta\Big(\frac{bcx}{aq}\Big), & \Theta_3\Big(a,b,c,x,\frac{q}{acx}\Big) &= \theta\Big(\frac{bq}{acx}\Big),\\
\Theta_3\Big(a,b,c,x,\frac{1}{b}\Big) &= \theta\Big(\frac{acx}{bq}\Big), & \Theta_3\Big(a,b,c,x,\frac{q}{bcx}\Big) &= \theta\Big(\frac{aq}{bcx}\Big),\\
\Theta_3\Big(a,b,c,x,\frac{1}{c}\Big) &= \theta\Big(\frac{abx}{cq}\Big), & \Theta_3\Big(a,b,c,x,\frac{q}{abcx}\Big) &= \theta\Big(\frac{q}{abcx}\Big).
\end{aligned}$$

$\Theta_3 \in T_4\big((abc)^2x^2/q\big)$ and

$$\Theta_3(a,b,c;x,y) = \theta\big(\tfrac{abcxy^2}{q}\big)$$

for 8 different values of y. By Appell's theorem, these two functions are identically equal. $\square$

Appell's formula

Theorem (Appell, 1884)

Let $f \in T_n(C)$. Let x_1, x_2, $\ldots$, x_n be non-zero complex numbers such that

(i) None of the ratios x_m/x_j is an integer power of q.

(ii) The product $Cx_1x_2\ldots x_n$ is not an integer power of q.

Then

$$f(x) = \frac{1}{\theta(Cx_1x_2\ldots x_n)} \sum_{m=1}^{n} f(x_m)\,\theta(Cx_1\ldots x_{m-1}x_{m+1}\ldots x_n x) \times \frac{\theta(\frac{x}{x_1})\ldots\theta(\frac{x}{x_{m-1}})\theta(\frac{x}{x_{m+1}})\ldots\theta(\frac{x}{x_n})}{\theta(\frac{x_m}{x_1})\ldots\theta(\frac{x_m}{x_{m-1}})\theta(\frac{x_m}{x_{m+1}})\ldots\theta(\frac{x_m}{x_n})}.$$

An analogue of Bailey's ${}_6\psi_6$ identity.

Theorem

Let

$$\Theta_4(a,b,c,d;x,y)$$
$$= \frac{1}{\theta\big(\frac{abcdxy^3}{q}\big)}\Bigg(\theta\Big(\frac{abcxy^2}{q}\Big)\theta\Big(\frac{abdxy^2}{q}\Big)\theta\Big(\frac{acdxy^2}{q}\Big)\theta\Big(\frac{bcdxy^2}{q}\Big)\theta\Big(\frac{abcdxy^2}{q}\Big)\theta\Big(\frac{abcdx^2y}{q^2}\Big)$$
$$+\frac{abcdx^2y}{q^2}\theta(y)\theta(ay)\theta(by)\theta(cy)\theta(dy)\theta\Big(\frac{(abcd)^2x^3y^4}{q^3}\Big)\Bigg).$$

Then, for $0<|x|<1$,

$$\sum_{n=-\infty}^{\infty}\frac{\big(ay,by,cy,dy,\frac{abcxy}{q},\frac{abdxy}{q},\frac{acdxy}{q},\frac{bcdxy}{q}\big)_n\big(\frac{abcdx^2y^2}{q^2}\big)_{2n}}{\big(qy,abxy,acxy,adxy,bcxy,bdxy,cdxy,\frac{abcdx^2y}{q}\big)_n(abcdxy^2)_{2n}}p_4\big(a,b,c,d,x,yq^n\big)x^n$$

$$=\frac{\big(\frac{q}{a},\frac{q}{b},\frac{q}{c},\frac{q}{d},ax,bx,cx,dx\big)_\infty\Theta_4\big(a,b,c,d,x,y\big)}{(q)_\infty^4\big(x,qy,\frac{q}{ay},\frac{q}{by},\frac{q}{cy},\frac{q}{dy},abxy,acxy,adxy,bcxy,bdxy,cdxy,\frac{q^2}{abcxy},\frac{q^2}{abdxy},\frac{q^2}{acdxy},\frac{q^2}{bcdxy},\frac{abcdx^2y}{q},abcdxy^2,\frac{q^3}{abcdx^2y^2}\big)_\infty}.$$

Method of proof: define Θ_4 by the second equation and then show that it can be expressed in terms of theta functions as in the first equation.

An analogue of Bailey's ${}_6\psi_6$ identity.

Proof.

The function Θ_4 satisfies the two relations

$$\Theta_4(a,b,c,d;x,y) = -\frac{(abcd)^4x^6y^8}{q^6}\Theta_4\Big(a,\frac{q}{bx},\frac{q}{cx},\frac{q}{dx};x,\frac{1}{ay}\Big),$$

$$\Theta_4(a,b,c,d,x,qy) = \frac{q^7}{(abcd)^6x^9y^{12}}\Theta_4(a,b,c,d,x,y).$$

The second of these asserts that, as a function of y, Θ_4 belongs to the 12-dimensional space $T_{12}\big((abcd)^6x^9/q^7\big)$. It also satisfies

$$\Theta_4(a,b,c,d,x,1) = \theta\Big(\frac{abcx}{q}\Big)\theta\Big(\frac{abdx}{q}\Big)\theta\Big(\frac{acdx}{q}\Big)\theta\Big(\frac{bcdx}{q}\Big)\theta\Big(\frac{abcdx^2}{q^2}\Big).$$

This gives rise to 16 special values — enough to determine Θ_4 in closed form via Appell's formula.

An analogue of Bailey's ${}_6\psi_6$ identity.

Proof.

$$\Theta_4(a,b,c,d,x,y)$$

$$= \frac{\theta(y)\theta(ay)\theta(by)\theta(cy)\theta(dy)\theta\big(\frac{abxy}{q}\big)\theta\big(\frac{acxy}{q}\big)\theta\big(\frac{adxy}{q}\big)\theta\big(\frac{bcxy}{q}\big)\theta\big(\frac{bdxy}{q}\big)\theta\big(\frac{cdxy}{q}\big)\theta\big(\frac{abcdx^2y}{q^2}\big)}{\theta(abcdqx)}$$

$$\times\Bigg(\frac{\theta\big(\frac{abcx}{q}\big)\theta\big(\frac{abdx}{q}\big)\theta\big(\frac{acdx}{q}\big)\theta\big(\frac{bcdx}{q}\big)\theta(abcdqxy)}{\theta(a)\theta(b)\theta(c)\theta(d)\theta\big(\frac{abx}{q}\big)\theta\big(\frac{acx}{q}\big)\theta\big(\frac{adx}{q}\big)\theta\big(\frac{bcx}{q}\big)\theta\big(\frac{bdx}{q}\big)\theta\big(\frac{cdx}{q}\big)\theta(y)}$$

$$+\frac{\theta\big(\frac{bcdx}{aq}\big)\theta\big(a^2bcdqxy\big)}{\theta\big(\frac{1}{a}\big)\theta\big(\frac{b}{a}\big)\theta\big(\frac{c}{a}\big)\theta\big(\frac{d}{a}\big)\theta\big(\frac{bx}{q}\big)\theta\big(\frac{cx}{q}\big)\theta\big(\frac{dx}{q}\big)\theta(ay)}+\frac{\theta\big(\frac{acdx}{bq}\big)\theta\big(ab^2cdqxy\big)}{\theta\big(\frac{1}{b}\big)\theta\big(\frac{a}{b}\big)\theta\big(\frac{c}{b}\big)\theta\big(\frac{d}{b}\big)\theta\big(\frac{ax}{q}\big)\theta\big(\frac{cx}{q}\big)\theta\big(\frac{dx}{q}\big)\theta(by)}$$

$$+\frac{\theta\big(\frac{abdx}{cq}\big)\theta\big(abc^2dqxy\big)}{\theta\big(\frac{1}{c}\big)\theta\big(\frac{a}{c}\big)\theta\big(\frac{b}{c}\big)\theta\big(\frac{d}{c}\big)\theta\big(\frac{ax}{q}\big)\theta\big(\frac{bx}{q}\big)\theta\big(\frac{dx}{q}\big)\theta(cy)}+\frac{\theta\big(\frac{abcx}{dq}\big)\theta\big(abcd^2qxy\big)}{\theta\big(\frac{1}{d}\big)\theta\big(\frac{b}{d}\big)\theta\big(\frac{c}{d}\big)\theta\big(\frac{a}{d}\big)\theta\big(\frac{ax}{q}\big)\theta\big(\frac{bx}{q}\big)\theta\big(\frac{cx}{q}\big)\theta(dy)}$$

$$+\frac{\theta\big(\frac{cd}{a}\big)\theta\big(\frac{cd}{b}\big)\theta\big(a^2b^2cdx^2y\big)}{\theta(ax)\theta(bx)\theta(abx)\theta\big(\frac{c}{a}\big)\theta\big(\frac{c}{b}\big)\theta\big(\frac{d}{a}\big)\theta\big(\frac{d}{b}\big)\theta\big(\frac{cdx}{q}\big)\theta\big(\frac{abxy}{q}\big)}+\frac{\theta\big(\frac{bd}{a}\big)\theta\big(\frac{bd}{c}\big)\theta\big(a^2bc^2dx^2y\big)}{\theta(ax)\theta(cx)\theta(acx)\theta\big(\frac{b}{a}\big)\theta\big(\frac{b}{c}\big)\theta\big(\frac{d}{a}\big)\theta\big(\frac{d}{c}\big)\theta\big(\frac{bdx}{q}\big)\theta\big(\frac{acxy}{q}\big)}$$

$$+\frac{\theta\big(\frac{bc}{a}\big)\theta\big(\frac{bc}{d}\big)\theta\big(a^2bcd^2x^2y\big)}{\theta(ax)\theta(dx)\theta(adx)\theta\big(\frac{b}{a}\big)\theta\big(\frac{b}{d}\big)\theta\big(\frac{c}{a}\big)\theta\big(\frac{c}{d}\big)\theta\big(\frac{bcx}{q}\big)\theta\big(\frac{adxy}{q}\big)}+\frac{\theta\big(\frac{ad}{b}\big)\theta\big(\frac{ad}{c}\big)\theta\big(ab^2c^2dx^2y\big)}{\theta(bx)\theta(cx)\theta(bcx)\theta\big(\frac{a}{b}\big)\theta\big(\frac{a}{c}\big)\theta\big(\frac{d}{b}\big)\theta\big(\frac{d}{c}\big)\theta\big(\frac{adx}{q}\big)\theta\big(\frac{bcxy}{q}\big)}$$

$$+\frac{\theta\big(\frac{ac}{b}\big)\theta\big(\frac{ac}{d}\big)\theta\big(ab^2cd^2x^2y\big)}{\theta(bx)\theta(dx)\theta(bdx)\theta\big(\frac{a}{b}\big)\theta\big(\frac{a}{d}\big)\theta\big(\frac{c}{b}\big)\theta\big(\frac{c}{d}\big)\theta\big(\frac{acx}{q}\big)\theta\big(\frac{bdxy}{q}\big)}+\frac{\theta\big(\frac{ab}{c}\big)\theta\big(\frac{ab}{d}\big)\theta\big(abc^2d^2x^2y\big)}{\theta(cx)\theta(dx)\theta(cdx)\theta\big(\frac{a}{c}\big)\theta\big(\frac{a}{d}\big)\theta\big(\frac{b}{c}\big)\theta\big(\frac{b}{d}\big)\theta\big(\frac{abx}{q}\big)\theta\big(\frac{cdxy}{q}\big)}$$

$$+\frac{\theta\big(\frac{a^2b^2c^2d^2x^3y}{q}\big)}{\theta(abx)\theta(acx)\theta(adx)\theta(bcx)\theta(bdx)\theta(cdx)\theta\big(\frac{abcdx^2y}{q^2}\big)}\Bigg).$$

Proof.

From this formula for Θ_4, it follows that

$$\Theta_4(aq,b,c,d;x,y) = \frac{q^5}{a^4(bcd)^3x^5y^6}\Theta_4(a,b,c,d;x,y),$$
$$\Theta_4(a,b,c,d;qx,y) = \frac{q^7}{(abcd)^5x^8y^9}\Theta_4(a,b,c,d;x,y).$$

Consider now the function

$$\Phi(t) = \Theta_4\big(\tfrac{a}{t},b,c,d;tx,y\big).$$

From the relations above, it follows that

$$\Phi(tq) = \frac{q^3}{a(bcd)^2t^2(xy)^3}\Phi(t).$$

Proof.

Hence

$$\Phi(t) = A\,\theta\Big(\frac{t}{ay}\Big)\theta\Big(\frac{(abcd)^2x^3y^4}{q^3}t\Big) + B\,\theta\Big(\frac{abcdx^2y}{q^2}t\Big)\theta\Big(\frac{bcdxy^2}{q}t\Big)$$

for some A, B independent of t. Set $t = ay$ and $t = q^2/abcdx^2y$ to determine the values of A and B. $\square$

A property of the functions p_4 and Θ_4.

The polynomial $p_4(a,b,c,d;x,y)$ is of degree 8 in its final argument, y. The coefficients of y, y^4 and y^7 are zero.

The function Θ_4 can be expanded as a Laurent series in powers of y:

$$\Theta_4(a,b,c,d;x,y) = \sum_{n=-\infty}^{\infty} u_n y^n.$$

The coefficients u_n satisfy $u_{3n+1} = 0$.

Compare this with Bailey's ${}_6\psi_6$ identity, in which the polynomial factor is $p_3(a,b,c;x,y) = 1 - abcxy^2/q$ and the theta function is $\Theta_3(a,b,c;x,y) = \theta(abcxy^2/q)$.

The ordinary hypergeometric limit.

$$\sum_{n=-\infty}^{\infty} \frac{[a+y,b+y,c+y,d+y]_n[a+b+c+x+y,a+b+d+x+y,a+c+d+x+y,b+c+d+x+y]_{n-1}}{[1+y,a+b+x+y,a+c+x+y,a+d+x+y,b+c+x+y,b+d+x+y,c+d+x+y]_n}$$

$$\times\frac{[a+b+c+d+2x+2y]_{2n-2}}{[a+b+c+d+2x+y]_{n-1}[a+b+c+d+x+2y]_{2n}}\times\frac{1}{a+b+c+d+x+3y+3n-1}$$

$$\times\Big((a+b+c+x+2y+2n-1)(a+b+d+x+2y+2n-1)(a+c+d+x+2y+2n-1)$$

$$\times(b+c+d+x+2y+2n-1)(a+b+c+d+x+2y+2n-1)(a+b+c+d+2x+y+n-2)$$

$$+(y+n)(a+n)(b+y+n)(c+y+n)(d+y+n)(2a+2b+2c+2d+3x+4y+4n-3)\Big)$$

$$=\frac{\Gamma(x)\Gamma(1+y)\Gamma(1-a-y)\Gamma(1-b-y)\Gamma(1-c-y)\Gamma(1-d-y)}{\pi^5\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)\Gamma(1-d)\Gamma(a+x)\Gamma(b+x)\Gamma(c+x)\Gamma(d+x)}$$

$$\times\Gamma(a+b+x+y)\Gamma(a+c+x+y)\Gamma(a+d+x+y)\Gamma(b+c+x+y)\Gamma(b+d+x+y)\Gamma(c+d+x+y)$$

$$\times\Gamma(1-a-b-c-x-y)\Gamma(1-a-b-d-x-y)\Gamma(1-a-c-d-x-y)\Gamma(1-b-c-d-x-y)$$

$$\times\Gamma(a+b+c+d+2x+y)\Gamma(a+b+c+d+x+2y)\Gamma(1-a-b-c-d-2x-2y)$$

$$\times\frac{\mathrm{i}}{32}e^{-2\pi\mathrm{i}(2(a+b+c+d)+3x+4y)}p_4\big(e^{2\pi\mathrm{i}a},e^{2\pi\mathrm{i}b},e^{2\pi\mathrm{i}c},e^{2\pi\mathrm{i}d},e^{2\pi\mathrm{i}x},e^{2\pi\mathrm{i}y}\big).$$

Lambert series.

As observed by Andrews, Lewis and Liu (2001), Bailey's ${}_6\psi_6$ identity reduces to a Lambert series identity when $x = q$, $c = 1$,

$$\sum_{n=-\infty}^{\infty} \left(\frac{q^n}{1 - yq^n} - \frac{aq^n}{1 - ayq^n} - \frac{bq^n}{1 - byq^n} + \frac{abq^n}{1 - abyq^n} \right) = \frac{\theta(a)\theta(b)\theta(aby^2)(q)_\infty^3}{\theta(y)\theta(ay)\theta(by)\theta(aby)}.$$

This formula is originally due to Jacobi (1836); it can be written in the alternative form

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\theta(ax)\theta(bx)}{\theta(x)\theta(abx)} \right) = \frac{\theta(a)\theta(b)\theta(abx^2)(q)_\infty^3}{\theta(x)^2\theta(abx)^2}.$$

Analogous to this, the special case $x = q$, $d = 1$ of the bilateral summation with symmetry in four variables is (with $y \mapsto x$)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\theta(ax)\theta(bx)\theta(cx)\theta(abcx)}{\theta(x)\theta(abx)\theta(acx)\theta(bcx)} \right) \\ &= \frac{\theta(a)\theta(b)\theta(c)(q)_\infty^3}{\theta(abx)^2\theta(acx)^2\theta(bcx)^2\theta(abcx^3)} \left(\frac{\theta(abx^2)\theta(acx^2)\theta(bcx^2)\theta(abcx)\theta(abcx^2)}{\theta(x)^2} \right. \\ &\qquad \left. + abcx \frac{\theta(ax)\theta(bx)\theta(cx)\theta\big((abc)^2x^4\big)}{\theta(abcx^2)} \right). \end{aligned}$$

Watson's LMS presidential address.

The following two partial fraction expansions appear in Watson's paper (1936):

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(qx, q/x)_n} = \frac{1-x}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1-xq^n},$$

$$\sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(x, q/x)_n} = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{\frac{3n(n+1)}{2}}}{1-xq^n}.$$

He derived them from his q-analogue of Whipple's formula, but added in a footnote

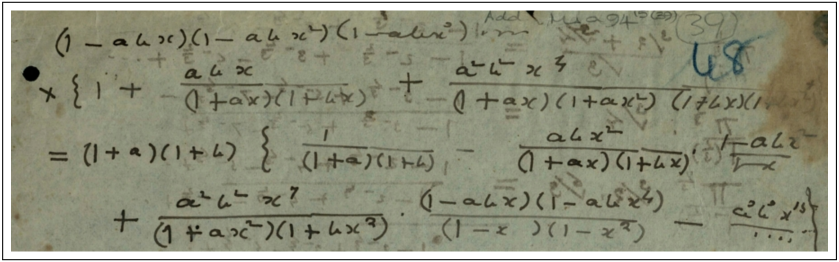
> This formula may also be obtained by expressing the series on the right (*qua* function of $\cos\theta$) as a sum of partial fractions.

The calculation of the residues uses the formula

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)} x^n}{(q, x)_n} = \frac{1}{(x)_\infty},$$

which is a limiting case of the q-Gauss identity.

The following entry appears in Ramanujan's lost notebook:



Source: `https://archives.trin.cam.ac.uk/index.php/the-so-called-lost-notebook-2`

Changing the notation and rearranging slightly, it can be written as

$$(xy)_\infty \sum_{n=0}^{\infty} \frac{x^n y^n q^{n^2}}{(x,y)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(3n+1)}{2}} x^n y^n (xy)_n (1-xyq^{2n})}{(q)_n (1-xq^n)(1-yq^n)}.$$

This is a partial fraction expansion with respect to the variable z introduced through the substitution $(x,y) \mapsto (xz, y/z)$.

Partial fraction expansions from q-Gauss and the ${}_6\phi_5$ identity.

This last equation is a limiting case of the expansion

$$\sum_{n=0}^{\infty} \frac{(a,b)_n (\frac{qxy}{ab})^n}{(x,y)_{n+1}} = \frac{(\frac{qxy}{a},\frac{qxy}{b})_\infty}{(xy,\frac{qxy}{ab})_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n+3)}{2}} (xy)^n (a,b,xy)_n (1-xyq^{2n})}{(ab)^n (q,\frac{qxy}{a},\frac{qxy}{b})_n (1-xq^n)(1-yq^n)},$$

for which the calculation of the residues uses the general case of the q-Gauss identity.

This in turn is a limiting case of

$$\sum_{n=0}^{\infty} \frac{(a,b,cz,\frac{abcx}{qz})_n (1-abcxq^{2n-1})}{(z,abx/qz)_{n+1}(acx,bcx)_n} x^n =$$
$$\frac{(ax,bx,cx,abcx)_\infty}{(x,abx,acx,bcx)_\infty} \sum_{n=0}^{\infty} \frac{(a,b,\frac{q}{c})_n (abx)_{n-1} (1-abxq^{2n-1})(cx)^n}{(q,ax,bx)_n (abcx)_{n-1} (1-zq^n)(1-abxq^{n-1}/z)},$$

which follows from the ${}_6\phi_5$ identity.

The most interesting special case is

$$\sum_{n=0}^{\infty} \frac{(\frac{q}{a}, \frac{q}{b}, cz, \frac{cq}{z})_n (1-cq^{2n+1})(ab)^n}{(ac, bc, z, \frac{q}{z})_{n+1}} = \frac{(a, b, c, abc)_\infty}{(q, ab, ac, bc)_\infty} \sum_{n=-\infty}^{\infty} \frac{(\frac{q}{a}, \frac{q}{b}, \frac{q}{c})_n (abc)^n}{(a, b, c)_{n+1}(1-zq^n)}.$$

When $c = 1/a$ and $b \to 0$, the right-hand side is a theta function.

When $a = -1$ and $b, c \to 0$, the right-hand side is the universal mock theta function g_2.

When $a, b, c \to 0$, the right-hand side is the universal mock theta function g_3.

An expansion with symmetry in four variables.

Theorem

For $|xz| < 1$,

$$(abx, acx, adx, bcx, bdx, cdx)_\infty \sum_{n=0}^{\infty} \frac{(a, b, c, d, \frac{abcx}{qz}, \frac{abdx}{qz}, \frac{acdx}{qz}, \frac{bcdx}{qz})_n (abcdx^2)_{2n-2}}{(abx, acx, adx, bcx, bdx, cdx)_n (z, \frac{abcdx^2}{q^2 z})_{n+1} (\frac{abcdx}{q^2 z})_{2n+2}} p_4\Big(\frac{a}{z}, \frac{b}{z}, \frac{c}{z}, \frac{d}{z}, xz, zq^n\Big)(xz)^n$$

$$= \frac{(ax, bx, cx, dx, abcx, abdx, acdx, bcdx)_\infty}{(x, abcdx)_\infty} \sum_{n=0}^{\infty} \frac{(a, b, c, d)_n (abcdx, abcdx^2)_{n-2} (1 - abcdxq^{2n-2}) q^n}{(q, \frac{q}{x})_n (abcx, abdx, acdx, bcdx)_{n-1} (1 - zq^n)(1 - \frac{abcdxq^{n-2}}{z})}$$

$$+ \frac{(a, b, c, d, abcx^2, abdx^2, acdx^2, bcdx^2)_\infty}{(\frac{1}{x}, abcdx^3)_\infty} \sum_{n=0}^{\infty} \frac{(ax, bx, cx, dx)_n (abcdx^2, abcdx^3)_{n-2} (1 - abcdx^3 q^{2n-2}) q^n}{(q, qx)_n (abcx^2, abdx^2, acdx^2, bcdx^2)_{n-1} (1 - xzq^n)(1 - \frac{abcdx^2 q^{n-2}}{z})}.$$

It is necessary to form an analytic continuation of the left-hand side in order to see that it does indeed have four sets of poles.

The theorem has a couple of special cases in which the partial fraction expansion on the right-hand side can be written in a bilateral form.

Rewriting mock theta functions: an example.

The second-order mock theta function

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n(n+1)}}{1-q^{2n+1}}$$

can be expressed in the alternative form

$$B(q) = \sum_{n=0}^{\infty} \frac{q^{n(3n+2)}(-q)_{2n}(1+q^{4n+2}-q^{4n+3}+q^{6n+4})}{(q;q^2)_{n+1}^2 (q^{2n+3};q^2)_{n+1}}.$$

Open problems.

- Jackson's problem in the case $k \geq 5$: can the unilateral summation formula with symmetry in four variables be extended to symmetry in five or more variables?
- Do those formulae (if they exist) extend to bilateral summations via the same techniques which work for $k = 3$ and $k = 4$?
- Do the theta functions in those bilateral formulae (if they exist) have coefficients which vanish in arithmetic progressions?
- Do the unilateral summation formulae have finite analogues? For the case $k = 3$, the relevant formula is Jackson's terminating ${}_8\phi_7$ summation.
- Do the finite analogues (if they exist) extend to identities of elliptic hypergeometric series? In the case $k = 3$, the terminating ${}_8\phi_7$ summation is a special case of a ${}_{10}V_9$ summation formula due to Frenkel and Turaev.

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