Gupta, Ramanujan, Dyson, and Ehrhart: Formulas for Partition Functions, Congruences, Cranks and Polyhedral Geometry

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Abstract

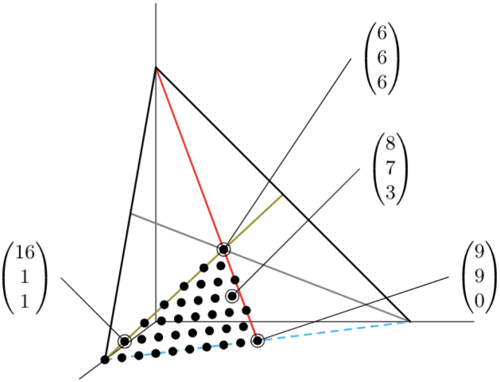
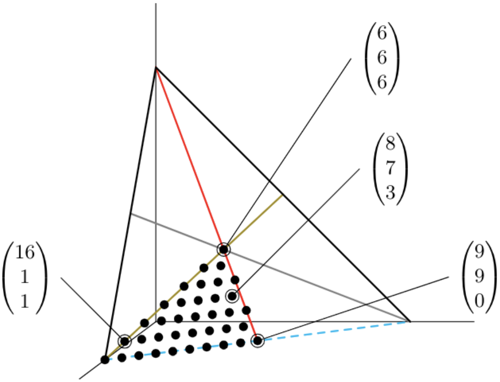


Figure 1: 37 integer lattice points in the set $P(18, \{1, 2, 3\})$.

We will revisit Gupta's result regarding properties of a formula for restricted partitions and generalize this. We will then use this result to prove an infinite family of congruences for a certain restricted partition function. We find and prove combinatorial witnesses, also known as cranks, for the congruences using polyhedral geometry.



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For any odd number $\ell \geq 3$ *and* $k \geq 0$*, we have*

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- *For* $\ell \equiv 1 \pmod 4$ *in Theorem 1, the crank* $4\lambda_2 - 3\lambda_3$*, witnesses the divisibility.*
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Partitions

- A partition of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$.
  The λ_i are called the parts of the partition.
  These are the partitions of the number 4:

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Generating Functions for Partitions

The generating function for the general partition function, $p(n)$, is

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i} = \frac{1}{(q;q)_\infty}.$$

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p(12k+1, 4) &= 1\binom{k+3}{3} + 35\binom{k+2}{3} + 35\binom{k+1}{3} + 1\binom{k}{3} = && 12k^3 + 18k^2 + 8k + 1\\
p(12k+2, 4) &= 2\binom{k+3}{3} + 39\binom{k+2}{3} + 30\binom{k+1}{3} + 1\binom{k}{3} = && 12k^3 + 21k^2 + 12k + 2\\
p(12k+3, 4) &= 3\binom{k+3}{3} + 42\binom{k+2}{3} + 27\binom{k+1}{3} = && 12k^3 + 24k^2 + 15k + 3\\
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Quasipolynomial for $p(n, 4)$

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\begin{aligned}
p(12k,4) &= 1\tbinom{k+3}{3} + 30\tbinom{k+2}{3} + 39\tbinom{k+1}{3} + 2\tbinom{k}{3} &= 12k^3 + 15k^2 + 6k + 1 \\
p(12k+1,4) &= 1\tbinom{k+3}{3} + 35\tbinom{k+2}{3} + 35\tbinom{k+1}{3} + 1\tbinom{k}{3} &= 12k^3 + 18k^2 + 8k + 1 \\
p(12k+2,4) &= 2\tbinom{k+3}{3} + 39\tbinom{k+2}{3} + 30\tbinom{k+1}{3} + 1\tbinom{k}{3} &= 12k^3 + 21k^2 + 12k + 2 \\
p(12k+3,4) &= 3\tbinom{k+3}{3} + 42\tbinom{k+2}{3} + 27\tbinom{k+1}{3} &= 12k^3 + 24k^2 + 15k + 3 \\
p(12k+4,4) &= 5\tbinom{k+3}{3} + 44\tbinom{k+2}{3} + 23\tbinom{k+1}{3} &= 12k^3 + 27k^2 + 20k + 5 \\
p(12k+5,4) &= 6\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 18\tbinom{k+1}{3} &= 12k^3 + 30k^2 + 24k + 6 \\
p(12k+6,4) &= 9\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 15\tbinom{k+1}{3} &= 12k^3 + 33k^2 + 30k + 9 \\
p(12k+7,4) &= 11\tbinom{k+3}{3} + 50\tbinom{k+2}{3} + 11\tbinom{k+1}{3} &= 12k^3 + 36k^2 + 35k + 11 \\
p(12k+8,4) &= 15\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 9\tbinom{k+1}{3} &= 12k^3 + 39k^2 + 42k + 15 \\
p(12k+9,4) &= 18\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 6\tbinom{k+1}{3} &= 12k^3 + 42k^2 + 48k + 18 \\
p(12k+10,4) &= 23\tbinom{k+3}{3} + 44\tbinom{k+2}{3} + 5\tbinom{k+1}{3} &= 12k^3 + 45k^2 + 56k + 23 \\
p(12k+11,4) &= 27\tbinom{k+3}{3} + 42\tbinom{k+2}{3} + 3\tbinom{k+1}{3} &= 12k^3 + 48k^2 + 63k + 27
\end{aligned}
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p(12k+2,4) &= 2\tbinom{k+3}{3}+39\tbinom{k+2}{3}+30\tbinom{k+1}{3}+1\tbinom{k}{3} &= 12k^3+21k^2+12k+2\\
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Quasipolynomials

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Definition 3

A function $f(n)$ is a quasipolynomial if there exists polynomials $f_0(n)$, $f_1(n)$,..., $f_{d-1}(n)$, called constituents, such that for all $n \in \mathbb{Z}$

$$f(n) = \begin{cases} \mathrm{f}_0(n) & \text{if } n \equiv 0 \mod d \\ \mathrm{f}_1(n) & \text{if } n \equiv 1 \mod d \\ \vdots & \\ \mathrm{f}_{d-1}(n) & \text{if } n \equiv d-1 \mod d \end{cases}$$

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p(12k,4) &= 1\tbinom{k+3}{3} + 30\tbinom{k+2}{3} + 39\tbinom{k+1}{3} + 2\tbinom{k}{3} &&= 12k^3 + 15k^2 + 6k + 1\\
p(12k+1,4) &= 1\tbinom{k+3}{3} + 35\tbinom{k+2}{3} + 35\tbinom{k+1}{3} + 1\tbinom{k}{3} &&= 12k^3 + 18k^2 + 8k + 1\\
p(12k+2,4) &= 2\tbinom{k+3}{3} + 39\tbinom{k+2}{3} + 30\tbinom{k+1}{3} + 1\tbinom{k}{3} &&= 12k^3 + 21k^2 + 12k + 2\\
p(12k+3,4) &= 3\tbinom{k+3}{3} + 42\tbinom{k+2}{3} + 27\tbinom{k+1}{3} &&= 12k^3 + 24k^2 + 15k + 3\\
p(12k+4,4) &= 5\tbinom{k+3}{3} + 44\tbinom{k+2}{3} + 23\tbinom{k+1}{3} &&= 12k^3 + 27k^2 + 20k + 5\\
p(12k+5,4) &= 6\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 18\tbinom{k+1}{3} &&= 12k^3 + 30k^2 + 24k + 6\\
p(12k+6,4) &= 9\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 15\tbinom{k+1}{3} &&= 12k^3 + 33k^2 + 30k + 9\\
p(12k+7,4) &= 11\tbinom{k+3}{3} + 50\tbinom{k+2}{3} + 11\tbinom{k+1}{3} &&= 12k^3 + 36k^2 + 35k + 11\\
p(12k+8,4) &= 15\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 9\tbinom{k+1}{3} &&= 12k^3 + 39k^2 + 42k + 15\\
p(12k+9,4) &= 18\tbinom{k+3}{3} + 48\tbinom{k+2}{3} + 6\tbinom{k+1}{3} &&= 12k^3 + 42k^2 + 48k + 18\\
p(12k+10,4) &= 23\tbinom{k+3}{3} + 44\tbinom{k+2}{3} + 5\tbinom{k+1}{3} &&= 12k^3 + 45k^2 + 56k + 23\\
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Quasipolynomial for $p(n,4)$

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The polynomial $E_S(q)$ is defined to be

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Theorem 5 (Gupta, 1975)

For the set A and $0 \leq r < lcm(A)$,

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Since there are a total of $\frac{\operatorname{lcm}(S)}{s_i} \times \frac{\operatorname{lcm}(S)}{s_j}$ terms, including multiplicity, each term has a multiplicity of exactly $\frac{\operatorname{lcm}(S)}{\operatorname{lcm}(s_i,s_j)}$.

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Since there are a total of $\frac{\text{lcm}(S)}{s_i} \times \frac{\text{lcm}(S)}{s_j}$ terms, including multiplicity, each term has a multiplicity of exactly $\frac{\text{lcm}(S)}{\text{lcm}(s_i,s_j)}$.

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Ramanujan

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$$\begin{aligned} p(5k+4) &\equiv 0 \pmod{5} \\ p(7k+5) &\equiv 0 \pmod{7} \\ p(11k+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{6}$$

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An Infinite Family of Partition Congruences

Theorem 8 (R.)

Let $S = \{a, b, c\}$ be a set of three relatively prime numbers, with one of them being an even integer. For $j \in \mathbb{N}$, we define the set $S_j = \{ja, jb, jc\}$. Then,

$$p\left(jabck + \frac{2jabc - ja - jb - jc}{2}, S_j\right) \equiv 0 \left(\text{mod}\, \frac{abc}{2}\right). \tag{7}$$

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The proof of this theorem can be broken up into four steps.

1. Show that $E_{S_j}(q)$ is a reciprocal polynomial.
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Ranks and Cranks

In 1919, Ramanujan proved the following congruences:

$$\begin{aligned} p(5k+4) &\equiv 0 \pmod{5} \\ p(7k+5) &\equiv 0 \pmod{7} \\ p(11k+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{9}$$

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p(5), Rank, Rank modulo 7

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Table 1: $p(5) = 7$

$\lambda \vdash 5$	rank(λ)	rank(λ) (mod 7)
1+1+1+1+1	-4	3
2+1+1+1	-2	5
2+1+1+1	-1	6
3+1+1	0	0
3+2	1	1
4+1	2	2
5	4	4

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Divisibility and Cranks

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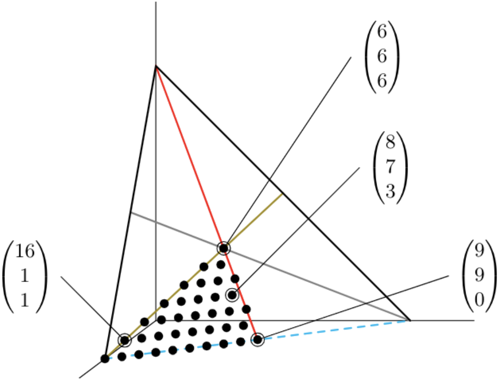
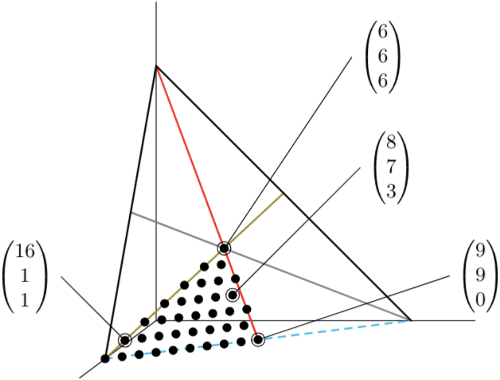


Figure 2: 37 integer lattice points in the set $P(18, \{1, 2, 3\})$.



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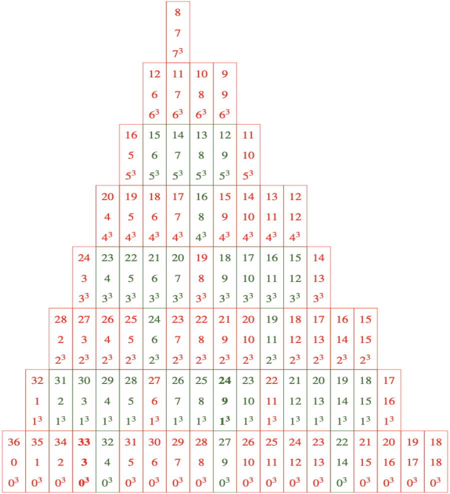

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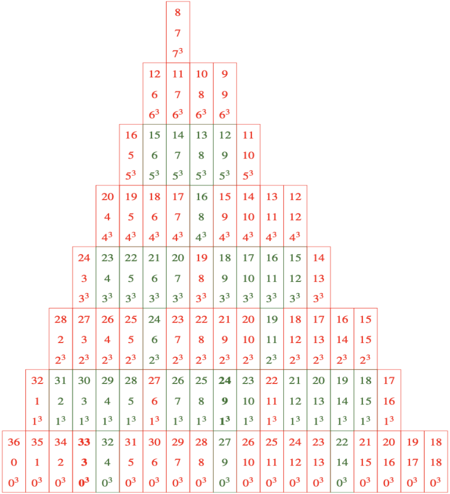

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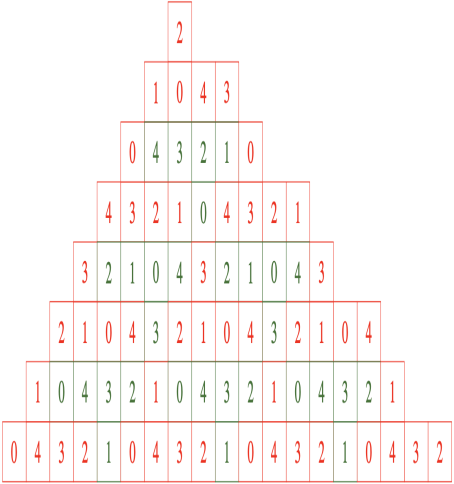

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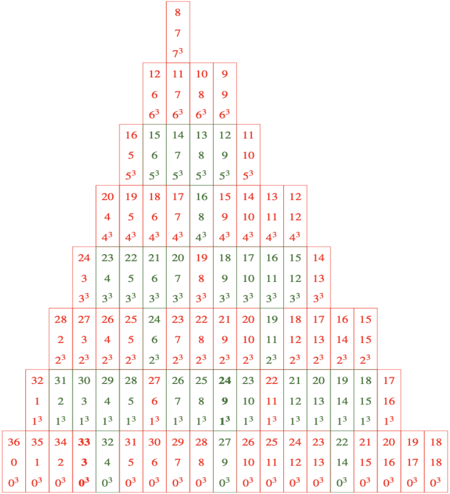
Polyhedral Geometry



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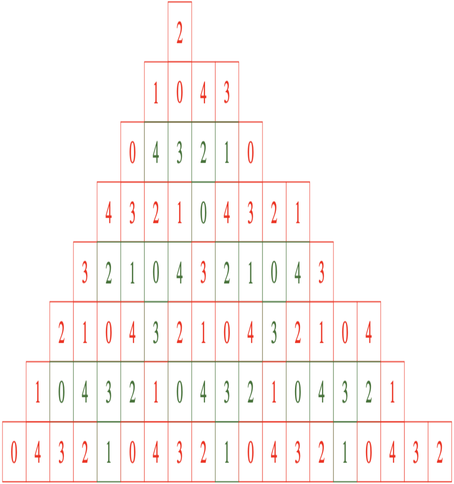

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Mathematica Code to Search for Cranks for Theorem 2

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ell := 3;
numberbeingpartitioned = (3 * ell - 3) / 2;  (* Theorem 2 with k=0 *)
z = E ^ (2 * Pi * I / ell);
For[m = 0, m < ell, m ++,
  For[j = 0, j < ell, j ++,
   For[i = 0, i < ell, i ++,
    If[0 == (*generating function weighted by crank = i*(# of 1's) + j*(# of 2's) + m*(# of ell's) *)
      FullSimplify[
       Coefficient[Expand[Sum[(z ^ i * q) ^ n, {n, 0, 2 * ell ^ 2 - 1}] * Sum[(z ^ j * q ^ 2) ^ n, {n, 0, ell ^ 2 - 1}] *
          Sum[(z ^ m * q ^ ell) ^ n, {n, 0, 2 * ell - 1}]], q, numberbeingpartitioned]], Print["(", i, j, m, ")"]]]]]
(201)
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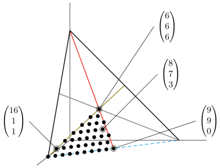

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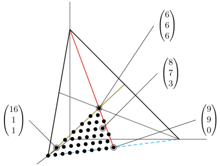

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2. We then show that we retain a complete set of residues modulo ℓ with any translation of these two slices.

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Theorem 2 (R.)

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Theorem 2 Proof for $\ell \equiv 1 \pmod{4}$ for $\lambda_3 = 0$

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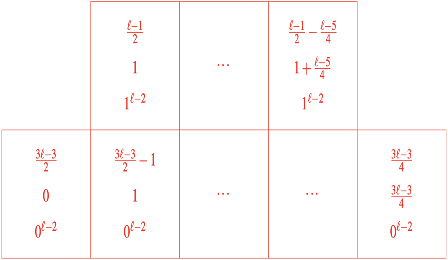
	$\frac{\ell-1}{2}$ 1 $1^{\ell-2}$	$\cdots$	$\frac{\ell-1}{2} - \frac{\ell-5}{4}$ $1 + \frac{\ell-5}{4}$ $1^{\ell-2}$	
$\frac{3\ell-3}{2}$ 0 $0^{\ell-2}$	$\frac{3\ell-3}{2} - 1$ 1 $0^{\ell-2}$	$\cdots$	$\cdots$	$\frac{3\ell-3}{4}$ $\frac{3\ell-3}{4}$ $0^{\ell-2}$

Figure 6: This is a slice of the fundamental parallelepiped with $k = 0$ for $p\left(2\ell k + \frac{3\ell-3}{2}, \{1, 2, \ell\}\right) \equiv 0 \pmod{\ell}$ at height $\frac{3\ell-3}{2}$. Note: The slices represent the conjugates of the partitions of n into part sizes $1, 2, \ell$. The crank is $4\lambda_2 - 3\lambda_3$.



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1. There are $\frac{3\ell+1}{4}$ vectors in the $\lambda_3^{\ell-2} = 0^{\ell-2}$ row.

	$\frac{\ell-1}{2}$ 1 $1^{\ell-2}$	$\cdots$	$\frac{\ell-1}{2}-\frac{\ell-5}{4}$ $1+\frac{\ell-5}{4}$ $1^{\ell-2}$	
$\frac{3\ell-3}{2}$ 0 $0^{\ell-2}$	$\frac{3\ell-3}{2}-1$ 1 $0^{\ell-2}$	$\cdots$	$\cdots$	$\frac{3\ell-3}{4}$ $\frac{3\ell-3}{4}$ $0^{\ell-2}$

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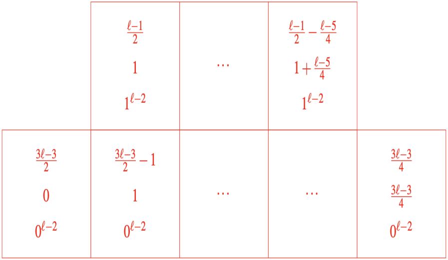

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3. The crank value increases by $4 \pmod{\ell}$ as we move to the right.

	$\frac{\ell-1}{2}$ 1 $1^{\ell-2}$	$\cdots$	$\frac{\ell-1}{2} - \frac{\ell-5}{4}$ $1+\frac{\ell-5}{4}$ $1^{\ell-2}$	
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3. The crank value increases by $4 \pmod{\ell}$ as we move to the right.
4. The first vector here has $c(\lambda) = 0$.

Theorem 2 Proof for $\ell \equiv 1 \pmod 4$ for $\lambda_3 = 0$

	$\frac{\ell-1}{2}$ 1 $1^{\ell-2}$	$\cdots$	$\frac{\ell-1}{2}-\frac{\ell-5}{4}$ $1+\frac{\ell-5}{4}$ $1^{\ell-2}$	
$\frac{3\ell-3}{2}$ 0 $0^{\ell-2}$	$\frac{3\ell-3}{2}-1$ 1 $0^{\ell-2}$	$\cdots$	$\cdots$	$\frac{3\ell-3}{4}$ $\frac{3\ell-3}{4}$ $0^{\ell-2}$

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3. The crank value increases by 4 $(\text{mod } \ell)$ as we move to the right.
4. The first vector here has $c(\lambda) = 0$.
5. The extra vector $\begin{pmatrix} \frac{3\ell-7}{4} \\ \frac{3\ell+1}{4} \\ 0^{\ell-2} \end{pmatrix}$ would have had crank value of $c(\lambda) = 1$.

Theorem 2 Proof for $\ell \equiv 1 \pmod 4$ for $\lambda_3 = 0$

	$\frac{\ell-1}{2}$ 1 $1^{\ell-2}$	$\cdots$	$\frac{\ell-1}{2}-\frac{\ell-5}{4}$ $1+\frac{\ell-5}{4}$ $1^{\ell-2}$	
$\frac{3\ell-3}{2}$ 0 $0^{\ell-2}$	$\frac{3\ell-3}{2}-1$ 1 $0^{\ell-2}$	$\cdots$	$\cdots$	$\frac{3\ell-3}{4}$ $\frac{3\ell-3}{4}$ $0^{\ell-2}$

Figure 6: This is a slice of the fundamental parallelepiped with $k = 0$ for $p\left(2\ell k + \frac{3\ell-3}{2}, \{1,2,\ell\}\right) \equiv 0 \pmod{\ell}$ at height $\frac{3\ell-3}{2}$. Note: The slices represent the conjugates of the partitions of n into part sizes $1, 2, \ell$. The crank is $4\lambda_2 - 3\lambda_3$.

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3. The crank value increases by $4 \pmod{\ell}$ as we move to the right.
4. The first vector here has $c(\lambda) = 0$.
5. The extra vector $\begin{pmatrix} \frac{3\ell-7}{4} \\ \frac{3\ell+1}{4} \\ 0^{\ell-2} \end{pmatrix}$ would have had crank value of $c(\lambda) = 1$.
6. The last crank value in this row is $4\lambda_2 - 3\lambda_3 \equiv -3 \pmod{\ell}$.

Theorem 2 Proof for $\ell \equiv 1 \pmod 4$ for $\lambda_3 = 0$

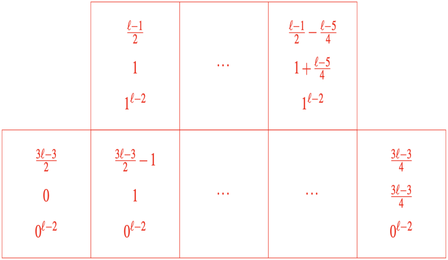

Figure 6: This is a slice of the fundamental parallelepiped with $k = 0$ for $p\left(2\ell k + \frac{3\ell-3}{2}, \{1, 2, \ell\}\right) \equiv 0 \pmod \ell$ at height $\frac{3\ell-3}{2}$. Note: The slices represent the conjugates of the partitions of n into part sizes $1, 2, \ell$. The crank is $4\lambda_2 - 3\lambda_3$.

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6. The last crank value in this row is $4\lambda_2 - 3\lambda_3 \equiv -3 \pmod \ell$.
7. Because $\gcd(4, \ell) = 1$,the crank value of each vector must be distinct.

Theorem 2 Proof for $\ell \equiv 1 \pmod 4$ for $\lambda_3 = 0$

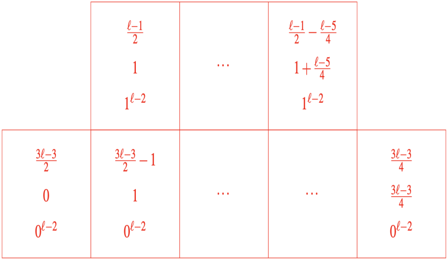

Figure 6: This is a slice of the fundamental parallelepiped with $k = 0$ for $p\left(2\ell k + \frac{3\ell-3}{2}, \{1, 2, \ell\}\right) \equiv 0 \pmod{\ell}$ at height $\frac{3\ell-3}{2}$. Note: The slices represent the conjugates of the partitions of n into part sizes $1, 2, \ell$. The crank is $4\lambda_2 - 3\lambda_3$.

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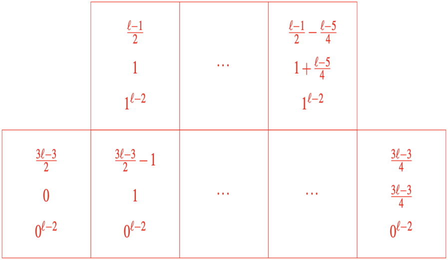
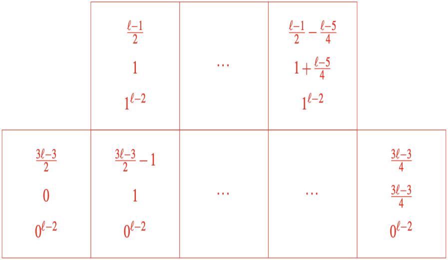


Figure 7: This is a slice of the fundamental parallelepiped with $k = 0$ for $p\left(2\ell k + \frac{3\ell-3}{2}, \{1, 2, \ell\}\right) \equiv 0 \pmod{\ell}$ at height $\frac{3\ell-3}{2}$.
The crank is $4\lambda_2 - 3\lambda_3$.



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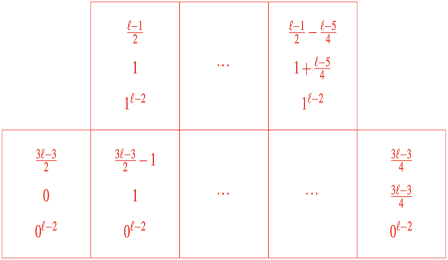

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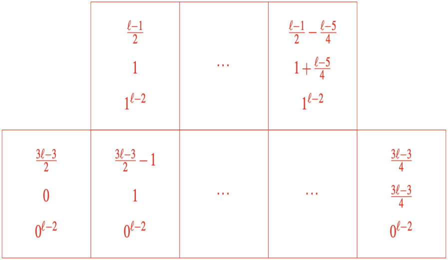
Theorem 2 Proof for $\ell \equiv 1 \pmod 4$ for $\lambda_3 = 1$



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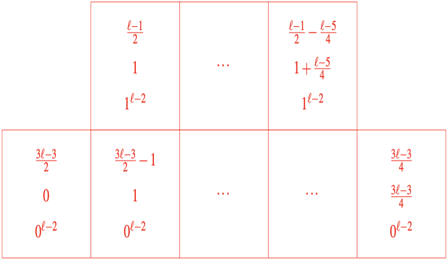
Theorem 2 Proof for $\ell \equiv 1 \pmod 4$ for $\lambda_3 = 1$



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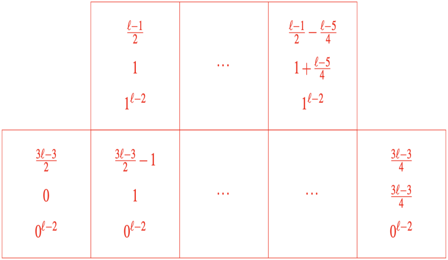
5. Together, the crank values of both rows make a complete set of residues modulo ℓ.



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5. The vector λ gives the coordinates of the new partition after the translation.

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Symmetry of the Fundamental Parallelepiped

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Figure 8: This is a slice of the fundamental parallelepiped with $k=1$ for $p\left(2\ell k+\frac{3\ell-3}{2},\{1,2,\ell\}\right)\equiv 0 \pmod{\ell}$ at height $2\ell k+\frac{3\ell-3}{2}$.

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Theorem 2 Example $p(36, \{1, 2, 5\})$

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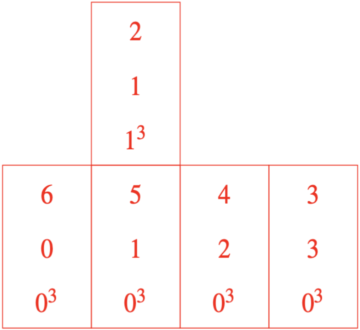

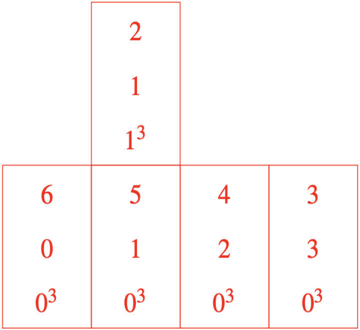

Theorem 1 (Kronholm, R.)

For any odd number $\ell \geq 3$ and $k \geq 0$, we have

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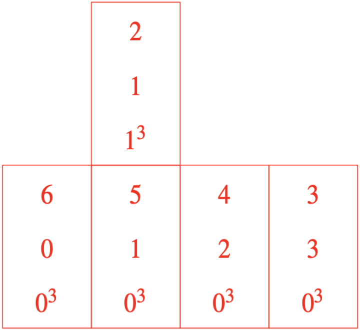

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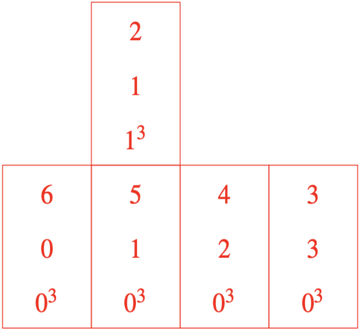

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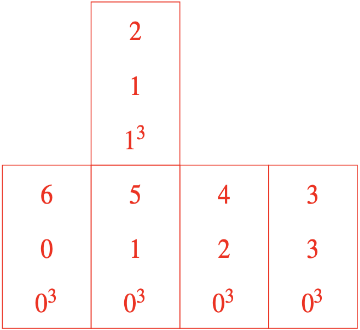

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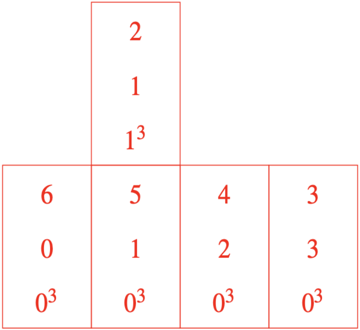

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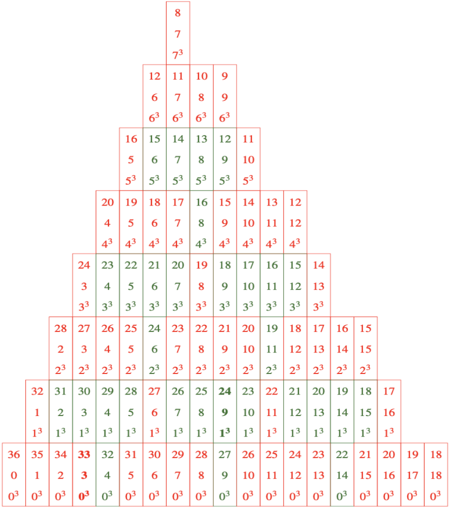
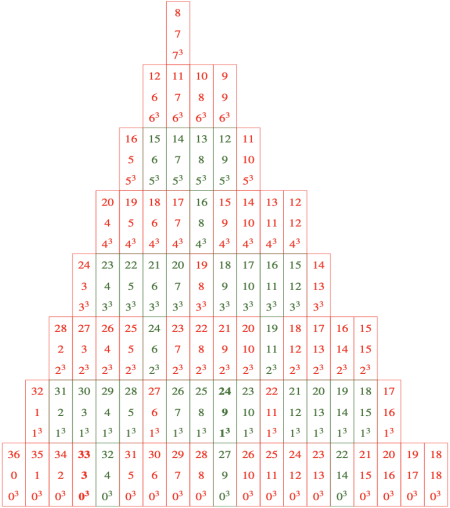


Figure 9: Slice of the partition cone at height 36.



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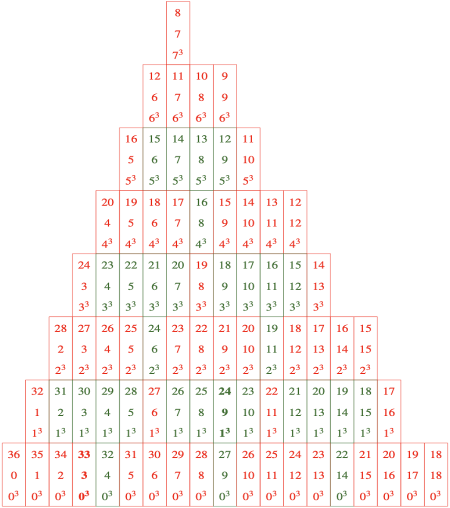

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$$\begin{pmatrix} 10 & 5 & 2 \\ 0 & 5 & 2 \\ 0^3 & 0^3 & 2^3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0^3 \end{pmatrix} = \begin{pmatrix} 33 \\ 3 \\ 0^3 \end{pmatrix}$$

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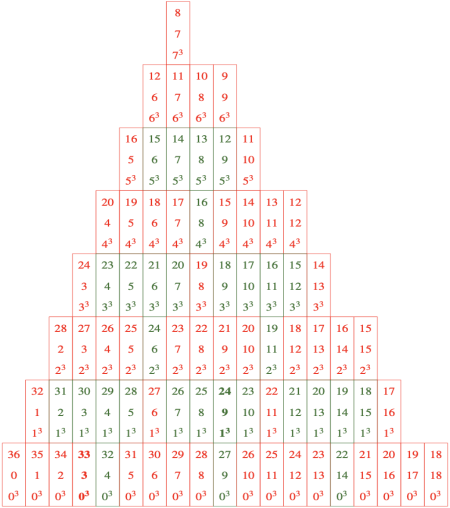

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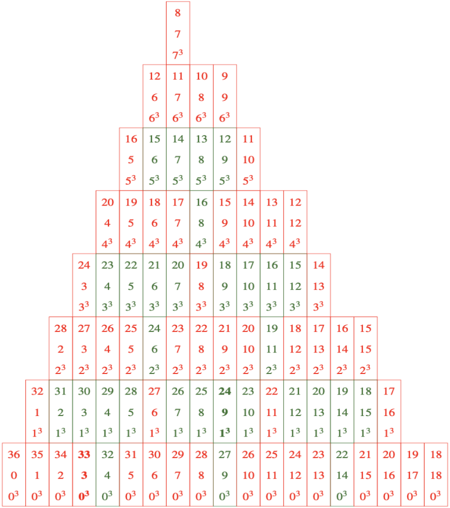
$$\begin{pmatrix} 10 & 5 & 2 \\ 0 & 5 & 2 \\ 0^3 & 0^3 & 2^3 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \\ 0^3 \end{pmatrix} = \begin{pmatrix} 33 \\ 3 \\ 0^3 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 5 & 2 \\ 0 & 5 & 2 \\ 0^3 & 0^3 & 2^3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 9 \\ 4 \\ 1^3 \end{pmatrix} = \begin{pmatrix} 24 \\ 9 \\ 1^3 \end{pmatrix}$$



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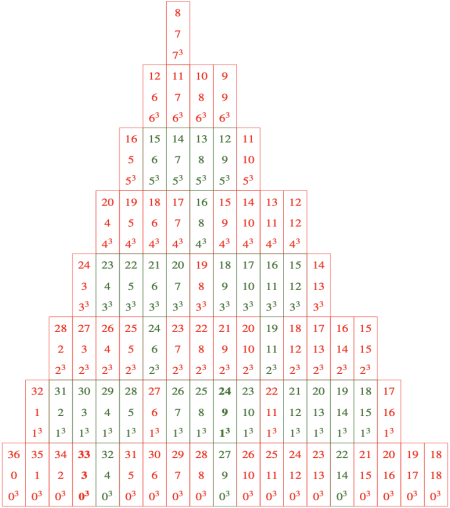
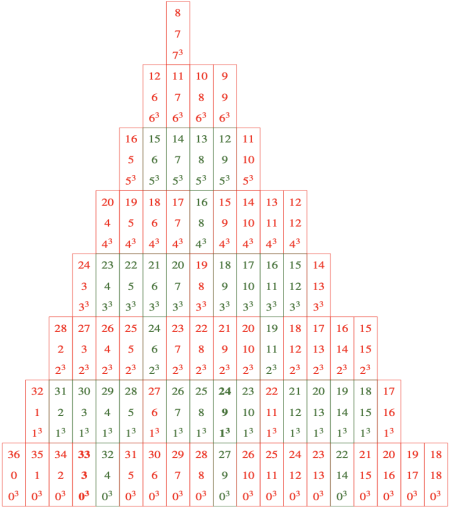


Figure 10: Slice of the partition cone at height 36.



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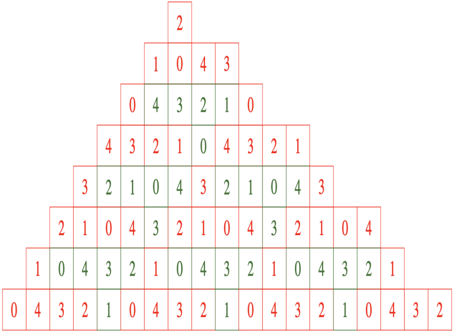

Figure 11: Crank values modulo 5 of the slice of the partition at height 36.

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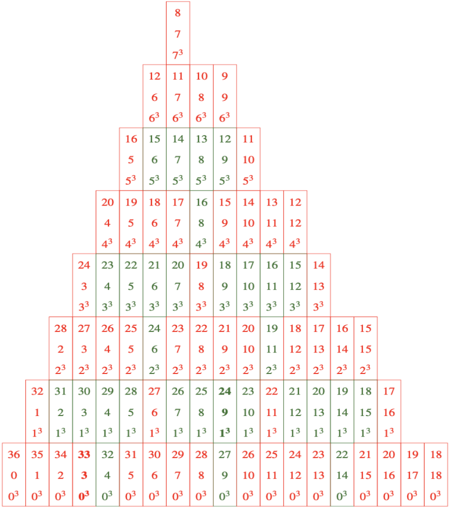

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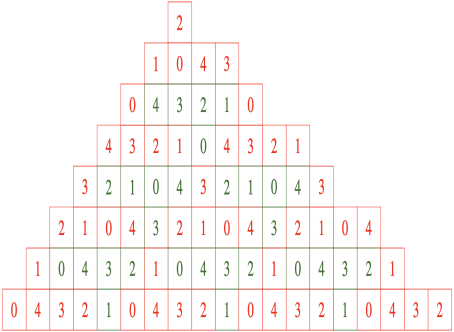

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Combinatorial Witness- Big Picture

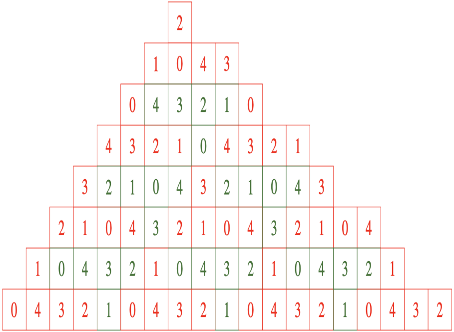
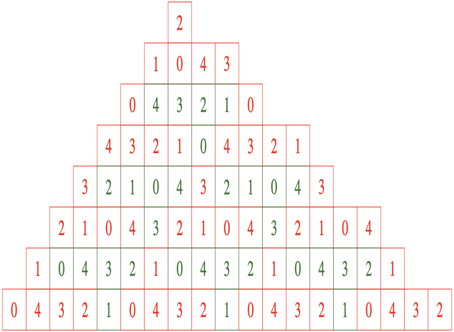


Figure 12: Crank values modulo 5 of the slice of the partition cone at height 36.



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1. Each slice of F_ℓ, and their translations, represent an ℓ-cycle of partitions.

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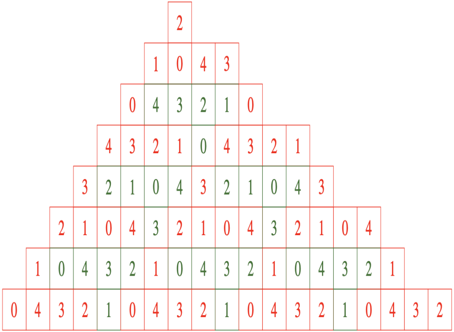

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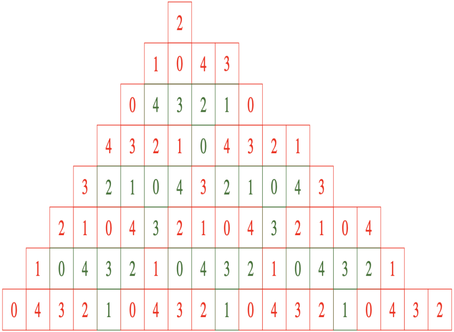
Combinatorial Witness- Big Picture



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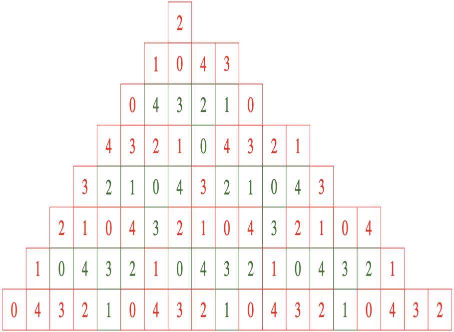

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2. λ' and λ are contained in the same translated slice, $V_\ell \times \tau + H_n$ for specified values of n.
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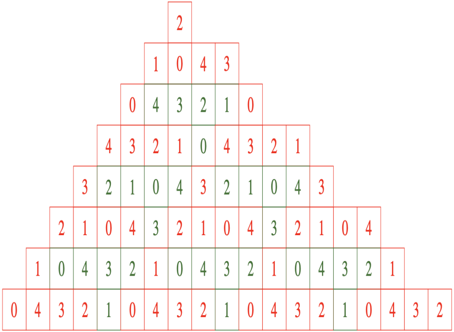

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Further Research Goals

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- To obtain similar results for the full generalization

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  theorem.
- To extend that theorem to include more than three parts in the set.
  - Equal Values of Certain Partition Functions via Diophantine Equations. SZ. Tengley, M. Ulas.

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References

- Gupta, Hansraj. "A TECHNIQUE IN PARTITIONS." Publikacije Elektrotehničkog Fakulteta. Serija Matematika i Fizika, no. 498/541, 1975, pp. 73–76.
- Breuer, F., Eichhorn, D., Kronholm, B. (2017). Polyhedral Geometry, supercranks, and combinatorial witnesses of congruences for partitions into three parts. European Journal of Combinatorics.
- Breuer, F. An Invitation to Ehrhart Theory: Polyhedral Geometry and its Applications in Enumerative Combinatorics
- Andrews, G. E., Eriksson, K. (2004). Integer Partitions. Cambridge: Cambridge University Press.