Forgotten conjectures of Andrews for Nahm-type sums

Partitions and q-Series Seminar

Joshua Males, University of Manitoba. Joint work with Amanda Folsom, Larry Rolen, and Matthias Storzer.

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The University of Manitoba campuses are located on original lands of Anishinaabeg, Cree, Oji-Cree, Dakota and Dene peoples, and on the homeland of the Métis Nation. We respect the Treaties that were made on these territories, we acknowledge the harms and mistakes of the past, and we dedicate ourselves to move forward in partnership with Indigenous communities in a spirit of reconciliation and collaboration.

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40	37,338
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Want a method to either approximate (well) or count exactly the number of partitions.

A wild modular form appears!

It turns out that by setting $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$ we get

$$\sum_{n\geq 1} p(n)q^n = \frac{q^{\frac{1}{24}}}{\eta(\tau)} = \frac{1}{\prod_{n\geq 1}(1-q^n)}$$

where $\eta(\tau)$ is the Dedekind eta-function, a prototypical example of a modular form of weight $\frac{1}{2}$.

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Theorem (Hardy–Ramanujan)
As $n \to \infty$ we have

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}$$

The Circle Method

The coefficients $c(n)$ of a Fourier expansion $C(q) = \sum_{n\geq 0} c(n)q^n$ can be recovered as

$$c(n) = \frac{1}{2\pi i}\int_C C(q)q^{-n}\frac{dq}{q}$$

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In many applications, the pole at $q = 1$ gives the largest growth and we call it the dominant pole. No particular need for this to be at 1.

Rademacher's exact formula

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Theorem
For all n we have

$$p(n) = \frac{\pi}{2^{\frac{5}{4}} 3^{\frac{3}{4}} N^{\frac{3}{4}}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}} \left(\frac{\pi}{k} \sqrt{\frac{2N}{3}} \right),$$

where I_ν is the usual I-Bessel function and

$$A_k(n) := \sum_{\substack{0 \leq h < k \\ \gcd(h,k)=1}} e^{\pi i s(h,k) - \frac{2\pi i n h}{k}}$$

is a Kloosterman sum with $s(h, k)$ the usual Dedekind sum.

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Wright's Circle Method is a trade-off. We throw all poles that do not give the largest growth (i.e. all poles that are not dominant) into an error term along minor arcs.

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Works really well when there are finitely many dominant main terms that always beat error terms for large n.

Works really well for modular objects and objects arising from infinite products. What about other objects?

Nahm sums

A Nahm sum is a sum of the form

$$\sum_{n_1, n_2, \ldots, n_r \geq 0} \frac{q^{\frac{1}{2} n^T A n}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_r}}$$

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They appear in many places throughout mathematics. For example in conformal field theory, algebraic K-theory, and of course number theory. Examples include many of Ramanujan's mock theta functions.

$\sigma(q)$

One of the most famous examples of a Nahm-type sum is

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n} =: \sum_{n=0}^{\infty} S(n) q^n,$$

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The coefficients $S(n)$ of $\sigma(q)$ count the difference between the number of partitions into distinct parts with even and odd rank. Andrews conjectured

Conjecture (Conjecture 1)
$\limsup |S(n)| = +\infty$.

Conjecture (Conjecture 2)
$S(n) = 0$ *for infinitely many n.*

The sequence $S(n)$

The sequence $S(n)$ are relatively integers, beginning with
$1, 1, -1, 2, -2, 1, 0, 1, -2, 0, 2, 0, -1, -2, 2, 1, 0, -2, 2, -2, \ldots$

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While the $\limsup$ is growing, it is doing so at a pretty slow speed. For example, $S(45) = 4$, and $S(1609) = 6$.

Compare this with the exponential growth of partitions; $S(100) = 1$ while $p(100) = 190, 569, 292$.

$\sigma(q)$

By showing a deep connection between $\sigma(q)$ and its so-called companion $\sigma^*(q)$ along with the arithmetic of $\mathbb{Q}(\sqrt{6})$, extending beyond their combinatorial interpretations, Andrews-Dyson-Hickerson succeeded in proving Andrews' two conjectures on $\sigma(q)$.

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For example, we now know that $S(n)$ may also be defined by a Hecke L-function, a certain sum over ideals in $\mathbb{Z}[\sqrt{6}]$. The coefficients were also very important in Cohen/Zwegers' construction of an important new class of objects - mock Maass waveforms.

A "forgotten Nahm-type sum"

In the same paper as σ appears, we see the function

$$v_1(q) := \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(-q^2;q^2)_n} \; =: \; \sum_{n\geq 0} V_1(n) q^n,$$

(alongside similar functions v_2, v_3, v_4).

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The function $v_1(q)$ admits a similar combinatorial interpretation to $\sigma(q)$: its coefficients $V_1(n)$ count the difference between the number of odd-even partitions of n with rank $\equiv 0 \pmod 4$ and $\equiv 2 \pmod 4$.

Four conjectures of Andrews

Conjecture
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Conjecture
For $n \geq 5$ *there is an infinite sequence* $N_5 = 293, N_6 = 410, N_7 = 545, N_8 = 702, \ldots, N_n \geq 10n^2, \ldots$ *such that* $V_1(N_n), V_1(N_n+1), V_1(N_n+2)$ *all have the same sign.*

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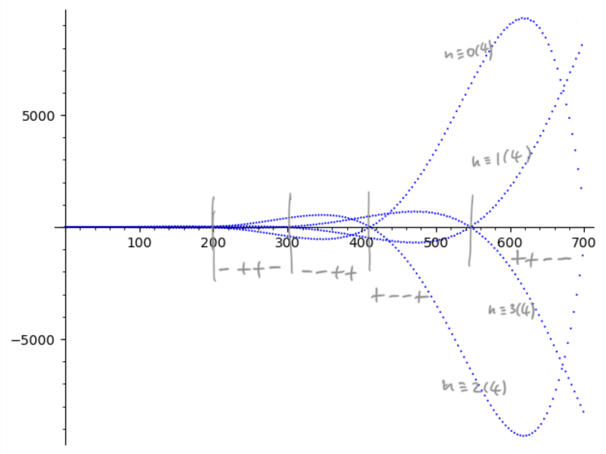
Conjecture
The numbers $|V_1(N_n)|$, $|V_1(N_n+1)|$, $|V_1(N_n+2)|$ *contain a local minimum of the sequence* $|V_1(j)|$.


5000
−5000
100
200
300
400
500
600
700
$n\equiv 0(4)$
$n\equiv 1(4)$
$n\equiv 3(4)$
$n\equiv 2(4)$
$-++-$
$--++$
$+--+$
$++--$

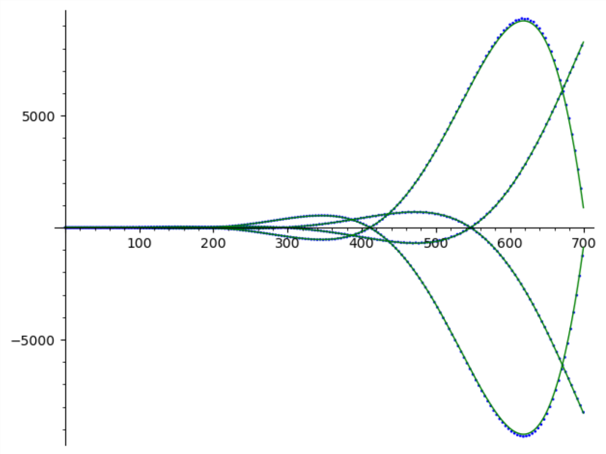


Figure 1: Our conjectured approximation

Projected theorems

We believe we can prove the following.

Theorem (Folsom, M., Rolen, Storzer)
The first two conjectures of Andrews are true.

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Theorem (Folsom, M., Rolen, Storzer)
The first two conjectures of Andrews are true.

We believe a slight modification of the first conjecture is needed, to say instead that "as $n \to \infty$, almost all values of n are such that $|V_1(n)| \to \infty$".

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Our explanation of the third conjecture relies on irrationality properties of $\zeta_{\mathbb{Q}(\sqrt{-3})}(2)$. With an assumption on this, we are able to make progress on the third conjecture.

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This means our usual techniques will not work, and we need new approaches. As with all Cirlce Method approaches, we want to know the behaviour of $v_1(q)$ toward roots of unity.

Asymptotics

Lemma
Let $\zeta_N := e^{2\pi i/N}$. *For any root of unity* ζ_m^ℓ *with* $\gcd(\ell, m) = 1$ *and* $4 \nmid m$, *we have that*

$$v_1(\zeta_m^\ell) = 2 \sum_{s=0}^{m-1} \frac{\zeta_{2m}^{\ell s(s+1)}}{(-\zeta_m^{2\ell}; \zeta_m^{2\ell})_s}.$$

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This is just some number, so we only need to worry about fourth roots of unity.

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If $4|n$, write $m = n/4$. Then as $z \to 0$, on a ray in the right half-plane with $0 \neq \arg z \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$v_1(\zeta_n e^{-z}) = \begin{cases} e^{\frac{V}{zm^2}} \left(\dfrac{z}{2\pi i}\right)^{-1/2} (\gamma_1^{(\alpha)} + O(z)) & \text{if } \arg(z) > 0 \\ e^{\frac{-V}{zm^2}} \left(\dfrac{-z}{2\pi i}\right)^{-1/2} (\gamma_2^{(\alpha)} + O(z)) & \text{if } \arg(z) < 0 \end{cases}$$

where, with the Bloch-Wigner dilogarithm D,

$$V = \mathrm{D}(e(1/6))i/8 = 0.1268877\ldots i,$$

and $\gamma_1^{(\alpha)}, \gamma_2^{(\alpha)} \in \mathbb{C}$.

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and $\gamma_1^{(\alpha)}, \gamma_2^{(\alpha)} \in \mathbb{C}$.

Note that $m = 1$ (so $n = 4$) is meant to give the largest growth, i.e. toward $\pm i$ our function grows the quickest.

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After a lot more hunting, results of Milnor give

$$|V| = \frac{9\sqrt{3}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)}{2\pi^2},$$

where ζ_K is the usual Dedekind zeta function associated with the field K.

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- Taking care of various branch cuts and poles/residues, make some changes of variable to massage the integral into a nicer form.
- Split into three integral pieces, each of which should have different properties.
- Use a precise version of the stationary phase method (saddle-point method) to determine the asymptotic behaviour of the function toward fourth roots of unity

Circle Method time

Now we want to apply Wright's Circle Method. We have two options;

- Major arcs around $\pm i$ and minor arcs everywhere else
- Major arcs around all $4m$-th roots of unity, minor arcs elsewhere.

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This justifies placing major arcs around $4m$-th roots of unity, and minor arcs elsewhere. For now, just think about major arcs around $\pm i$.

Circle Method time

Write

$$V_1(n) = \frac{1}{2\pi i} \int_C \frac{v_1(q)}{q^n} \frac{dq}{q}.$$

Now let

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C - C_1 - C_2},$$

where C_1 is a major arc around i, C_2 is a major arc around $-i$, and everything else is a minor arc.

Circle Method time

Consider the term $M_1(n) := \frac{1}{2\pi i} \int_{C_1} \frac{v_1(q)}{q^{n+1}} dq$.

Choose the radius of the circle C to be $e^{-\lambda}$ with $\lambda := \sqrt{\frac{|V|}{n}}$. Then the arc C_1 is described by $ie^{-\lambda + i\theta}$ with $\theta \in (-\delta, \delta)$.

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Make the change of variable $q = ie^{-z}$ and parameterize where z runs from $\lambda + i\delta$ to $\lambda - i\delta$, to obtain

$$M_1(n) = -\frac{(-i)^n}{2\pi i} \int_{\lambda+i\delta}^{\lambda-i\delta} \frac{v_1\left(ie^{-z}\right)}{e^{-zn}} dz = \frac{(-i)^n}{2\pi i} \int_{\lambda-i\delta}^{\lambda+i\delta} \frac{v_1\left(ie^{-z}\right)}{e^{-zn}} dz.$$

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Letting $\delta = \lambda$ and making a change of variable, plugging in and rearranging (and ignoring some constants) should give us combinations of integrals of the shape

$$\int_{\sqrt{|V|}(1-i)}^{\sqrt{|V|}(1+i)} e^{\sqrt{n}\left(\frac{V}{z}+z\right)} z^{-\frac{1}{2}}\, dz.$$

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Looks more complicated! But now this integral is amenable to the saddle-point method again.

The output on the major arcs

Ignoring all the horrible details, doing this for $\pm i$ we should obtain that

$V_1(n) \sim$

$$\Big(\frac{(-i)^n\beta_1}{2\sqrt{\pi n}}e^{2\sqrt{nV}}+\frac{i^{n-1}\beta_1}{2\sqrt{\pi n}}e^{2\sqrt{-nV}}+\frac{(-i)^n\beta_2}{2\sqrt{n\pi}}e^{2\sqrt{-nV}}+\frac{i^{n+1}\beta_2}{2\sqrt{\pi n}}e^{2\sqrt{nV}}\Big)(1+O(n^{-\frac{1}{2}})).$$

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Ignoring all the horrible details, doing this for $\pm i$ we should obtain that

$V_1(n) \sim$

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Not particularly satisfying or useful yet, we need an error term from minor arcs on the right. Luckily, the error term is much easier. Just plug in an estimate of $v_1(q)$ near 8-th order roots of unity and crudely estimate to get $O\left(n^{-\frac{1}{2}}e^{\sqrt{\frac{n|V|}{2}}}\right)$.

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No! This asymptotic has oscillation. This becomes more clear if we tidy things up a bit to get

$$V_1(n) \sim \gamma \frac{e^{\sqrt{2|V|n}}}{\sqrt{\pi n}} (-1)^{\lfloor \frac{n}{2} \rfloor} \left(\cos(\sqrt{2|V|n}) + (-1)^{n+1} \sin(\sqrt{2|V|n}) \right)$$

for some particular $\gamma \in \mathbb{R}$.

The useful asymptotic

Collecting things together, we believe that we can prove

$$
\begin{aligned}
V_1(n) =& \frac{e^{\sqrt{2|V|n}}}{\sqrt{\pi n}}(-1)^{\lfloor \frac{n}{2} \rfloor}\gamma\left(\cos(\sqrt{2|V|n}) + (-1)^{n+1}\sin(\sqrt{2|V|n})\right)\left(1 + O(n^{-\frac{1}{2}})\right) \\
&+ O\left(n^{-\frac{1}{2}} e^{\sqrt{\frac{n|V|}{2}}}\right) \\
=& M(n) + E(n).
\end{aligned}
$$

Sign patterns

We have the following table of signs for $(-1)^{\lfloor \frac{n}{2} \rfloor}$:

$n \pmod 4$	$(-1)^{\lfloor \frac{n}{2} \rfloor}$
0	$+$
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With the observation that $\frac{e^{\sqrt{2|V|n}}}{\sqrt{\pi n}}$ is exponentially positive, our investigation boils down to the function

$$\cos(\sqrt{2|V|n}) + (-1)^{n+1} \sin(\sqrt{2|V|n})$$

for $n, n+1, n+2, n+3$.

Conjectures 1 and 2

Conjecture
As $n \to \infty$, almost all values of n are such that $|V_1(n)| \to \infty$

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Heuristically, when n gets large the values $\cos(\sqrt{2|V|(n+j)})$ (resp. $\sin(\sqrt{2|V|(n+j)})$) for $j \in \{0,1,2,3\}$ are close to each other. To see this, for $a \in \mathbb{R}$ consider

$$\lim_{x\to\infty} \frac{\cos(a\sqrt{x+1})}{\cos(a\sqrt{x})} = 1 = \lim_{x\to\infty} \frac{\sin(a\sqrt{x+1})}{\sin(a\sqrt{x})}.$$

Equidistribution

Label the roots of $\cos(x) + (-1)^{n+1} \sin(x)$ by ϑ_j modulo 2π for $j = 1, 2, 3, 4$. They occur at $\pi\left(\ell \pm \frac{1}{4}\right)$.

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Weyl's criterion states that a sequence s_n is equidistributed modulo 1 if and only if for all $h \in \mathbb{Z}$ with $h \neq 0$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i h s_j} = 0.$$

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To prove conjecture 2, it should be clear from the above that almost always $n, n+1, n+2, n+3$ have two plus signs and two minus signs, since the trig term is not small, the exponential dominates with a sign dictated by $(-1)^{\lfloor \frac{n}{2} \rfloor}$.

Andrews' third conjecture

Conjecture
For $n \geq 5$ *there is an infinite sequence*
$N_5 = 293, N_6 = 410, N_7 = 545, N_8 = 702, \ldots, N_n \geq 10n^2, \ldots$ *such that*
$V_1(N_n), V_1(N_n + 1), V_1(N_n + 2)$ *all have the same sign.*

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Seems "clear" that these must be points where $\cos \pm \sin$ is very close to 0, and so when $\sqrt{2|V|n}$ is close to $\pi\left(\ell \pm \frac{1}{4}\right)$.

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Solving directly, we want to choose infinitely many $n \in \mathbb{N}$ to be arbitrarily close to

$$\frac{\pi^2\left(m \pm \frac{1}{4}\right)^2}{2|V|}, \qquad m \in \mathbb{Z}.$$

A problem!

If we assume that $\frac{\pi^2}{|V|} = \frac{h}{k} \in \mathbb{Q}$ we see that one would need to choose infinitely many positive integers n that are arbitrarily close to the points

$$\frac{h}{2k}\left(m \pm \frac{1}{4}\right)^2,$$

which cannot happen as the right-hand side has a fixed denominator $32k$.

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But the sequence N_j appears to be infinite.

Conjecture
The value $\frac{\pi^2}{|V|} = \frac{2\pi^3}{9\sqrt{3}\zeta_{\mathbb{Q}(\sqrt{-3})}(2)}$ *is irrational.*

Conditional (partial) result

Assume the previous conjecture. We want to determine whether there are infinitely many choices of positive integers m, n such that

$$\frac{2n}{\left(m \pm \frac{1}{4}\right)^2} = \frac{32n}{(4m \pm 1)^2}$$

is arbitrarily close to $\frac{\pi^2}{|V|}$.

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Let $\|\cdot\|$ denote the distance to the nearest integer.

Theorem (Baker–Harman)
Let α be irrational and $k \geq 1$. Then there are infinitely many primes p such that

$$\|\alpha p^k\| < p^{-\rho(k)+\varepsilon}$$

for every $\varepsilon > 0$, where $\rho(2) = \frac{3}{20}$ and $\rho(k) = (3 \cdot 2^{k-1})^{-1}$ for $k \geq 3$.

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Another problem

Simply apply this theorem, and we win for the main term - that is, the main term is arbitrarily small. However, we have a pesky error term from the Circle Method of $O\left(n^{-\frac{1}{2}}e^{\sqrt{\frac{n|V|}{2}}}\right)$.

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This leads to a question of infinite simultaneous Diophantine approximation, which I have not been able to find in the literature (yet).

Andrews' fourth conjecture

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The numbers $|V_1(N_n)|$, $|V_1(N_n+1)|$, $|V_1(N_n+2)|$ *contain a local minimum of the sequence* $|V_1(j)|$.

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Seems just as difficult as the third conjecture, as we still need to know about the sequence N_n. Perhaps this becomes apparent if one is able to prove the third conjecture (just like parts 1 and 2 paired up).

Wrap-up

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Two of Andrews' conjectures appear to be extraordinarily deep, relying on irrationality properties of $\zeta_{\mathbb{Q}(\sqrt{-3})}(2)$ (at least, using this method). Is there a different way to approach these conjectures?

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Many follow-up questions could be asked. Probably the easiest will be regarding the functions v_2, v_3, v_4 from the same paper of Andrews.

Thank you!