

# AN ALTERNATIVE GENERATING FUNCTION FOR $k$ -REGULAR PARTITIONS

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An integer partition of a non-negative integer  $n$  is a non-decreasing sequence of positive integers whose sum is  $n$ .

4 has the following five partitions.

$$4, \quad 2 + 2, \quad 1 + 3, \quad 1 + 1 + 2, \quad 1 + 1 + 1 + 1.$$

$$\sum_{n \geq 0} p(n|A)q^n \quad \text{or} \quad \sum_{m, n \geq 0} p(m, n|A)x^m q^n$$

are called a partition generating functions.

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are called a partition generating functions.

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_\infty},$$

$$\sum_{n \geq 0} p(n | \text{distinct parts}) q^n = (-q; q)_\infty,$$

$$\sum_{m, n \geq 0} p(m, n) x^m q^n = \frac{1}{(xq; q)_\infty},$$

$$\sum_{m, n \geq 0} p(m, n | \text{distinct parts}) x^m q^n = (-xq; q)_\infty,$$

where

$$(a; q)_n = \prod_{j=1}^n (1 - aq^{j-1}), \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n, \quad \text{and } |q| < 1.$$

# PRELIMINARIES

$k$ -regular partitions are those in which no part is repeated more than  $k$  times.

1-regular partitions are partitions into distinct parts.

The seven 2-regular partitions of 6 are

6, 1 + 5, 2 + 4, 1 + 1 + 4, 3 + 3, 1 + 2 + 3, 1 + 1 + 2 + 2.

We can also write partitions as 1 1 2 2 for brevity,

or 

1	2
1	2

 for emphasis on regularity.

Various combinatorial and arithmetic properties of  $k$ -regular partitions are being studied.

(just search for “regular partitions” in google scholar)

Our motivation (obsession?) has been to find an evidently positive  $q$ -series for 2-regular partitions together with a combinatorial explanation via *base partitions* and *moves*.

The problem arose while studying Kanade and Russell’s paper “Staircases to Analytic Sum-Sides for Many New Integer Partition Identities of Rogers-Ramanujan Type”.

A partial answer was given in Halime Ömrüüzun Seyrek’s PhD thesis. We will present another partial answer now.

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# MOTIVATION

Consider the partition  $3 + 6 + 10 + 15 + 19$  into distinct parts.

1 4 8 13 17

1 2 6 11 15

1 2 3 8 12

1 2 3 4 8

1 2 3 4 5

$$\sum_{n \geq 0} \binom{n+1}{2} x^n = \frac{x}{(1-x)^2}$$

a base ptn.  
 $\binom{5+1}{2}$

Money  
 $(3, 4, 3, 2, 2)$   
 $(1, 2, 3, 4, 5)$   
 a ptn into parts  $\leq 5$

5  
 2  
 4  
 3  
 3  
 2  
 4  
 1  
 3

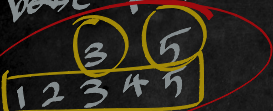


# THE 2-REGULAR CASE

Now, consider the 2-regular partition

	10	19
3	6	10
15	19	

base ptn:



$n$ : # of singletons

$m$ : # of pairs

$$q^{\binom{n+m}{2}} q^{\square + \square + \dots + \square}$$

$$(q; q)_{2m+n}$$

no 4's.

$$q^{\binom{n+1}{2}} q^{3+5}$$

$$(1-q^4)(1-q^5)$$

$$(q; q)_7$$

no 1's



How to write a generating function?

# THE 2-REGULAR CASE

The generating function is:

$$\sum_{m,n \geq 0} \frac{q^{\binom{m+n+1}{2}} x^{2m+n}}{(q; q)_{2m+n}} \cdots$$

$$\times \sum_{1 \leq i_1 < \cdots < i_m \leq m+n} q^{i_1 + \cdots + i_m} (1 - q^{2m+n+1-i_1-1}) \cdots (1 - q^{2m+n+1-i_m-m})$$

Call the highlighted inner sum  $a(m, n)$ .

Take  $a(m, n) = 0$  for  $m < 0$  or  $n < 0$ .

Standard manipulations yield  $a(0, 0) = 1$  and

$$a(m, n) = q^m (1 - q^{2m+n-1}) a(m-1, n) + q^m a(m, n-1).$$

# THE 2-REGULAR CASE

The generating function is:

$$\sum_{m,n \geq 0} \frac{q^{\binom{m+n+1}{2}} x^{2m+n}}{(q; q)_{2m+n}} \dots$$

$a(m, n)$ , always a polynomial

$$\times \sum_{1 \leq i_1 < \dots < i_m \leq m+n} q^{i_1 + \dots + i_m} (1 - q^{2m+n+1-i_1-1}) \dots (1 - q^{2m+n+1-i_m-m})$$

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# THE 2-REGULAR CASE

Some  $a(m, n)$ 's:

$n \setminus m$	0	1	2	3	4
0	1	1	1	1	1
1	$q - q^2$	$q + q^2 - 2q^3$	$q + q^2 + q^3 - 3q^4$	$q + q^2 + q^3$	
2	$q^3 - q^4 - q^6 + q^7$	$q^3 + q^4 - q^5 - q^6 - q^7 - 2q^8 + 3q^9$	$q^3 + q^4 + 2q^5 - 2q^6 - q^7 - 2q^8 - 2q^9 - 3q^{10} + 6q^{11}$		
3	$q^6 - q^7 - q^9 + q^{10} - q^{11} + q^{12} + q^{14} - q^{15}$	$q^6 + q^7 - q^8 - 2q^{10} - 2q^{11} + q^{12} + 2q^{15} + q^{16} + 3q^{17} - 4q^{18}$			
4	$q^{10} - q^{11} - q^{13} + q^{14} - q^{15} + q^{16} - q^{17} + 2q^{18} - q^{19} + q^{20} - q^{21} + q^{22} - q^{23} - q^{25} + q^{26}$				

# THE 2-REGULAR CASE

When we substitute  $a(m, n) = q^{\binom{m+1}{2}} (1-q)^m b(m, n)$ , we obtain:

**Theorem:**

$$\frac{(x^3 q^3; q^3)_\infty}{(xq; q)_\infty} = \sum_{m, n \geq 0} \frac{q^{\binom{m+n+1}{2} + \binom{m+1}{2}} (1-q)^m x^{2m+n}}{(q; q)_{2m+n}} b(m, n),$$

where  $b(0, 0) = 1$ ,  $b(m, n) = 0$  if  $m < 0$  or  $n < 0$ , and

$$b(m, n) = (1 + q + q^2 + \cdots + q^{2m+n-2}) b(m-1, n) + q^m b(m, n-1).$$

Some  $b(m, n)$ 's:

$n \setminus m$	0	1	2	3	4
0	1	1	1	1	1
1	1	$1 + 2q$	$1 + 2q + 3q^2$	$1 + 2q + 3q^2 + 4q^3$	
2	$1 + q + q^2$	$1 + 3q + 4q^2 + 4q^3 + 3q^4$	$1 + 3q + 7q^2 + 9q^3 + 10q^4 + 9q^5 + 6q^6$		
3	$1 + 2q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6$	$1 + 4q + 8q^2 + 13q^3 + 17q^4 + 17q^6 + 14q^7 + 9q^8 + 4q^9$			
4	$1 + 3q + 6q^2 + 9q^3 + 12q^4 + 14q^5 + 15q^6 + 14q^7 + 12q^8 + 9q^9 + 6q^{10} + 3q^{11} + q^{12}$				

Any comments?



Some  $b(m, n)$ 's:

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# CONNECTION TO BESSEL POLYNOMIALS

$$\lim_{q \rightarrow 1^-} b(m, n) = \frac{(2m+n)!}{m! (m+n)! 2^m}$$

Change parameters:  $m+n \leftarrow n$  and  $m \leftarrow k$ :

$$y_n(x) = \sum_{k=0}^n \frac{(n+k)!}{n! k!} \left(\frac{x}{2}\right)^k, \text{ the Bessel polynomials.}$$

$b(m, n)$ 's are different from the existing  $q$ -analogs of these coefficients.

# THE 3-REGULAR CASE

**Theorem:**

$$\frac{(x^4 q^4; q^4)_\infty}{(xq; q)_\infty} = \sum_{l, m, n \geq 0} \frac{q^{\binom{l+m+n+1}{2} + \binom{l+m+1}{2} + \binom{l+1}{2}} (1-q)^{2l+m} x^{3l+2m+n}}{(q; q)_{3l+2m+n}} b(l, m, n)$$

where  $b(0, 0, 0) = 1$ ,  $b(l, m, n) = 0$  if  $l < 0$  or  $m < 0$  or  $n < 0$ , and

$$\begin{aligned} b(l, m, n) = & \left(1 + q + q^2 + \dots + q^{3l+2m+n-2}\right) \left(1 + q + q^2 + \dots + q^{3l+2m+n-3}\right) b(l-1, m, n) \\ & + \left(1 + q + q^2 + \dots + q^{3l+2m+n-2}\right) q^l b(l, m-1, n) \\ & + q^{2l+m} b(l, m, n-1). \end{aligned}$$

**Theorem:** For any positive integer  $k$ ,

$$\begin{aligned}
 & \frac{(x^{k+1}q^{k+1}; q^{k+1})_{\infty}}{(xq; q)_{\infty}} \\
 = & \sum_{n_k, n_{k-1}, \dots, n_1 \geq 0} \frac{q^{\binom{n_k+n_{k-1}+\dots+n_1+1}{2} + \binom{n_k+n_{k-1}+\dots+n_2+1}{2} + \dots + \binom{n_k+1}{2}}{(q; q)_{kn_k+(k-1)n_{k-1}+\dots+2n_2+n_1}} \dots \\
 & \times x^{kn_k+(k-1)n_{k-1}+\dots+2n_2+n_1} b(n_k, n_{k-1}, \dots, n_1),
 \end{aligned}$$

where  $b(0, \dots, 0) = 1$ ,  $b(n_k, n_{k-1}, \dots, n_1) = 0$  if  $n_j < 0$  for any  $j = 1, 2, \dots, k$ , and

## Theorem: (cont'd)

$$\begin{aligned}
 & b(n_k, n_{k-1}, \dots, n_1) \\
 &= \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 2}\right) \dots \\
 &\times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 3}\right) \dots \\
 &\vdots \\
 &\times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - k}\right) b(n_k - 1, n_{k-1}, n_{k-2}, \dots, n_1) \\
 &+ \dots \\
 &+ \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 2}\right) \dots \\
 &\times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - 3}\right) \dots \\
 &\vdots \\
 &\times \left(1 + q + q^2 + \dots + q^{kn_k + (k-1)n_{k-1} + \dots + 2n_2 + n_1 - j}\right) \dots \\
 &\times q^{(k-j)n_k + (k-1-j)n_{k-1} + \dots + 2n_{j+2} + n_{j+1}} b(n_k, \dots, n_{j+1}, n_j - 1, n_{j-1}, \dots, n_1) \\
 &+ \dots \\
 &+ q^{(k-1)n_k + (k-2)n_{k-1} + \dots + 2n_3 + n_2} b(n_k, \dots, n_2, n_1 - 1).
 \end{aligned}$$

- ▶ **Conjecture:** For any integer  $k \geq 2$ , and any non-negative  $n_k, n_{k-1}, \dots, n_1$ ,  $b(n_k, \dots, n_1)$  is unimodal.  
(This is different from the unimodality properties of the coefficients of Bessel polynomials. )
- ▶ For  $k \geq 3$ ,  $\lim_{q \rightarrow 1^-} b(n_k, \dots, n_1)$  does not seem to be in OEIS.
- ▶ It appears that there are no closed formulas for  $a(\dots)$ 's or  $b(\dots)$ 's.

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Thanks for listening!

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