An Alternative Generating Function for k-Regular Partitions

Kağan Kurşungöz

Sabancı University, İstanbul
kursungoz@sabanciuniv.edu

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An integer partition of a non-negative integer n is a non-decreasing sequence of positive integers whose sum is n.
4 has the following five partitions.

$$4, \quad 2+2, \quad 1+3, \quad 1+1+2, \quad 1+1+1+1.$$

$$\sum_{n \geq 0} p(n|A) q^n \qquad \text{or} \qquad \sum_{m,n \geq 0} p(m,n|A) x^m q^n$$

are called a partition generating functions.

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$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty},$$

$$\sum_{n\geq 0} p(n| \text{ distinct parts })q^n = (-q;q)_\infty,$$

$$\sum_{m,n\geq 0} p(m,n)x^m q^n = \frac{1}{(xq;q)_\infty},$$

$$\sum_{m,n\geq 0} p(m,n| \text{ distinct parts })x^m q^n = (-xq;q)_\infty,$$

where

$$(a;q)_n = \prod_{j=1}^{n} \left(1 - aq^{j-1}\right), \quad (a;q)_\infty = \lim_{n\to\infty} (a;q)_n, \quad \text{and } |q| < 1.$$

Preliminaries

k-regular partitions are those in which no part is repeated more than k times.

1-regular partitions are partitions into distinct parts.

The seven 2-regular partitions of 6 are

$$6, \quad 1+5, \quad 2+4, \quad 1+1+4, \quad 3+3, \quad 1+2+3, \quad 1+1+2+2.$$

We can also write partitions as 1 1 2 2 for brevity,

or $\begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix}$ for emphasis on regularity.

Motivation

Various combinatorial and arithmetic properties of k-regular partitions are being studied.
(just search for "regular partitions" in google scholar)

Our motivation (obsession?) has been to find an evidently positive q-series for 2-regular partitions together with a combinatorial explanation via *base partitions* and *moves*.

The problem arose while studying Kanade and Russell's paper "Staircases to Analytic Sum-Sides for Many New Integer Partition Identities of Rogers-Ramanujan Type".

A partial answer was given in Halime Ömrüuzun Seyrek's PhD thesis. We will present another partial answer now.

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Motivation

Consider the partition $3 + 6 + 10 + 15 + 19$ into distinct parts.

Now, consider the 2-regular partition

		10		19
3	6	10	15	19

How to write a generating function?

The generating function is:

$$\sum_{m,n\geq 0} \frac{q^{\binom{m+n+1}{2}} x^{2m+n}}{(q;q)_{2m+n}} \cdots$$

$$\times \sum_{1\leq i_1 < \cdots < i_m \leq m+n} q^{i_1+\cdots+i_m} \left(1 - q^{2m+n+1-i_1-1}\right) \cdots \left(1 - q^{2m+n+1-i_m-m}\right)$$

Call the highlighted inner sum $a(m,n)$.
Take $a(m,n) = 0$ for $m < 0$ or $n < 0$.
Standard manipulations yield $a(0,0) = 1$ and

$$a(m,n) = q^m(1 - q^{2m+n-1})a(m-1,n) + q^m a(m,n-1).$$

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Some $a(m,n)$'s:

$n\backslash m$	0	1	2	3	4
0	1	1	1	1	1
1	$q - q^2$	$q + q^2 - 2q^3$	$q + q^2 + q^3 - 3q^4$	$q + q^2 + q^3 + q^4 - 4q^5$	
2	$q^3 - q^4 - q^6 + q^7$	$q^3 + q^4 - q^5 - q^6 - q^7 - 2q^8 + 3q^9$	$q^3 + q^4 + 2q^5 - 2q^6 - q^7 - 2q^8 - 2q^9 - 3q^{10} + 6q^{11}$		
3	$q^6 - q^7 - q^9 + q^{10} - q^{11} + q^{12} + q^{14} - q^{15}$	$q^6 + q^7 - q^8 - 2q^{10} - 2q^{11} + q^{12} + 2q^{15} + q^{16} + 3q^{17} - 4q^{18}$			
4	$q^{10} - q^{11} - q^{13} + q^{14} - q^{15} + q^{16} - q^{17} + 2q^{18} - q^{19} + q^{20} - q^{21} + q^{22} - q^{23} - q^{25} + q^{26}$				

When we substitute $a(m,n) = q^{\binom{m+1}{2}}(1-q)^m b(m,n)$, we obtain:

Theorem:

$$\frac{(x^3q^3;q^3)_\infty}{(xq;q)_\infty} = \sum_{m,n\geq 0} \frac{q^{\binom{m+n+1}{2}+\binom{m+1}{2}}(1-q)^m x^{2m+n}}{(q;q)_{2m+n}} b(m,n),$$

where $b(0,0) = 1$, $b(m,n) = 0$ if $m < 0$ or $n < 0$, and

$$b(m,n) = \left(1 + q + q^2 + \cdots + q^{2m+n-2}\right) b(m-1,n) + q^m b(m,n-1).$$

Some $b(m, n)$'s:

$n \backslash m$	0	1	2	3	4
0	1	1	1	1	1
1	1	$1 + 2q$	$1 + 2q + 3q^2$	$1 + 2q + 3q^2 + 4q^3$	
2	$1 + q + q^2$	$1 + 3q + 4q^2 + 4q^3 + 3q^4$	$1 + 3q + 7q^2 + 9q^3 + 10q^4 + 9q^5 + 6q^6$		
3	$1 + 2q + 3q^2 + 3q^3 + 3q^4 + 2q^5 + q^6$	$1 + 4q + 8q^2 + 13q^3 + 17q^4 + 18q^5 + 17q^6 + 14q^7 + 9q^8 + 4q^9$			
4	$1 + 3q + 6q^2 + 9q^3 + 12q^4 + 14q^5 + 15q^6 + 14q^7 + 12q^8 + 9q^9 + 6q^{10} + 3q^{11} + q^{12}$				

Any comments?

Some $b(m, n)$'s:

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$$\lim_{q \to 1^-} b(m, n) = \frac{(2m+n)!}{m!\,(m+n)!\,2^m}.$$

Change parameters: $m + n \leftarrow n$ and $m \leftarrow k$:

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{n!\,k!} \left(\frac{x}{2}\right)^k,$$ the Bessel polynomials.

$b(m, n)$'s are different from the existing q-analogs of these coefficients.

Theorem:

$$\frac{(x^4q^4;q^4)_\infty}{(xq;q)_\infty} = \sum_{l,m,n\geq 0} \frac{q^{\binom{l+m+n+1}{2}+\binom{l+m+1}{2}+\binom{l+1}{2}}(1-q)^{2l+m}x^{3l+2m+n}}{(q;q)_{3l+2m+n}} b(l,m,n)$$

where $b(0,0,0)=1$, $b(l,m,n)=0$ if $l<0$ or $m<0$ or $n<0$, and

$$\begin{aligned} b(l,m,n) = {} & \left(1+q+q^2+\cdots+q^{3l+2m+n-2}\right)\left(1+q+q^2+\cdots+q^{3l+2m+n-3}\right) b(l-1,m,n) \\ & + \left(1+q+q^2+\cdots+q^{3l+2m+n-2}\right) q^l b(l,m-1,n) \\ & + q^{2l+m} b(l,m,n-1). \end{aligned}$$

Theorem: For any positive integer k,

$$\frac{\left(x^{k+1}q^{k+1};q^{k+1}\right)_\infty}{(xq;q)_\infty}$$

$$= \sum_{n_k, n_{k-1}, \ldots, n_1 \geq 0} \frac{q^{\binom{n_k+n_{k-1}+\cdots+n_1+1}{2}+\binom{n_k+n_{k-1}+\cdots+n_2+1}{2}+\cdots\binom{n_k+1}{2}}(1-q)^{(k-1)n_k+(k-2)n_{k-1}+\cdots 2n_3+n_2}}{(q;q)_{kn_k+(k-1)n_{k-1}+\cdots+2n_2+n_1}} \cdots$$

$$\times \; x^{kn_k+(k-1)n_{k-1}+\cdots+2n_2+n_1} b(n_k, n_{k-1}, \ldots, n_1),$$

where $b(0, \ldots, 0) = 1$, $b(n_k, n_{k-1}, \ldots, n_1) = 0$ if $n_j < 0$ for any j =1, 2, ..., k, and

Theorem: (cont'd)

$$
\begin{aligned}
&b(n_k, n_{k-1}, \ldots, n_1) \\
&= \left(1 + q + q^2 + \cdots + q^{kn_k + (k-1)n_{k-1} + \cdots + 2n_2 + n_1 - 2}\right) \cdots \\
&\times \left(1 + q + q^2 + \cdots + q^{kn_k + (k-1)n_{k-1} + \cdots + 2n_2 + n_1 - 3}\right) \cdots \\
&\vdots \\
&\times \left(1 + q + q^2 + \cdots + q^{kn_k + (k-1)n_{k-1} + \cdots + 2n_2 + n_1 - k}\right) b(n_k - 1, n_{k-1}, n_{k-2}, \ldots, n_1) \\
&+ \cdots \\
&+ \left(1 + q + q^2 + \cdots + q^{kn_k + (k-1)n_{k-1} + \cdots + 2n_2 + n_1 - 2}\right) \cdots \\
&\times \left(1 + q + q^2 + \cdots + q^{kn_k + (k-1)n_{k-1} + \cdots + 2n_2 + n_1 - 3}\right) \cdots \\
&\vdots \\
&\times \left(1 + q + q^2 + \cdots + q^{kn_k + (k-1)n_{k-1} + \cdots + 2n_2 + n_1 - j}\right) \cdots \\
&\times q^{(k-j)n_k + (k-1-j)n_{k-1} + \cdots + 2n_{j+2} + n_{j+1}} b(n_k, \ldots, n_{j+1}, n_j - 1, n_{j-1}, \ldots, n_1) \\
&+ \cdots \\
&+ q^{(k-1)n_k + (k-2)n_{k-1} + \cdots + 2n_3 + n_2} b(n_k, \ldots, n_2, n_1 - 1).
\end{aligned}
$$

- **Conjecture:** For any integer $k \geq 2$, and any non-negative n_k, n_{k-1}, $\ldots n_1$, $b(n_k, \ldots, n_1)$ is unimodal.
  (This is different from the unimodality properties of the coefficients of Bessel polynomials.)
- For $k \geq 3$, $\lim_{q \to 1^-} b(n_k, \ldots, n_1)$ does not seem to be in OEIS.
- It appears that there are no closed formulas for $a(\cdots)$'s or $b(\cdots)$'s.

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