Modulo d extension of parity results in Rogers-Ramanujan-Gordon type overpartition identities

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in Partition Theory, q-Series and Related Topics
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Definition (Euler)

An integer partition is an unordered finite sum of positive integers (parts) ($\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$).

Example

$n = 4$ has 5 partitions ($p(4) = 5$):

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

We will use the frequency notation:

$$3+1 \qquad \leftrightarrow \qquad f_1 = 1, \quad f_2 = 0, \quad f_3 = 1, \quad f_4 = f_5 = \cdots = 0.$$

Definitions

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Definition (Corteel and Lovejoy)

An overpartition is a partition such that the first occurrence of each part may be overlined.

Example

$n = 3$ has 8 overpartitions ($\overline{p}(3) = 8$):

$$3, \quad \overline{3}, 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1},$$

$$1+1+1, \quad \overline{1}+1+1.$$

The frequency notation is used for overpartitions, too.

$$2+\overline{1} \qquad \leftrightarrow \qquad \begin{array}{c} f_1 = 0, \quad f_{\overline{1}} = 1, \quad f_2 = 1, \quad f_{\overline{2}} = 0, \\ f_i = f_{\overline{i}} = 0 \text{ for } i > 2. \end{array}$$

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q-Pochhammer Symbol

Definition

For $n \in \mathbb{N}$ *,*

$$(a;q)_n = \prod_{j=1}^{n} (1 - aq^{j-1}),$$

$$(a_1, \ldots, a_k; q)_n = (a_1;q)_n \cdots (a_k;q)_n,$$

and for $|q| < 1$

$$(a;q)_\infty = \lim_{n\to\infty} (a;q)_n = \prod_{j=1}^{\infty} (1 - aq^{j-1}).$$

(*sine qua non* of partition generating functions)

The Partition and Overpartition Generating Functions

Theorem (Euler)

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = \frac{1}{(q;q)_\infty}$$

Theorem (Corteel and Lovejoy)

$$\sum_{n\geq 0} \overline{p}(n)q^n = \frac{(1+q)(1+q^2)(1+q^3)\cdots}{(1-q)(1-q^2)(1-q^3)\cdots} = \frac{(-q;q)_\infty}{(q;q)_\infty}$$

The First Rogers-Ramanujan Identity

Theorem (Rogers-Ramanujan I, combinatorial version)

For any $n \in \mathbb{N}$,
the number of partitions of n
into distinct and non-consecutive parts
equals the number of partitions into parts
$\not\equiv 0, \pm 2 \pmod 5$.

In frequency notation,

$$p(n \quad | \quad f_i + f_{i+1} < 2) = p(n \quad | \quad f_{5j} = f_{5j \pm 2} = 0).$$

A general theme: multiplicity/frequency/gap conditions on parts
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EXAMPLE

For $n = 9$,

$$9, \quad 8+1, \quad 7+2, \quad 6+3, \quad 5+3+1$$

vs.

$$9, \quad 6+1+1+1, \quad 4+4+1, \quad 4+1+\cdots+1, \quad 1+\cdots+1.$$

Theorem (Rogers-Ramanujan I, q-series version)

$$\sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)\cdots} = \frac{1}{(q,q^4;q^5)_\infty}$$

Theorem (Rogers-Ramanujan-Gordon Identities, in frequency notation)

Let a, k be integers such that $k \geq 2$ and $1 \leq a \leq k$. Then, for all non-negative integers n,

$$p(n \mid f_i + f_{i+1} < k, f_1 < a) = p(n \mid f_{(2k+1)j} = f_{(2k+1)j \pm a} = 0).$$

Remarks:

- $k = 2$ and $a = 1, 2$ cases are the Rogers-Ramanujan identities.
- These identities were independently found by Andrews.
- In Andrews's analytic proof, $a = 0$ is allowed for convenience.

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Example

Let $k = a = 5$.
(we just display 3's and 4's)

$$\cdots + \textit{ (no 3's or 4's here) } + \cdots,$$

$$\cdots + 3 + \cdots, \quad \cdots + 4 + \cdots,$$

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Theorem (Rogers-Ramanujan-Gordon Identities, using infinite products)

Let a, k be integers such that $k \geq 2$ and $1 \leq a \leq k$. Then,

$$\sum_{n \geq 0} p(n \mid f_i + f_{i+1} < k, f_1 < a)\, q^n = \frac{(q^a, q^{2k+1-a}, q^{2k+1}; q^{2k+1})_\infty}{(q; q)_\infty}$$

Remark: With the (evidently) positive multiple series generating function, the identities are called Andrews-Gordon identities.

THEOREM (ROGERS-RAMANUJAN-GORDON IDENTITIES, USING INFINITE PRODUCTS)

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Gordon's Theorem for Overpartitions

Theorem (Gordon's theorem for overpartitions, Lovejoy, Chen and Sang and Shi)

Let a, k be integers such that $k \geq 2$ and $1 \leq a \leq k$. Then,

$$\sum_{n \geq 0} p(n \mid f_i + f_{\bar{i}} + f_{i+1} < k, f_1 < a)\, q^n = \frac{(-q;q)_\infty (q^a, q^{2k-a}, q^{2k}; q^{2k})_\infty}{(q;q)_\infty}$$

Remark: The original version is $p(n|\text{ multiplicity condition }) = p(n|\text{ congruence condition })$, but the correspondence is straightforward.

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$p(n | \text{ multiplicity condition }) = p(n | \text{ congruence condition })$, but the correspondence is straightforward.

Example

Let $k = a = 5$.
Begin with any of

$$\begin{gathered}\vdots \\ \cdots + 4 + 3 + 3 + 3 + \cdots , \\ \cdots + 4 + \overline{3} + 3 + 3 + \cdots , \\ \cdots \overline{4} + 4 + 3 + 3 + 3 + \cdots , \\ \cdots \overline{4} + 4 + \overline{3} + 3 + 3 + \cdots , \\ \vdots\end{gathered}$$

and delete parts as desired.

For us, a Rogers-Ramanujan generalization is:

$$p(n|\ \text{multiplicity condition}^*) = p(n|\ \text{congruence condition}\ )$$

* The multiplicity condition relates consecutive frequencies ONLY.

mise en scene credit: Steven Crowder. Also see https://imgflip.com/i/71la16 and https://knowyourmeme.com/memes/steven-crowders-change-my-mind-campus-sign.

The parchment figure is from twinkl.com

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One of many results in Andrews' "Parity in Partitions"

Theorem (Andrews)

Let a, k be integers such that $k \geq 2$, $1 \leq a \leq k$, and $k \equiv a \pmod 2$. Consider partitions satisfying
(G) $f_i + f_{\bar{i}} + f_{i+1} < k, f_1 < a$,
(P) $f_{2i} \equiv 0 \pmod 2$. Then,

$$\sum_{n \geq 0} p(n \mid \ (G) \text{ and } (P)\ )\, q^n = \frac{(-q;q^2)_\infty (q^a, q^{2k+2-a}, q^{2k+2}; q^{2k+2})_\infty}{(q^2;q^2)_\infty}$$

Remarks:

The missing $k \not\equiv a \pmod 2$ case was first found by Kim and Yee.

Andrews also asked for an overpartition analog.

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EXAMPLE

Let $k = a = 6$.
Begin with any of

$$\begin{gathered}\vdots \\ \cdots + 3 + 3 + 3 + 3 + 3 + \cdots, \\ \cdots + 4 + 4 + 3 + 3 + 3 + \cdots, \\ \cdots + 4 + 4 + 4 + 4 + 3 + \cdots, \\ \vdots\end{gathered}$$

and delete 3's at will, and 4's only in pairs.

One of Sang, Shi and Yee's Parity in Overpartitions theorems

Theorem (Sang, Shi and Yee)

Let a, k be integers such that $k \geq 2$, $1 \leq a \leq k$, and $k \equiv a \pmod 2$. Let $U_{k,a}(n)$ be the number of overpartitions satisfying

- $f_1 \leq a - 1 + f_{\overline{1}}$,
- $f_{2l-1} \geq f_{\overline{2l-1}}$,
- $f_{2l} + f_{\overline{2l}} \equiv 0 \pmod 2$,
- $f_l + f_{\overline{l}} + f_{l+1} \leq k - 1 + f_{\overline{l+1}}$;

and let $\overline{U}_{k,a}(n)$ be the number of overpartitions of n satisfying

- $f_1 \leq a - 1 + f_{\overline{1}}$,
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Example

Let $2k = 2a = 6$. *Yes?*

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One of Sang, Shi and Yee's Parity in Overpartitions theorems

Theorem (Sang, Shi and Yee)

... Then,

$$\sum_{n\geq 0} U_{2k,2a}(n)q^n = \frac{(-q;q)_\infty (q^{2a},q^{4k-2a},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty}$$

$$\sum_{n\geq 0} U_{2k,2a-1}(n)q^n = \frac{1}{(1+q)}\frac{(-q;q)_\infty (q^{2a},q^{4k-2a},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty} + \frac{q}{(1+q)}\frac{(-q;q)_\infty (q^{2a-2},q^{4k-2a+2},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty}$$

$$\sum_{n\geq 0} \overline{U}_{2k,2a-1}(n)q^n = \sum_{n\geq 0} \overline{U}_{2k,2a}(n)q^n = \frac{(-q^2;q^2)_\infty^2 (q^{2a},q^{4k-2a},q^{4k};q^{4k})_\infty}{(q^2;q^2)_\infty}$$

Setup:
Hereafter, let d, k, a, e, and f be non-negative parameters such that

$$d \geq 1, \quad k \geq 1, \quad 0 \leq a \leq k, \quad \text{and} \quad 1 \leq e, f \leq d.$$

(we will allow $f = 0$, as necessary, in the proofs)

Let $U_{dk+e,da+f}(n)$ be the number of overpartitions of n satisfying

(i) $f_1 \leq da + f - 1 + (d-1) f_{\overline{1}}$,

(ii) $f_{2l-1} \geq (d-1) f_{\overline{2l-1}}$,

(iii) $f_{2l} + f_{\overline{2l}} \equiv 0 \pmod{d}$,

(iv) $f_l + f_{\overline{l}} + f_{l+1} \leq dk + e - 1 + (d-1) f_{\overline{l+1}}$;

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Example

Let $d = 3$, $dk + e = da + f = 6$.

Our contribution

Theorem

For parameters as above with $e = d$ or $2e = d$

$$\sum_{n \geq 0} U_{dk+e,da+f}(n) q^n$$

$$= \begin{cases} \frac{(1-q^{d+f-e})}{(1-q^d)} \frac{(q^{da+e}, q^{2dk-da+e}, q^{2dk+2e}; q^{2dk+2e})_\infty}{(q;q^2)_\infty (q^d;q^d)_\infty} \\ + \frac{(q^{d+f-e} - q^d)}{(1-q^d)} \frac{(q^{da-d+e}, q^{2dk-da+d+e}, q^{2dk+2e}; q^{2dk+2e})_\infty}{(q;q^2)_\infty (q^d;q^d)_\infty}, & \text{if } f < e, \\ \frac{(q^{da+e}, q^{2dk-da+e}, q^{2dk+2e}; q^{2dk+2e})_\infty}{(q;q^2)_\infty (q^d;q^d)_\infty}, & \text{if } f = e, \\ \frac{(q^{f-e} - q^d)}{(1-q^d)} \frac{(q^{da+e}, q^{2dk-da+e}, q^{2dk+2e}; q^{2dk+2e})_\infty}{(q;q^2)_\infty (q^d;q^d)_\infty} \\ + \frac{(1-q^{f-e})}{(1-q^d)} \frac{(q^{da+d+e}, q^{2dk-da-d+e}, q^{2dk+2e}; q^{2dk+2e})_\infty}{(q;q^2)_\infty (q^d;q^d)_\infty}, & \text{if } f > e; \end{cases}$$

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$$\sum_{n\geq 0} \overline{U}_{dk+e,da+f}(n)q^n = \frac{(-q^d;q^d)_\infty (q^{da+d}, q^{2dk-da-d+2e}, q^{2dk+2e}; q^{2dk+2e})_\infty}{(q^2;q^2)_\infty (q^d;q^{2d})_\infty}.$$

Remarks:

- For $d = e = 1$, the theorem is Gordon's theorem for overpartitions.
- For $d = e = 2$, it is Sang, Shi and Yee's result.
- It is possible to eliminate the overlined parts in the proofs. (Then what?)
- It is possible (but painful) to interpret the infinite products as partition generating functions.

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Remember Andrews' analytic proof of the Rogers-Ramanujan-Gordon identities?

Let $b_{k,a}(m,n)$ be the number of partitions of n into m parts satisfying Gordon's condition.

Set

$$R_{k,a}(x) = \sum_{m,n \geq 0} b_{k,a}(m,n) x^m q^n$$

Observe/prove that

$$\begin{aligned} &R_{k,a}(x) - R_{k,a-1}(x) = (xq)^{a-1} R_{k,k-a+1}(xq), \\ &R_{k,0}(x) = 0, \\ &R_{k,a}(0) = 1. \end{aligned}$$

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Define

$$Q_{k,a}(x) = \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{(2k+1)\binom{n+1}{2}} q^{-an}}{(q;q)_n (xq^{n+1};q)_\infty} - \frac{(-1)^n x^{kn} q^{(2k+1)\binom{n+1}{2}} x^a q^{(n+1)a}}{(q;q)_n (xq^{n+1};q)_\infty}$$

Regard the series as

$$Q_{k,a}(x) = \sum_{n\geq 0} \alpha_n(x) \qquad + \beta_n(x)$$

Define

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Regard the series as

$$Q_{k,a}(x) = \sum_{n\geq 0} \alpha_n(x) q^{-an} + \beta_n(x) x^a q^{(n+1)a}$$

Verify that

$$Q_{k,a}(x) - Q_{k,a-1}(x) = (xq)^{a-1} Q_{k,k-a+1}(xq)$$

by showing that

$$\alpha_n(x) q^{-an} - \alpha_n(x) q^{-(a-1)n} = (xq)^{a-1} \beta_{n-1}(xq) x^{k-a+1} q^{(n+1)(k-a+1)}$$

and

$$\beta_n(x) x^a q^{(n+1)a} - \beta_n(x) x^{a-1} q^{(n+1)(a-1)} = (xq)^{a-1} \alpha_n(xq) q^{-n(k-a+1)}$$

for all n.

Remark: These are sufficient, but not necessary conditions!

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and

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Next, just observe that $Q_{k,0}(x) = 0$ and $Q_{k,a}(0) = 1$.

Because Q's and R's satisfy the same set of functional equations, and the set of functional equations determine double power series uniquely,

$$R_{k,a}(x) = \sum_{m,n \geq 0} b_{k,a}(m,n) x^m q^n = Q_{k,a}(x)$$

(defining q-equations principle as Andrews calls it).

Finally,

$$Q_{k,a}(1) = \sum_{n \geq 0} \left(\sum_{m \geq 0} b_{k,a}(m,n) \right) q^n,$$

which equals the desired infinite product by the Jacobi's Triple Product identity.

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$$R_{k,a}(x) = \sum_{m,n \geq 0} b_{k,a}(m,n) x^m q^n = Q_{k,a}(x)$$

(defining q-equations principle as Andrews calls it).

Finally,

$$Q_{k,a}(1) = \sum_{n \geq 0} \left(\sum_{m \geq 0} b_{k,a}(m,n) \right) q^n,$$

which equals the desired infinite product by the Jacobi's Triple Product identity.

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Panorama of the Proof

Now, slightly change the order of the above operations,
adjust as necessary,
and the proof writes itself.

Future Work

- There still are many missing cases.
- Sang, Shi and Yee have Andrews-Gordon type series as generating functions, those series are missing in the general case.
- Now that we have theorems, we can look for alternative proofs.
- The process begs for automation.

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Thank you for your attention.

Any questions?

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