Part Size Count in Cylindric Partitions and Bijections for Small Profiles

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The Main Definition

Definition (Gessel & Krattenthaler, 1997):
Let r and ℓ be positive integers. Let $c = (c_1, c_2, \ldots, c_r)$ be a composition, where $c_1 + c_2 + \cdots + c_r = \ell$. A *cylindric partition with profile* c is a vector partition $\Lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$, where each $\lambda^{(i)} = \lambda_1^{(i)} + \lambda_2^{(i)} + \cdots + \lambda_{s_i}^{(i)}$ is a partition, such that for all i and j,

$$\lambda_j^{(i)} \geq \lambda_{j+c_{i+1}}^{(i+1)} \quad \text{and} \quad \lambda_j^{(r)} \geq \lambda_{j+c_1}^{(1)}.$$

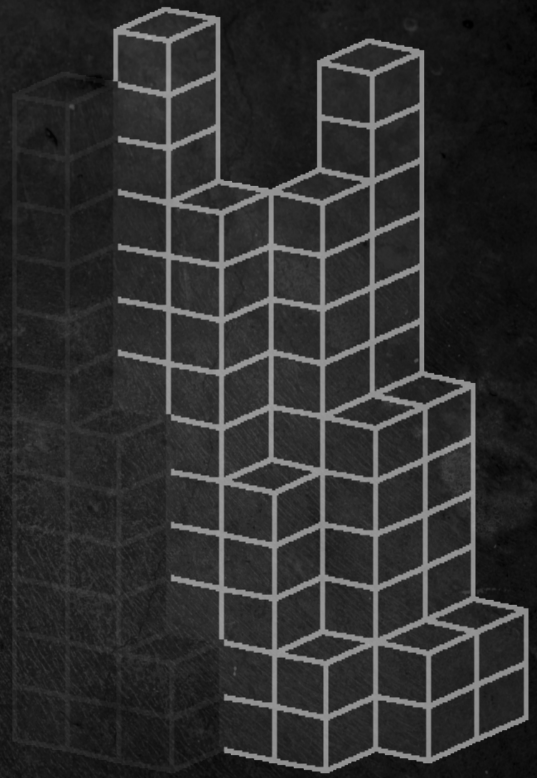
The integers r and ℓ are called the *rank*, and the *level*, of the cylindric partition, respectively.

background image credit: Peter Gargiulo on Unsplash

Example

$\Lambda = ((12, 6, 2), (10, 6, 2), (13, 10, 5, 2))$ is a cylindric partition with profile $(1, 0, 2)$. rank $r = 3$ and level $\ell = 3$.

$$\Lambda = \begin{array}{cccc} & & 12 & 6 & 2 \\ & & 10 & 6 & 2 \\ & 13 & 10 & 5 & 2 \\ 12 & 6 & 2 & & \end{array}$$



$|\Lambda| = 68$, $\max(\Lambda) = 13$, and $\mathrm{st}(\Lambda) = 6$.

The concept of part sizes in integer partitions goes back to MacMahon.

Borodin's Theorem

Set $F_c(z,u,q) := \sum_{\Lambda \in \mathcal{P}_c} z^{\max(\Lambda)} u^{\mathrm{st}(\Lambda)} q^{|\Lambda|}$,

where the sum is over all cylindric partitions with profile c.

Theorem (Borodin, 2007): Let r and ℓ be positive integers, and let $c = (c_1, c_2, \ldots, c_r)$ be a composition of ℓ. Define $t := r + \ell$ and $s(i,j) := c_i + c_{i+1} + \cdots + c_j$. Then,

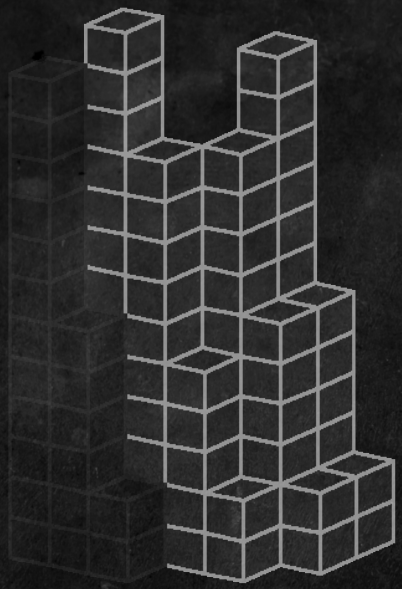
$$F_c(1,1,q) = \frac{1}{(q^t;q^t)_\infty} \prod_{i=1}^{r} \prod_{j=i}^{r} \prod_{m=1}^{c_i} \frac{1}{(q^{m+j-i+s(i+1,j)};q^t)_\infty} \cdots$$

$$\times \prod_{i=2}^{r} \prod_{j=2}^{i} \prod_{m=1}^{c_i} \frac{1}{(q^{t-m+j-i-s(j,i-1)};q^t)_\infty}.$$

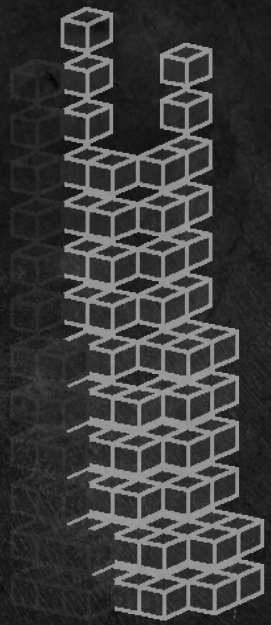
Cylindric partitions are studied a lot

- Decompositions of cylindric partitions into two or more components,
- Cylindric partitions into distinct parts,
- Evidently positive generating functions,
- Connections to lie algebras and their representations,
- etc.

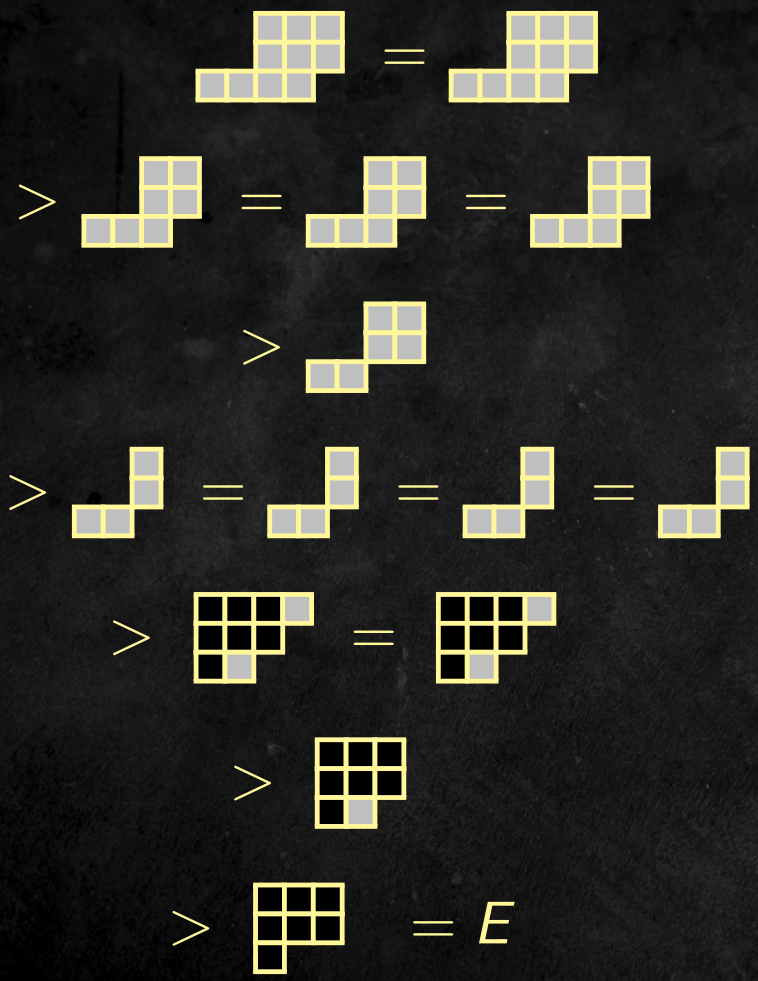
Slices of a cylindric partition:



→



We can order the slices in a chain:



$$= \quad > \quad = \quad = \quad > \quad = \quad = \quad = \quad > \quad = \quad > \quad = E$$

$$_6\Sigma = {}_6\Sigma$$

$$> {}_5\Sigma = {}_5\Sigma = {}_5\Sigma$$

$$> {}_4\Sigma$$

$$> {}_3\Sigma = {}_3\Sigma = {}_3\Sigma = {}_3\Sigma$$

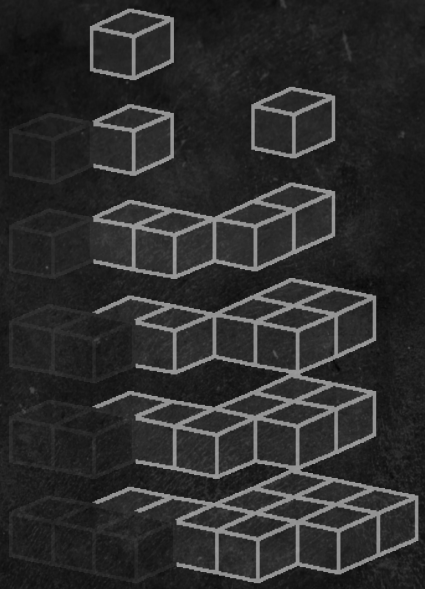
$$> {}_2\Sigma = {}_2\Sigma$$

$$> {}_1\Sigma$$

$$> {}_0\Sigma = E$$

The indices may be big-endian, little-endian, repeated, non-repeated, etc.

To determine the *pivot* slices in a chain,
eliminate the repetitions and list one slice of each size.



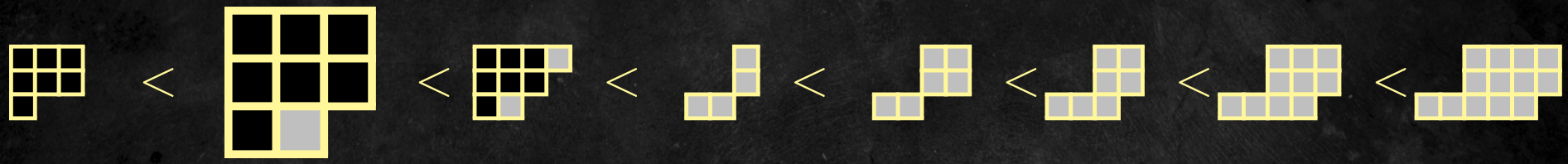


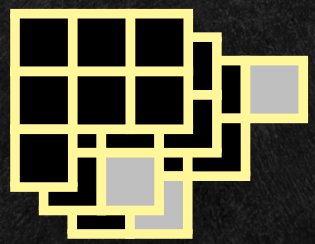
To determine the *pivot* slices in a chain,
Then, augment the chain by specified slices, one at each end.

$$E = {}_0\Sigma < {}_1\Sigma < {}_2\Sigma < {}_3\Sigma < {}_4\Sigma < {}_5\Sigma < {}_6\Sigma < {}_{6+1}\Sigma$$

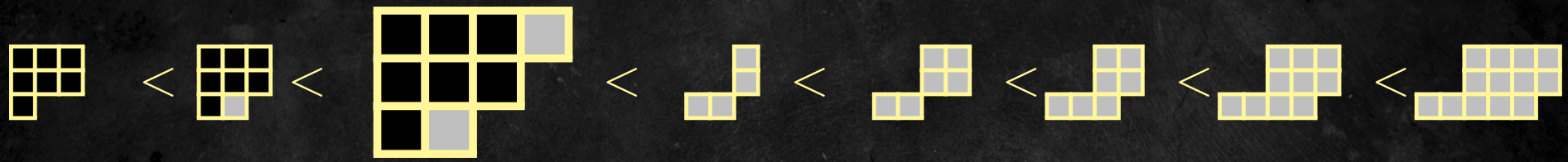


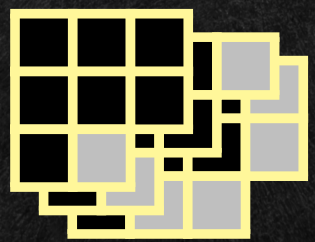
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Then, consider each slice together with the preceding and succeeding slices...





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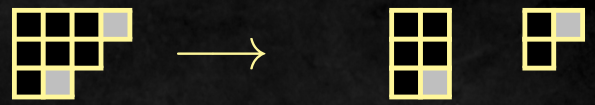


To determine the *pivot* slices in a chain,
Then, consider each slice together with the preceding and succeeding slices...



In the end,  and  are found to be the pivots in this chain.

When emphasis is needed, pivots are denoted by .Π instead of .Σ

To determine the shape of a slice: 

Discard the left piece, and read the remaining as a partition into $r-1$ parts.

So,  has shape $(2,1)$.

Denote it by $2^{(2,1)}$ i.e. $\text{weight}^{\text{shape}}$.

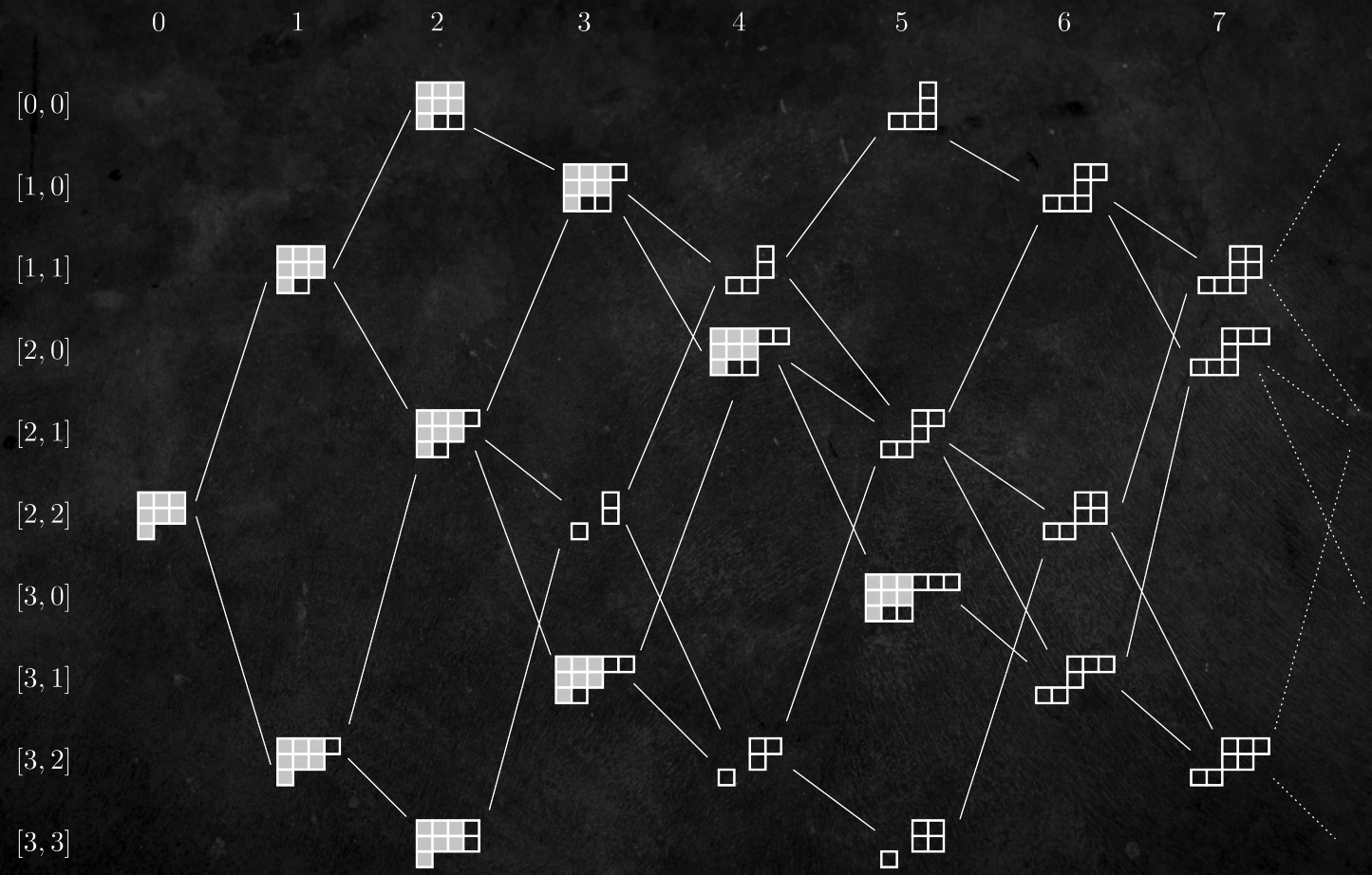
e.g., the chain



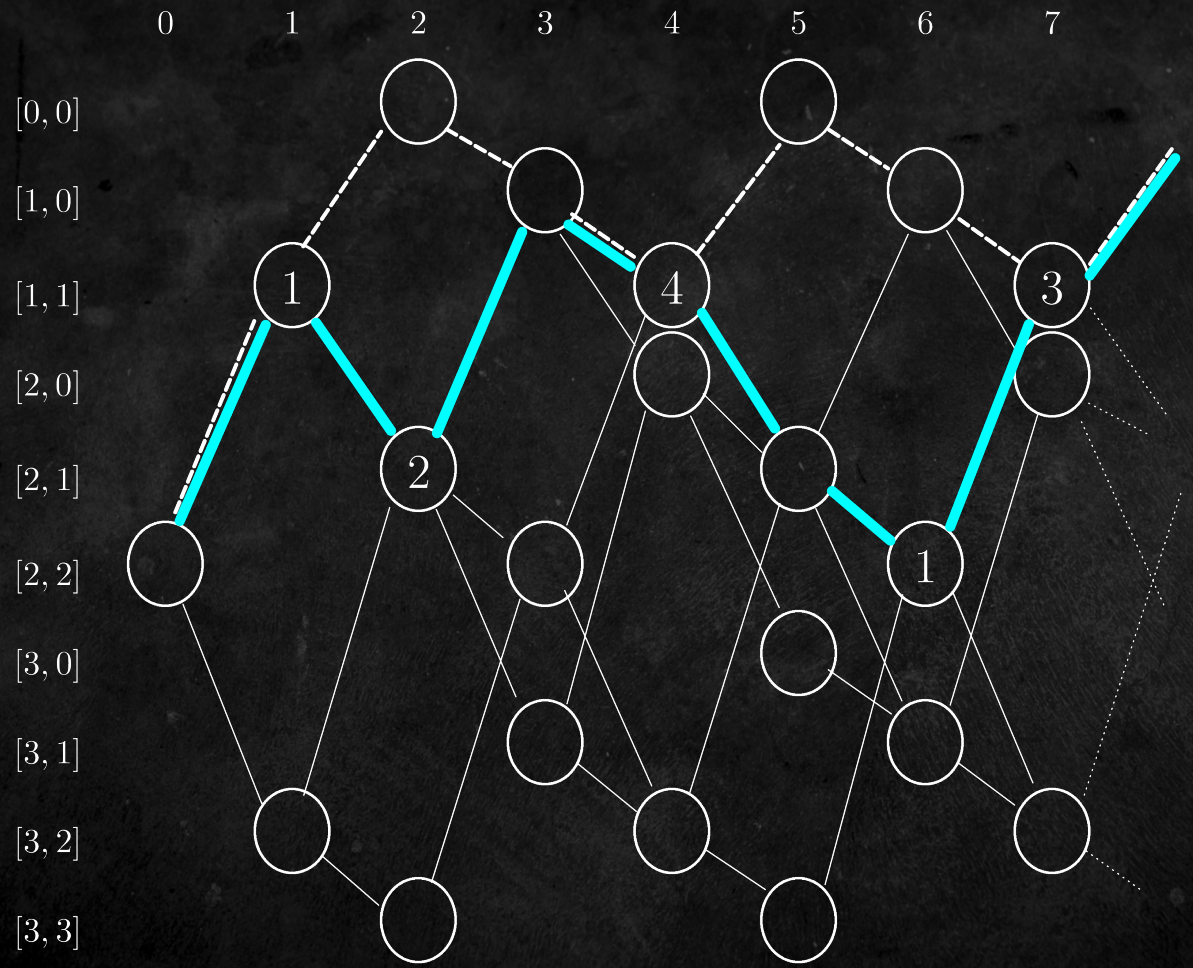
is the same as

$$0^{(2,2)} < 1^{(1,1)} < 2^{(2,1)} < 4^{(1,1)} < 6^{(2,2)} < 7^{(1,1)} < 10^{(1,1)} < 13^{(1,1)}.$$

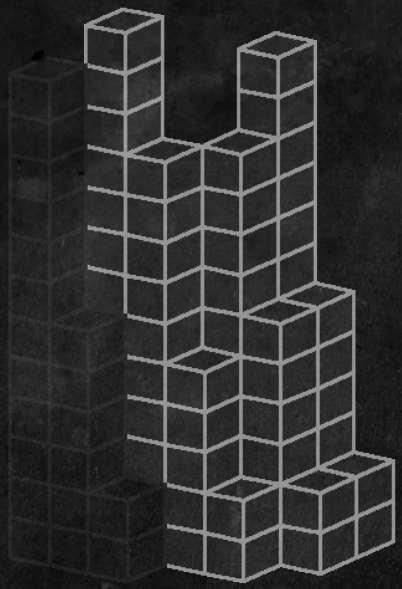
The (sideways) Hasse diagram of the slices (profile is $c = (1, 0, 2)$):



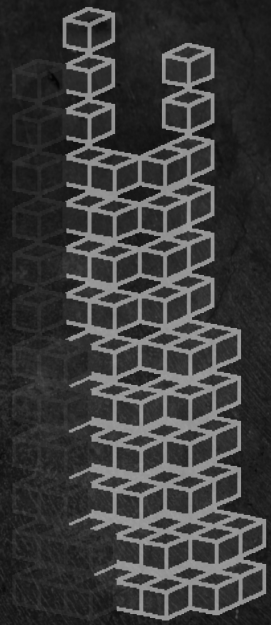
To represent a cylindric partition on a path diagram:



Let's remember the slices in the running example:



$\rightarrow$



Theorem (Kurşungöz, Ömrüuzun Seyrek, 2025): For any fixed profile c, the collection of cylindric partitions with profile c is in 1-1 correspondence with pairs

$$(\text{ an ordinary partition }, \text{ a chain of pivots }).$$

The chain of pivots is the "navigational data" on the path diagram.

Theorem: For a chosen and fixed profile c,

$$\sum_{\Lambda \in \mathcal{P}_c} q^{|\Lambda|} z^{\max(\Lambda)} u^{\mathrm{st}(\Lambda)}$$

$$= \frac{((1-u)zq;q)_\infty}{(zq;q)_\infty} \left[1 + \sum_{m \geq 1} \sum_{\substack{\Pi_1 < \Pi_2 < \cdots < \Pi_m \\ \text{a chain of } m \text{ pivots}}} \prod_{j=1}^{m} \frac{uzq^{|\Pi_j|}}{1-(1-u)zq^{|\Pi_j|}} \right].$$

Idea of proof:

In case of no pivots, for any weight n, the factor in the generating function will be:

$$1 + uzq^n + uz^2q^{2n} + \cdots = 1 + u\sum_{i\geq 0}(zq^n)^i = 1 + \frac{uzq^n}{1-zq^n} = \frac{1-(1-u)zq^n}{1-zq^n}.$$

Any pivot of length n must clear this factor first, and then replace it with

$$uzq^n + uz^2q^{2n} + \cdots = \frac{uzq^n}{1-zq^n}$$

Applications:

$$\sum_{\Lambda \in \mathcal{P}_{(1,1)}} q^{|\Lambda|} z^{\max(\Lambda)} u^{\mathrm{st}(\Lambda)} = \frac{\left((1-u)zq^2;q^2\right)_\infty \left((1-2u)zq;q^2\right)_\infty}{(zq;q)_\infty}$$

$$\sum_{\Lambda \in \mathcal{P}_{(2,0)}} q^{|\Lambda|} z^{\max(\Lambda)} u^{\mathrm{st}(\Lambda)} = \frac{\left((1-u)zq;q^2\right)_\infty \left((1-2u)zq^2;q^2\right)_\infty}{(zq;q)_\infty}.$$

Applications:

$$\sum_{\Lambda\in\mathcal{P}_{(2,1)}} q^{|\Lambda|} z^{\max(\Lambda)} u^{\mathrm{st}(\Lambda)} = 1 + \sum_{m\geq 1} \sum_{0<j_1<j_2<\cdots<j_m} g_{r_1} g_{r_2} \cdots g_{r_s} \prod_{k=1}^{m} \frac{uzq^{j_k}}{1-zq^{j_k}}$$

where the sequence $j_1 < j_2 < \ldots < j_m$ consists of exactly s maximal runs with lengths $r_1, r_2, \ldots, r_s$ in their respective order; and

$$g_1 = 2, \quad g_2 = 3, \quad \text{and} \quad g_r = g_{r-1} + g_{r-2} \text{ for } r > 2,$$

i.e. the shifted Fibonacci numbers.

e.g. $2, 3, 4, 5, 8, 10, 11, 12, 17, 20, 21$ has maximal runs

$$2, 3, 4, 5, \qquad 8, \qquad 10, 11, 12, \qquad 17, \qquad 20, 21,$$

so the associated factor of g's is

$$g_4\ g_1\ g_3\ g_1\ g_2 = 8 \cdot 2 \cdot 5 \cdot 2 \cdot 3$$

Bijections to ordinary partitions

Here is the simplest example:

Theorem: Let $\mathcal{C}^d_{(1,1)}$ denote the set of cylindric partitions $\Lambda = (\lambda^{(1)}, \lambda^{(2)})$ of profile $(1,1)$, written as pairs

$$P_k = [\lambda^{(2)}_k, \lambda^{(1)}_k] = [b_k, a_k], \qquad 1 \leq k \leq N,$$

where

$$N = \max\{\ell(\lambda^{(1)}), \ell(\lambda^{(2)})\},$$

with zeros appended to the shorter component when necessary, such that $a_k - b_k \in \{0,1\}$ for every k; and in which every part from 1 to $\max(\Lambda)$ appears. Then, $\mathcal{C}^d_{(1,1)}$ is in 1-1 correspondence with partitions into distinct parts.

The corresponding generating function is:

$$\sum_{\Lambda \in \mathcal{C}^d_{(1,1)}} q^{|\Lambda|} z^{\max(\Lambda)} u^{\mathrm{st}(\Lambda)} = (-uzq; q)_\infty$$

The approach:

Choose and fix a path on the path diagram
(not necessarily the default path)
so that no "navigational data" is needed.
Consequently, there is a unique shape for each weight.

Then, find out how to characterize this
on the cylindric partition side.

Proof by example:

Start with a partition into distinct parts.

$$(12, 11, 8, 6, 5, 3, 2)$$

$\longrightarrow$ 

Now stack them all:

$$\longrightarrow \Lambda = \begin{array}{cccccccc} & & & 7 & 6 & 5 & 3 & 2 & 2 \\ & & 7 & 5 & 4 & 3 & 2 & 1 & \\ & 7 & 6 & 5 & 3 & 2 & 2 & & \end{array}$$

Indeed, all parts from 1 to $\max(\Lambda) = 7$ appear, and $0 \leq a_k - b_k \leq 1$ for $k = 1, 2, \ldots, 6$.

We have two more theorems involving profiles $(1, 1)$ and $(2, 0)$.

Distinct Parts

How can we characterize cylindric partitions into distinct parts?

Observations:

- All slice sizes must appear from 1 to largest part, say n ($u^n\, q^{1+2+\cdots+n} = q^{\binom{n+1}{2}}$).
- They may be repeated as many times as we want ($\times \frac{1}{(q;q)_n}$, no more u's).
- Each such cylindric partition determines a path of length n in the path diagram, starting with the zero slice. ($\times a_n$, the number of paths of length n).

The problem reduces to finding $\displaystyle\sum_{n\geq 0} \frac{a_n u^n q^{\binom{n+1}{2}}}{(q;q)_n}$.

We show that $a_{rn+s} = \displaystyle\sum_{i,j} n^i \alpha_j^n$,

i.e., a_{rn+s} satisfies a linear recurrence with constant coefficients. (for fixed r and s)

Distinct Parts

Theorem: Let $\mathcal{D}_c(m,n)$ denote the number of cylindric partitions of n with profile c into m distinct parts. Set

$$D_c(u;q) = \sum_{m,n\geq 0} \mathcal{D}_c(m,n) u^m q^n.$$

Then, there exist a rational function $rg_c(u;q)$ which depends on the profile c, $k \in \mathbb{N}$ and doubly indexed complex numbers $\alpha_{\dots}^{\dots}$, $\beta_{\dots}^{\dots}$ such that $D_c(u;q)$ is the following finite linear combination of infinite products and infinite products multiplied by Lambert series, up to a correction by the rational function $rg_c(u;q)$.

$$\begin{aligned} D_c(u;q) = rg_c(u;q) &+ \sum_{i=1}^{B_0} \alpha_i^0 \ (-\beta_i^0 uq;q)_\infty \\ &+ \sum_{i=1}^{B_1} \alpha_i^1 \ \left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)(-\beta_i^1 uzq;q)_\infty\right]_{z=1} \\ &\vdots \\ &+ \sum_{i=1}^{B_k} \alpha_i^k \ \left[\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^k (-\beta_i^k uzq;q)_\infty\right]_{z=1} \end{aligned}$$

Here, $\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^m (-\beta zq;q)_\infty$ for $m > 0$ can be calculated as

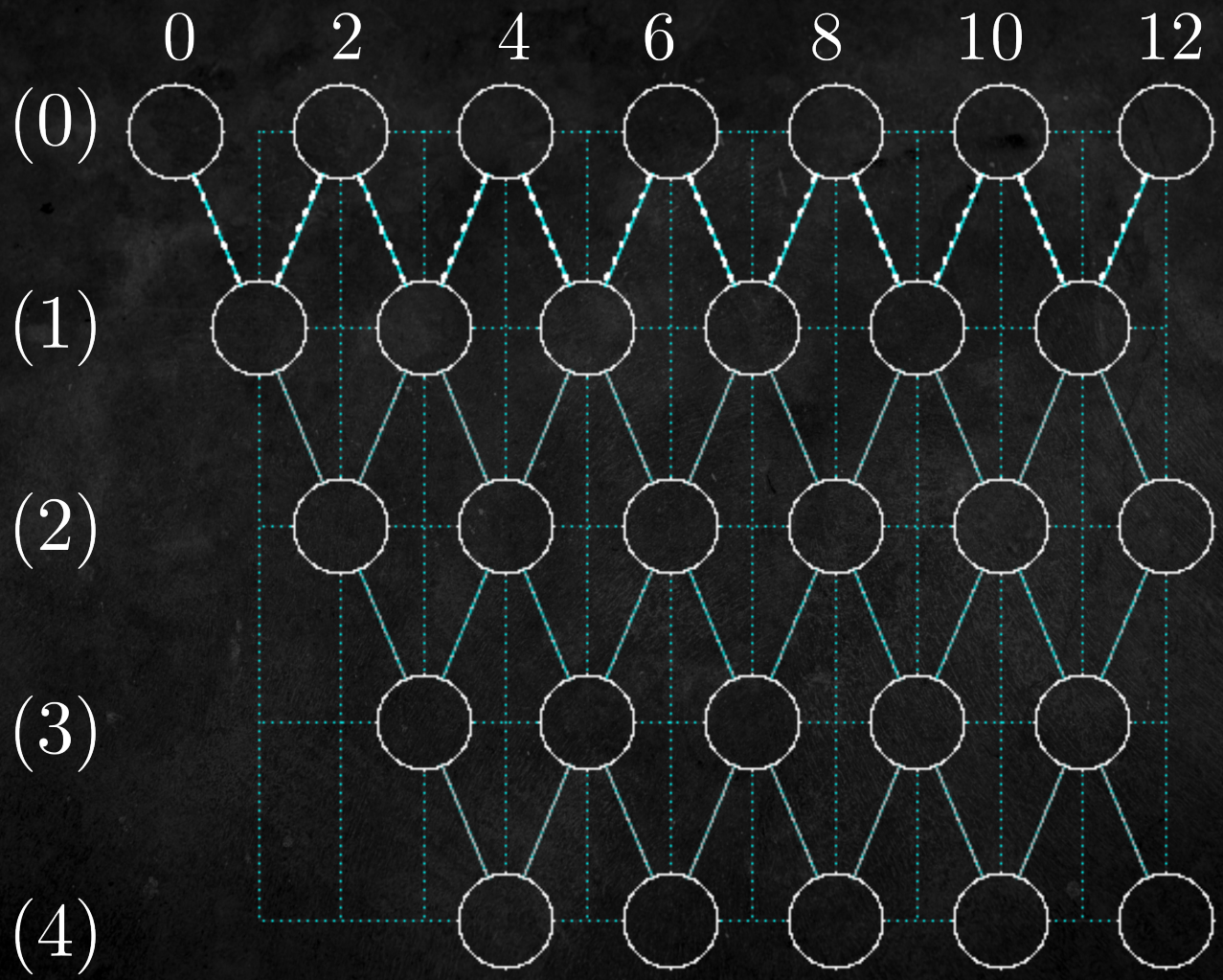
$(-\beta zq;q)_\infty \times$ a polynomial in $\displaystyle\sum_{n\geq 0}\left(\frac{\beta zq^n}{1+\beta zq^n}\right)^j$ for various j's.

The last kind of series is known as *Lambert series*.

Unfortunately, this is still an existence result.

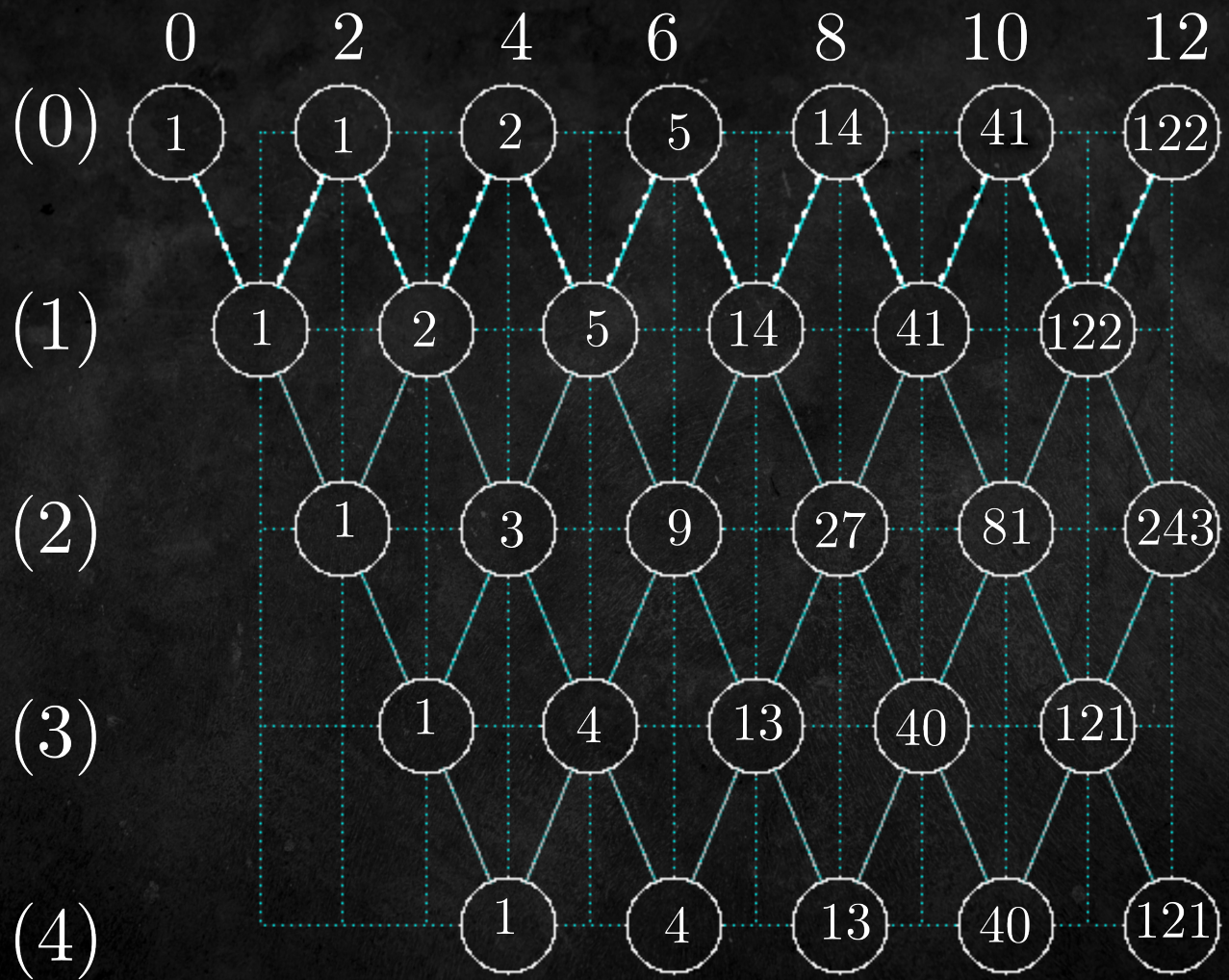
Examples

Concrete example, profile $c = (4, 0)$:



Examples

Concrete example, profile $c = (4, 0)$:



Examples

We see that $a_{2n} = 2 \cdot 3^{n-1}$ except for a_0, and $a_{2n+1} = 3^n$.
The rest is standard q-series manipulations.

$$D_{(4,0)}(u;q) = 1 + \sum_{n\geq 0} \frac{3^n u^{2n+1} q^{\binom{2n+2}{2}}}{(q;q)_{2n+1}} + 2\sum_{n\geq 1} \frac{3^{n-1} u^{2n} q^{\binom{2n+1}{2}}}{(q;q)_{2n}}$$

$$= \frac{1}{3} + \frac{1}{\sqrt{3}} \sum_{n\geq 0} \frac{(\sqrt{3}u)^{2n+1} q^{\binom{2n+2}{2}}}{(q;q)_{2n+1}} + \frac{2}{3}\left[-1 + \sum_{n\geq 0} \frac{(\sqrt{3}u)^{2n} q^{\binom{2n+1}{2}}}{(q;q)_{2n}}\right]$$

$$= 1 + \frac{1}{\sqrt{3}}\left[\frac{(\sqrt{3}uq;q)_\infty - (-\sqrt{3}uq;q)_\infty}{2}\right] + \frac{2}{3}\left[\frac{(\sqrt{3}uq;q)_\infty + (-\sqrt{3}uq;q)_\infty}{2} - 1\right]$$

$$= \frac{1}{3} + \left(\frac{2+\sqrt{3}}{6}\right)(\sqrt{3}uq;q)_\infty + \left(\frac{2-\sqrt{3}}{6}\right)(-\sqrt{3}uq;q)_\infty$$

Let $f_c(n;u) := f_c(n;u,q) = \sum_{\substack{\Lambda \text{ with profile } c \\ \max(\Lambda)\leq n}} q^{|\Lambda|}u^{\mathrm{st}(\Lambda)}$.

Theorem: For an arbitrary but fixed profile $c = (c_1, c_2, \ldots, c_r)$ with rank r and level $\ell = c_1 + c_2 + \cdots + c_r$,

$$f_c(0;u) = 1,$$

and for $n \geq 1$

$$f_c(n;u) = f_c(n-1;u) + \frac{uq^{nr}}{(1-q^{nr})} f_c(n-1;u) + \sum_{d\neq c} \frac{uq^{n\Delta(d,c)}}{(1-q^{nr})} f_d(n-1;u),$$

where the sum is over all profiles d except for c with rank r and level ℓ.

$\Delta(d,c)$: the smallest number of boxes needed to make a slice with shape d into a slice with shape c.

Application and correction

$$F_c(u,z) = \frac{(1-(1-u)zq^r)}{(1-zq^r)} F_c(u,zq^r) + uz \sum_{d \neq c} \frac{q^{\Delta(d,c)}}{1-zq^{\Delta(d,c)}} F_d(u, zq^{\Delta(d,c)}).$$

Future work

- Find other bijections, especially for larger profiles.
- Following "Partitions with designated summands" (Andrews, Lewis and Lovejoy), consider "cylindric partitions with designated pivots".
- Make the crude generating function look nicer.

Gratitude

Thank you!

Questions?

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