Palindrome Partitions and Calkin-Wilf Tree

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Partition Theory Seminar
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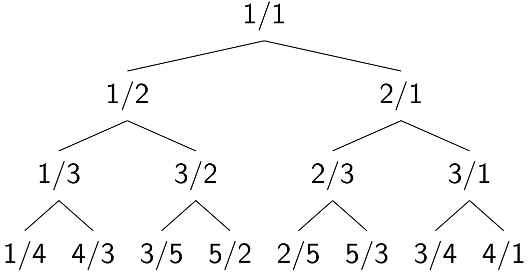
*This is joint work with my advisor David Hemmer

Background Information

In number theory, the Calkin–Wilf tree is a tree in which the vertices correspond one-to-one to the positive rational numbers. The tree is rooted at the number 1, and any rational number expressed in simplest terms as the fraction $\frac{a}{b}$ has as its two children the numbers $\frac{a}{a+b}$ (move to the left) and $\frac{a+b}{b}$. (move to the right)



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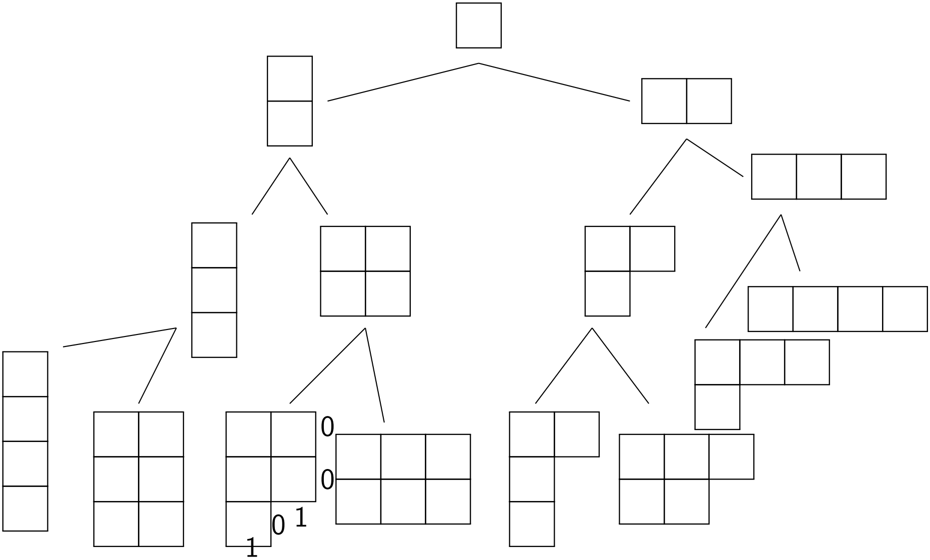

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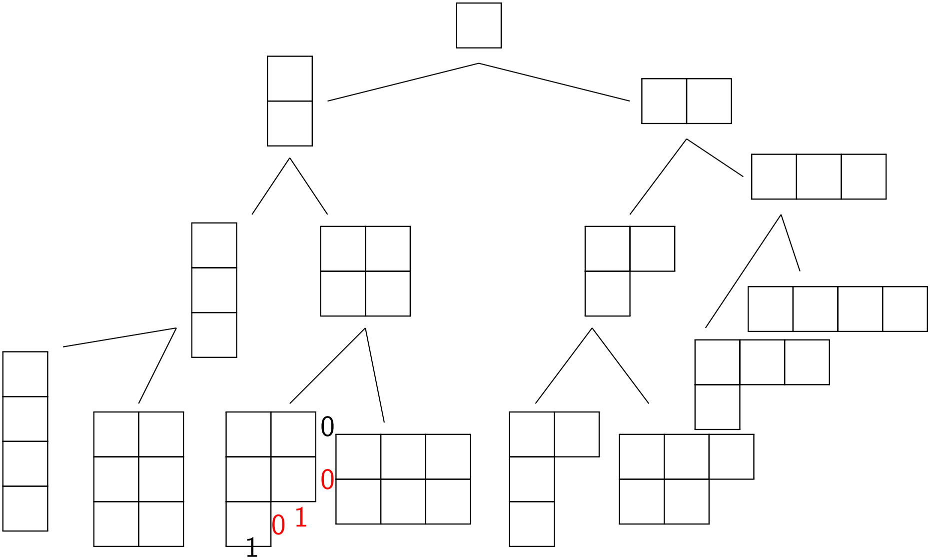


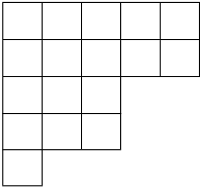
Partition Theory

Recall that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a partition of n, denoted $\lambda \vdash n$, if $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > 0$ are positive integers with $\sum_{i=1}^{r} \lambda_i = n$.

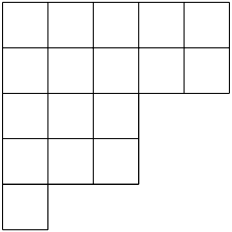
$\lambda = (5, 5, 3, 3, 1)$ is a partition of $n = 17$.

To λ we associate its *Young diagram* $[\lambda]$.

$[\lambda] =$



Partition Sequences



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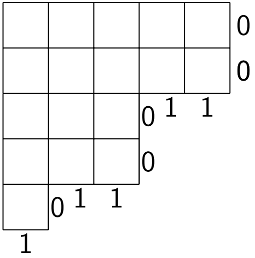
Partition Sequences



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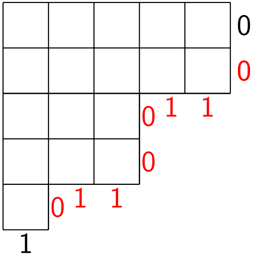

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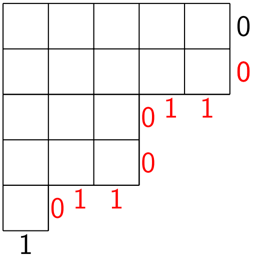

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For what partitions of n is the corresponding binary sequence a palindrome? How many are there for each n?

Palindrome Sequences

Definition

Let $PP(n)$ be the number of palindrome sequences whose corresponding integer partition is a partition of n.

n	$PP(n)$	n	$PP(n)$	n	$PP(n)$	n	$PP(n)$
1	1	11	10	21	12	31	38
2	2	12	2	22	2	32	34
3	2	13	8	23	36	33	18
4	2	14	10	24	12	34	46
5	4	15	10	25	14	35	104
6	2	16	2	26	24	36	2
7	4	17	18	27	36	37	20
8	4	18	2	28	2	38	46
9	6	19	20	29	60	39	108
10	2	20	16	30	2	40	2

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9	6	19	20	29	60	39	108
10	2	20	16	30	2	40	2

Palindrome Partitions

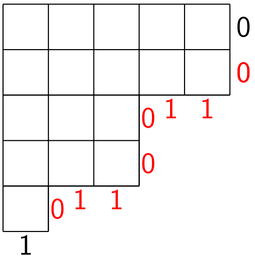
Theorem

The number $PP(n) = 2$ if and only if $n = 3$ or $n + 1$ is prime.

Which the two palindrome sequences are all zeros and all ones corresponding to (1^n) and (n) respectively.

Palindrome Partitions

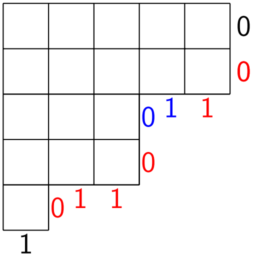
Q: How do we calculate the number of palindromes for n?



We have one of three cases, an odd # of 1's, an odd # of 0's, or both # of 1's and 0's is even. Given a palindrome sequence of n, swapping a "01" in one spot means swapping a "10" in another spot.

Palindrome Partitions

Q: If $n \neq 3$ or $n+1$ is not prime, how do we calculate the number of palindromes for n?



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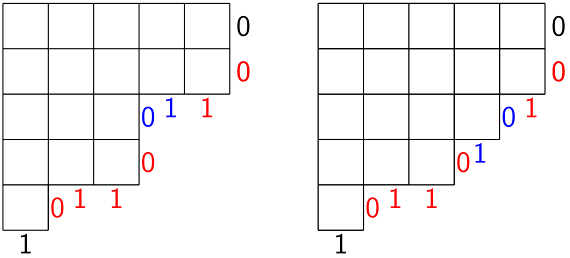

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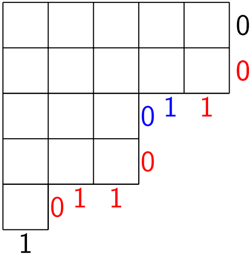

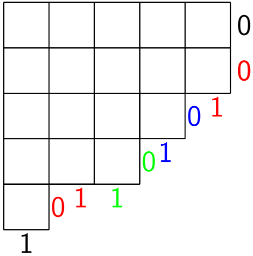

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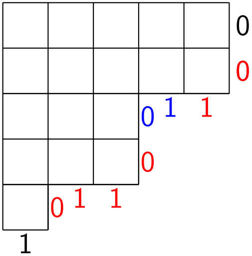
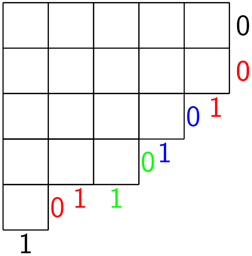
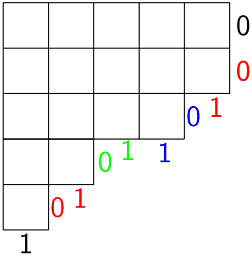




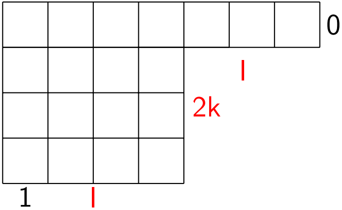


We have one of three cases, an odd number of 1's, an odd number of 0's, or an even number of 1's and 0's. Given a palindrome sequence of n, swapping a "01" in one spot means swapping a "10" in another spot.

Palindrome Partitions

We have one of three cases, an odd number of 1's, an odd number of 0's, or an even number of 1's and 0's.
Case 3: we can will assume that the palindrome sequence looks like

$$\underbrace{11\ldots1}_{l}\underbrace{00\ldots0}_{2k}\underbrace{11\ldots1}_{l}$$



$n = 2kl + l + 1 + l + 2k \quad$ and $2(n+1) = (2k+2)(2l+2)$

Palindrome Partitions

Case	$\lambda_1' - 1 = \#$ 0's	$\lambda_1 - 1 = \#$ 1's	2(n+1)
1	2k+1	2l	(2l+2)(2k+3)
2.	2k	2l+1	(2k+2)(2l+3)
3.	2k	2l	(2k+2)(2l+2)

If $n+1=p$ is prime, then solving for k, l for each case, we find that the only solutions are when $k=0, l=(p-3)/2$ and $k=(p-3)/2, l=0$.

Palindrome Partitions

Table: Palindrome partitions of $n = 11$

λ	11	$7,4$	$5^2,1$	$5,4,2$	$5,3^2$	$3^3,1^2$	$3^2,2^2,1$
Sequence	1^{10}	$1^3 0 1^3$	01^40	101101	110011	001100	010010
length	10	7	6	6	6	6	6

λ	$3,2^4$	$2^4,1^3$	1^{11}
Sequence	10^41	$0^3 1\, 0^3$	0^{10}
length	6	7	10

We only have 3 possible lengths.

Definition

Let $PL(n)$ be the number of lengths among all sequences as λ runs over all palindrome partitions of n. This is also the number of distinct perimeters among all palindrome partitions of n.

Palindrome Partitions

Theorem

$PL(n)$ is the number of factorizations $xy = 2(n+1)$ where $0 < x \leq y \leq n$. This is the sequence `https://oeis.org/A211270` *shifted by one. Moreover, suppose there is a palindrome partition $\lambda \vdash n$ with the palindrome sequence having length $m = a + b$ with a zeros and b ones. Then any other palindrome partition $\mu \vdash n$ with palindrome sequence of length m must have a zeros and b ones or b zeros and a ones.*

If $n = 11$, then $2(n+1) = 24$, which has 3 factorizations namely, $2 \cdot 12, 3 \cdot 8$, and $4 \cdot 6$.

We will use the factorizations of $2(n+1)$ to find all palindrome partitions for the given n.

Palindrome Partitions for n=29

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Palindrome Partitions for n=29

$2(n+1) = 60 = x \cdot y$				
$2 \cdot 30$	$2 = 2k + 2$	$30 = 2l + 2$	$2k = 0$	$2l = 28$
$3 \cdot 20$				
$4 \cdot 15$				
$5 \cdot 12$				
$6 \cdot 10$				

Palindrome Partitions for n=29

$2(n+1) = 60 = x \cdot y$	# of zeros	# of ones	# of λ	Example
$2 \cdot 30$	0	28	1	(29)
$2 \cdot 30$	28	0	1	(1^{29})
$3 \cdot 20$				
$4 \cdot 15$				
$5 \cdot 12$				
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$3 \cdot 20$	1	18	1	$(19, 10)$
$3 \cdot 20$	18	1	1	$(2^{10}, 1^9)$
$4 \cdot 15$	2	13	$\binom{7}{1}$	$(14, 14, 1)$
$4 \cdot 15$	13	2	$\binom{7}{1}$	$(3, 2^{13})$
$5 \cdot 12$	3	10	$\binom{6}{1}$	$(11, 11, 6, 1)$
$5 \cdot 12$	10	3	$\binom{6}{1}$	$(4, 3^5, 2^5)$
$6 \cdot 10$	4	8	$\binom{6}{2}$	$(9^3, 1, 1)$
$6 \cdot 10$	8	4	$\binom{6}{2}$	$(5^5, 1^4)$

Palindrome Partitions

Theorem

We have the following generating function:

$$\sum_{n=0}^{\infty} PP(n)q^n$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k+l}{k}q^{2kl+2k+2l+1} + 2\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{k+l}{k}q^{2kl+2k+3l+2}.$$

Palindrome Partitions

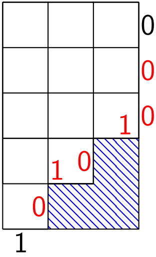
James Sellers provided this simplification to the last theorem using the binomial series $\frac{1}{(1-x)^{n+1}} = \sum_{j=0}^{\infty} \binom{n+j}{j} x^j$.

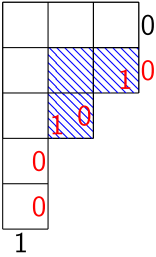
Theorem

$$\sum_{n=0}^{\infty} PP(n)q^n = \sum_{k=0}^{\infty} \frac{q^{2k+1}}{(1-q^{2k+2})^{k+1}} + 2\frac{q^{2k+2}}{(1-q^{2k+3})^{k+1}}$$

Conjugating inner Young diagram

Consider λ with sequence $B(\lambda)$ that has A zeros and B ones. Consider the boxes not in the first row or column as a partition $\tilde{\lambda}$ sitting inside an $A \times B$ rectangle. We will replace $\tilde{\lambda}$ with the partition obtained by taking its complement inside the $A \times B$ rectangle and rotating it 180 degrees, while preserving the first row and column of λ. Then the new diagram will be called $P(B(\lambda)^r)$.





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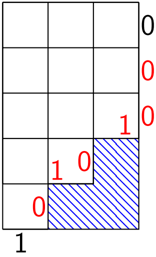

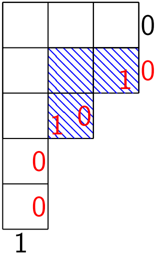

Let $R(n)$ be the number of partitions $\lambda \vdash n$ where $P(B(\lambda)^r) \vdash n$.

Partitions with weight fixed by reversal

We require that it fill exactly half of the boxes in the $2k \times 2l$ or $(2k+1) \times 2l$ rectangle. Recall that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the generating function for partitions that fit inside an $(n-k) \times k$ rectangle, and is called a Gaussian polynomial. The coefficient of q^m counts partitions of m fitting inside a $(n-k) \times k$ rectangle, so it has leading term $q^{k(n-k)}$.

Partitions with weight fixed by reversal

Definition

Let $T(n,k)$ be the number of nondecreasing sequences of length n, with integer entries in $[-k,k]$, summing to zero.

For example if $n=5$ and $k=4$ a possible sequence would be $\{-4,-2,1,1,4\}$. The array $T(n,k)$ is given as sequence A183917 in the OEIS.

Proposition

The coefficient of q^{kl} in the Gaussian polynomial $\left[{2k+l \atop l}\right]_q$ is $T(l,k)$.

We give a bijection $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \vdash kl$ to the corresponding sequence $\{\lambda_l - k, \lambda_{l-1} - k, \dots, \lambda_1 - k\}$.

Partitions with weight fixed by reversal

Theorem

We have the following generating function:

$$\sum_{n=0}^{\infty} R(n)q^n$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T(2l,k)q^{2kl+2k+2l+1} + 2\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T(k+1,l)q^{2kl+2k+3l+2}.$$

Future research

Problem

Can we write the generating functions for $R(n)$ in a more compact form $\sum f(n)q^n$ similar to the generating function for $PP(n)$? Can we see clearly from the generating function why $f(n) = 2$ when $n+1$ is prime?

Problem

The traditional Young's lattice has all partitions of n in row n with edges corresponding to removing and/or adding a single box. Classically this describes the branching of irreducible representations of the symmetric group. Does the branching diagram have any representation-theoretic interpretation?

Future research

Problem

Can we determine if a fraction in the Calkin-Wilf tree corresponds to a palindrome without doing the continued fraction expansion?

Thank you

And a big thanks to Dave, as well as Matthew Just and Robert Schneider for suggesting looking at this operation on partitions and William Keith for help with generating functions. Also, a thanks to James Sellers for calculating the simplification of the generating function for $PP(n)$.