Distribution of Partition Statistics in Arithmetic Progressions

K. Bringmann, W. Craig, J. Males, and K. Ono

The Partition function $p(n)$

Definition

A **partition** of an integer n is any nonincreasing sequence

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$$4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 \implies p(4) = 5.$$

Ramanujan's Legacy

Theorem (Hardy and Ramanujan)

We have that

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Theorem (Ramanujan)

For every n, we have that

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Dyson's Rank

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Example

The ranks of the partitions of 4:

Partition	Largest Part	# Parts	Rank
4	4	1	$3 \equiv 3 \pmod 5$
$3+1$	3	2	$1 \equiv 1 \pmod 5$
$2+2$	2	2	$0 \equiv 0 \pmod 5$
$2+1+1$	2	3	$-1 \equiv 4 \pmod 5$
$1+1+1+1$	1	4	$-3 \equiv 2 \pmod 5$

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Conjecture (Dyson, 1944)

For every n and every a, we have

$$N(a, 5; 5n + 4) = p(5n + 4)/5,$$

$$N(a, 7; 7n + 5) = p(7n + 5)/7.$$

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Remark

Consequences of harmonic Maass form theory ("mock modularity").

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$$p(n) = C(0, b; n) + C(1, b; n) + \cdots + C(b-1, b; n)?$$

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$$p(n) = C(0, b; n) + C(1, b; n) + \cdots + C(b-1, b; n)?$$

2. *For instance, do we have* **equidistribution**

$$\lim_{n \to +\infty} \frac{C(a, b; n)}{p(n)} = \frac{1}{b} \ ?$$

New Examples

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What can be said about the distributions of

- *Partition hook numbers?*

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What can be said about the distributions of

- *Partition hook numbers?*
- *Betti numbers of Hilbert schemes?*

Hook Numbers

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Each partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ has a *Ferrers-Young diagram*

$$\begin{array}{cccccc} \bullet & \bullet & \bullet & \cdots\bullet & \leftarrow & \lambda_1 \text{ nodes} \\ \bullet & \bullet & \dots & \bullet & \leftarrow & \lambda_2 \text{ nodes} \\ \vdots & \vdots & \vdots & & & \\ \bullet & \dots & \bullet & & \leftarrow & \lambda_m \text{ nodes}, \end{array}$$

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where λ'_j is the number of nodes in column j.
A t-**hook** is any hook number which is a multiple of t.

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The partition $\Lambda = 5 + 4 + 1$, has Young diagram

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Therefore, we have

$$\begin{aligned} &\text{(Hooks)} && \mathcal{H}(\Lambda) = \{1,1,1,2,3,3,4,5,5,7\} \\ &\text{(2 Hooks)} && \mathcal{H}_2(\Lambda) = \{2,4\} \\ &\text{(3 Hooks)} && \mathcal{H}_3(\Lambda) = \{3,3\}. \end{aligned}$$

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(1) The p-divisibility of hook numbers dictates divisibility of $\dim(V_\Lambda)$.
(2) Granville-O solved Brauer's Problem 19 by classifying $\mathcal{H}_p(\Lambda) = \emptyset$.

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Theorem (Nekrasov-Okounkov, 2003)

For any complex number z, we have

$$\prod_{n=1}^{\infty}(1-q^n)^{z-1}=\sum_{\Lambda} q^{|\Lambda|}\prod_{h\in\mathcal{H}(\Lambda)}\left(1-\frac{z}{h^2}\right).$$

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Theorem (Han, 2008)

For $t \in \mathbb{N}$, roots of unity ζ and $z \in \mathbb{C}$ we have

$$\prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{(1-(\zeta q^t)^n)^{t-z}(1-q^n)} = \sum_{\Lambda} q^{|\Lambda|} \prod_{h\in\mathcal{H}_t(\Lambda)} \left(\zeta-\frac{\zeta t z}{h^2}\right).$$

Eclipses classical theta function identities

- (Euler) $q\prod_{n=1}^{\infty}(1-q^{24n}) = q - q^{25} - q^{49} + q^{121} + q^{169} - \dots$.
- (Jacobi) $q\prod_{n=1}^{\infty}(1-q^{8n})^3 = q - 3q^{9} + 5q^{25} - 7q^{49} + 11q^{121} - \dots$.
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$$\sum_{\Lambda \vdash n} \prod_{h \in \mathcal{H}(\Lambda)} \left(1 - \frac{z}{h^2}\right).$$

(2) The N-O and Han q-series "are" modular forms when $z \in \mathbb{Z}$.

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Question

For large n, what is the distribution of

$$p(n) = p_t(0,b;n) + p_t(1,b;n) + \cdots + p_t(b-1,b;n) \ ?$$

Number of 3-hooks modulo 3

n	$\Psi_3(0,3;n)$	$\Psi_3(1,3;n)$	$\Psi_3(2,3;n)$
100	≈ 0.4356	≈ 0.1639	≈ 0.4003
$\vdots$	$\vdots$	$\vdots$	$\vdots$
500	≈ 0.3234	≈ 0.3670	≈ 0.3096
600	≈ 0.3318	≈ 0.3114	≈ 0.3567
$\vdots$	$\vdots$	$\vdots$	$\vdots$
2100	≈ 0.3320	≈ 0.3348	≈ 0.3332
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Remark

The number of 3*-hooks seems to be* **equidistributed** *modulo 3.*

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300	≈ 0.7347	≈ 0.2653	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
600	≈ 0.6977	≈ 0.3022	0
900	≈ 0.6837	≈ 0.3163	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
4500	≈ 0.6669	≈ 0.3330	0
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Questions

1. *What is going on?*
2. *Does it matter that the n in the table are multiples of 3?*

Number of 4-hooks modulo 3

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n	$\Psi_4(0,3;12n)$	$\Psi_4(1,3;12n)$	$\Psi_4(2,3;12n)$
10	≈ 0.4804	≈ 0.3373	≈ 0.1823
$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	≈ 0.4500	≈ 0.3381	≈ 0.2119
60	≈ 0.4485	≈ 0.3373	≈ 0.2142
$\vdots$	$\vdots$	$\vdots$	$\vdots$
180	≈ 0.4447	≈ 0.3340	≈ 0.2212
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Speculation

$$\lim_{n\to+\infty} \Psi_4(a,3;12n) = \begin{cases} 4/9 & \text{if } a=0 \\ 1/3 & \text{if } a=1 \\ 2/9 & \text{if } a=2 \end{cases} \quad ?$$

Distributions

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Theorem (B-C-M-O)

If $t > 1$ and $0 \leq a < b$, where b is an odd prime, then we have

$$p_t(a, b; n) \sim \frac{c_t(a, b; n)}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

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Moreover, the function $c_t(a,b;n)$ is **periodic in n modulo b.**

Remarks

1. *Equidistribution requires $c_t(a,b;n) = 1/b$.*
2. *Theorem known earlier for $t = 2$ by Craig and Pun.*

FORMULAS FOR $c_t(a,b;n)$

DEFINITION

If $0 \leq a < b$, where b is an odd prime, then

$$c_t(a,b;n) := \frac{1}{b} + \begin{cases} 0 & \text{if } b|t, \\ (-1)^{\frac{(1-t)(b-1)}{4}} \mathbb{I}(a,b,t,n) b^{-\frac{t+1}{2}} \left(\frac{t}{b}\right) & \text{if } b \nmid t \text{ and } t \text{ is odd}, \\ (-1)^{\frac{(1-t)(b-1)}{4}} b^{-\frac{t}{2}} \left(\frac{\frac{1}{24}(1-t^2)(1-b^2)+at-n}{b}\right) \varepsilon_b & \text{if } b \nmid t \text{ and } t \text{ is even}. \end{cases}$$

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Here $\mathbb{I}(a, b, t, n)$ is a certain residue class indicator function, and $\varepsilon_b := 1$ (resp, i) when $b \equiv 1 \pmod 4$ (resp. $b \equiv 3 \pmod 4$).

Formulas for $c_t(a,b;n)$

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Remark

We have **equidistribution** *precisely when* $b \mid t$.

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(2) *If* $\ell \equiv 2 \pmod 3$ *and* $\mathrm{ord}_\ell(-9a_1 + 3a_2 + 1) = 1$, *then we have*

$$p_3\left(a_1, \ell^2; \ell^2 n + a_2\right) = 0.$$

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(2) *If $\ell \equiv 2 \pmod 3$ and $\mathrm{ord}_\ell(-9a_1 + 3a_2 + 1) = 1$, then we have*

$$p_3\left(a_1, \ell^2; \ell^2 n + a_2\right) = 0.$$

Question

The proof uses theta functions and weight 1 Eisenstein series.

VANISHING FOR $t \in \{2,3\}$

THEOREM (B-C-M-O)

The following are true for primes ℓ.

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QUESTION

The proof uses theta functions and weight 1 Eisenstein series. Is there an "elementary combinatorial proof"?

Examples

Example (2 hooks)

For $\ell = 3$, part (1) implies

$$p_2(0,3;3n+2) = p_2(1,3;3n+1) = p_2(2,3;3n) = 0.$$

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Example (3 hooks)

For $\ell = 2$, part (2) gives

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For $\ell \equiv 2 \pmod{3}$ and each $0 \le a_1 < \ell^2$, there are $\ell - 1$ choices for a_2.

Hilbert schemes on n points

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Definition

The nth **Hilbert scheme of a projective variety** S is a "smoothed" version of the nth symmetric product of S.

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Denote the Hilbert scheme of n points of $\mathbb{C}^2$ by $(\mathbb{C}^2)^{[n]}$.
For $0 \leq a < b$, we define the modular sums of Betti numbers

$$B\left(a, b; (\mathbb{C}^2)^{[n]}\right) := \sum_{j \equiv a \pmod{b}} \dim\left(H_j\left((\mathbb{C}^2)^{[n]}, \mathbb{Q}\right)\right).$$

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The homology is **labelled by the partitions of** n, and so we have

$$p(n) = \sum_{a=0}^{b-1} B\left(a, b; \left(\mathbb{C}^2\right)^{[n]}\right).$$

Distribution Functions

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Question

For large n, what is the distribution of

$$p(n) =$$

$$B\left(0, b; \left(\mathbb{C}^2\right)^{[n]}\right) + B\left(1, b; \left(\mathbb{C}^2\right)^{[n]}\right) + \cdots + B\left(b-1, b; \left(\mathbb{C}^2\right)^{[n]}\right) \ ?$$

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Definition

If $0 \leq a < b$, then define the **Betti distribution functions**

$$\delta(a, b; n) := \frac{B\left(a, b; (\mathbb{C}^2)^{[n]}\right)}{p(n)}.$$

Numerical Example

n	$\delta(0,3;n)$	$\delta(1,3;n)$	$\delta(2,3;n)$
1	1	0	0
2	0.5000	0	0.500
$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	≈ 0.3377	≈ 0.3325	≈ 0.3299
19	≈ 0.3367	≈ 0.3306	≈ 0.3327
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Remark

This looks like equidistribution!

Betti number distribution

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Theorem (B-C-M-O)

As $n \to \infty$, we have

$$B\left(a,b;\left(\mathbb{C}^2\right)^{[n]}\right) \sim \frac{d(a,b)}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}},$$

where

$$d(a,b) := \begin{cases} \frac{1}{b} & \text{if } b \text{ is odd,} \\ \frac{2}{b} & \text{if } a \text{ and } b \text{ are even,} \\ 0 & \text{if } a \text{ is odd and } b \text{ is even.} \end{cases}$$

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Remarks

(1) We have **equidistribution** *for odd b.*

(2) We have **equidistribution over even classes** *modulo even b.*

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$$G_s(q) := \sum_{\Lambda} z^{s(\Lambda)} q^{|\Lambda|}.$$
2. For $0 \leq a < b$ and $\zeta_b := e^{2\pi i/b}$, we have
$$G_s(a, b; q) := \frac{1}{b} \sum_{r=0}^{b-1} \zeta_b^{-ar} G_s(\zeta_b^r; q) = \sum_{\substack{\Lambda \\ s(\Lambda) \equiv a \pmod{b}}} q^{|\Lambda|}.$$

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3. Therefore, by Cauchy's Theorem we get

$$\#\{\Lambda \vdash n \text{ with } s(\Lambda) \equiv a \pmod{b}\} = \frac{1}{2\pi i} \int_C \frac{G_s(a, b; q)}{q^{n+1}} dq.$$

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$$p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{k \geq 1} \frac{A_k(n)}{k} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

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2. "Wright's Variant" is sufficient for asymptotics for many other generating functions.

Theorem (Wright)

Suppose the following for $F(q) = \sum c(n)q^n = L(q)\xi(q)$.

(1) *For* $k \geq 1$, *as* $|z| \to 0$ *in the cone* $|\mathrm{Arg}(z)| < \frac{\pi}{2} - \delta$, *we have*

$$L(e^{-z}) = \frac{1}{z^B}\left(\sum_{j=0}^{k-1} \alpha_j z^j + O_\delta\left(z^k\right)\right)$$
$$\xi(e^{-z}) = z^\beta e^{\frac{c^2}{z}}\left(1 + O_\delta\left(e^{\frac{-\gamma}{z}}\right)\right).$$

(2) *As* $|z| \to 0$ *in the cone* $\frac{\pi}{2} - \delta \leq |\mathrm{Arg}(z)| < \frac{\pi}{2}$, *we have* $L(e^{-z}) \ll_\delta z^{-C}$.
(3) *As* $|z| \to 0$ *in the bounded cone* $\frac{\pi}{2} - \delta \leq |\mathrm{Arg}(z)| < \frac{\pi}{2}$, *we have*

$$|\xi(e^{-z})| \ll_\delta \xi(|e^{-z}|)e^{-\frac{\varrho}{z}}.$$

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If (1)-(3) *hold, then as* $n \to \infty$ *we have for any* $R \in \mathbb{R}^+$

$$c(n) = n^{\frac{1}{4}(2B-2\beta-3)} e^{2c\sqrt{n}}\left(\sum_{r=0}^{R-1} p_r N^{-\frac{r}{2}} + O\left(N^{-\frac{R}{2}}\right).\right),$$

t-HOOK GENERATING FUNCTION

t-hook generating function

Theorem (Han, 2008)

As formal power series, we have

$$H_t(z;q) := \sum_{\Lambda \in \mathcal{P}} z^{\#\mathcal{H}_t(\Lambda)} q^{|\Lambda|} = \frac{1}{F_2(z;q^t)^t} \cdot \prod_{n=1}^{\infty} \frac{\left(1-q^{tn}\right)^t}{1-q^n},$$

where $F_2(z;q) := \prod_{n=1}^{\infty} \left(1-(zq)^n\right).$

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1. *Note that* $H_t(1;q) = \sum_{n=0}^{\infty} p(n)q^n$.

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REMARKS

1. *Note that* $H_t(1;q) = \sum_{n=0}^{\infty} p(n) q^n.$
2. *Dedekind's eta-function* $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ *is a weight 1/2 modular form in* $q := e^{2\pi i \tau}.$

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2. *Dedekind's eta-function* $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ *is a weight 1/2 modular form in* $q := e^{2\pi i \tau}$.
3. *For roots of unity* $z = \zeta$, *we have that* $H_t(\zeta;q)$ *is "essentially" a weight -1/2 modular form.*

Betti Numbers for n point Hilbert schemes

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Theorem (Göttsche 1992)

As formal power series, we have that

$$G(z;q) := \sum_{n=0}^{\infty} P\left(\left(\mathbb{C}^2\right)^{[n]};z\right) q^n = \prod_{m=1}^{\infty} \frac{1}{1-z^{2m-2}q^m},$$

where $P\left(\left(\mathbb{C}^2\right)^{[n]};z\right)$ *is the Poincaré polynomial.*

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Remarks

1. *Note that* $G(1;q) = \sum_{n=0}^{\infty} p(n)q^n$.
2. *At roots of unity* $z=\zeta$, *this* q*-series is not a modular form.*

Some Infinite Products

Definition

For roots of unity z, we define

$$F_1(z;q) := \prod_{n=1}^{\infty} (1 - zq^n),$$

$$F_2(z;q) := \prod_{n=1}^{\infty} (1 - (zq)^n),$$

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Remarks

1. *We need to determine their behavior as $q \to$ roots of unity.*
2. $F_2(\zeta;q)$ *is a twisted Dedekind eta-function.*

Asymptotics of $F_2(\zeta; q)$

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Lemma

Let $\zeta = e^{\frac{2\pi i a}{b}}$ *and* $q = e^{\frac{2\pi i}{k}(h+iz)}$, *where* $\gcd(h,k) = 1$.

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Then as $z \to 0$ *we have*

$$F_2\left(\zeta; q^t\right) \sim \omega^{-1}_{\frac{ht\lambda_{h,a}}{k}+\frac{\lambda_{h,a}a}{b},\lambda_{h,a}} \left(\frac{\lambda_{h,a}tz}{k}\right)^{-\frac{1}{2}} e^{-\frac{\pi k}{12\lambda_{h,a}^2 tz}}.$$

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Remarks

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2. *RHS depends on automorphy factors (i.e. Dedekind sums).*

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(1) *Assume that $\zeta \neq 1$. As $x \to 0^+$, we have*

$$F_1\left(\zeta; e^{-x}\right) = \frac{1}{\sqrt{1-\xi}}\, e^{-\frac{\zeta\Phi(\zeta,2,1)}{x} - \frac{\zeta x}{12(1-\zeta)} + O\left(x^2\right)}.$$

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ASYMPTOTICS FOR $F_1(\zeta;q)$ AND $F_3(\zeta;q)$

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$$\sum_{n=a}^{b} f(n) \sim \int_a^b f(x)\,dx + \frac{f(b)+f(a)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right),$$

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where $I_h := \int_0^\infty h(u)du$.

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   where $I_h := \int_0^\infty h(u)du$.

3. Lots of little details to get right.

Leads to a Key Lemma

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Lemma 2.2. *Assume that* $f(x) \sim \sum_{n=n_0}^{\infty} c_n x^n$ *as* $x \to 0^+$. *For* $0 < a \leq 1$ *and any* $A \in \mathbb{R}^+$, *we have, as* $x \to 0^+$, *that*

$$\sum_{n=0}^{\infty} f((n+a)x) \sim \sum_{n=n_0}^{-2} c_n \zeta(-n,a) x^n + \frac{I_{f,A}^*}{x} - \frac{c_{-1}}{x}\left(\log(Ax) + \psi(a) + \gamma\right) - \sum_{n=0}^{\infty} c_n \frac{B_{n+1}(a)}{n+1} x^n,$$

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1. *Apply lemma to the logarithms of* $F_1(\zeta; q)$ *and* $F_3(\zeta; q)$.

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Remarks

1. *Apply lemma to the logarithms of* $F_1(\zeta; q)$ *and* $F_3(\zeta; q)$.
2. *Exponentiate to get the asymptotics.*
3. *Inserting into the Circle Method(s) gives our results!*

t-hook Distributions

Theorem (B-C-M-O)

If $t > 1$ and $0 \leq a < b$, where b is an odd prime, then we have

$$p_t(a,b;n) \sim \frac{c_t(a,b;n)}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}}.$$

Moreover, we have **equidistribution** *precisely when $b \mid t$.*

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Remark

For $t \in \{2, 3\}$ there is a web of arithmetic progressions with

$$p_2(a_1, \ell; \ell n + a_2) = 0 \quad p_3(a_1, \ell^2; \ell^2 n + a_2) = 0.$$

Betti numbers on n point Hilbert schemes

Theorem (B-C-M-O)

As $n \to \infty$, we have

$$B\left(a, b; \left(\mathbb{C}^2\right)^{[n]}\right) \sim \frac{d(a,b)}{4\sqrt{3}n} \cdot e^{\pi\sqrt{\frac{2n}{3}}},$$

where

$$d(a,b) := \begin{cases} \frac{1}{b} & \text{if } b \text{ is odd,} \\ \frac{2}{b} & \text{if } a \text{ and } b \text{ are even,} \\ 0 & \text{if } a \text{ is odd and } b \text{ is even.} \end{cases}$$

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Remarks

(1) We have **equidistribution for odd** *b.*

(2) We have **equidistribution over odd classes** *modulo even b.*