Euler's Partition Theorem and Refinements Without Appeal to Infinite Products*

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*Appeared in *Algorithmic Combinatorics - in honor of Peter Paule on his 60th birthday*, (V. Pillwein & C. Schneider Eds.), Springer 2020 9–23.

Outline of the talk

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1) Introduction

One of the first results an entrant to the theory of partitions encounters is Euler's fundamental and beautiful theorem:

Theorem E

Let $p_d(n)$ and $p_o(n)$ denote the number of partitions of n into distinct parts and odd parts respectively. Then

$$p_d(n) = p_o(n).$$

Euler's proof of Theorem E made use of product representations of the generating functions of $p_d(n)$ and $p_o(n)$:

$$\sum_{n=0}^{\infty} p_d(n)q^n = \prod_{m=1}^{\infty}(1+q^m) = \prod_{m=1}^{\infty}\frac{(1-q^{2m})}{(1-q^m)} = \prod_{m=1}^{\infty}\frac{1}{(1-q^{2m-1})} = \sum_{n=0}^{\infty} p_o(n)q^n. \tag{1}$$

Euler's theorem and his fundamental idea, namely the replacement of expressions of the form $1+y$ by $(1-y^2)/(1-y)$ in products and the study of the resulting the cancellations, plays a crucial role in the theory of partitions. Many proofs of Euler's theorem are known and a variety of important refinements of it have been obtained by Sylvester, Fine, Bessenrodt and others. Our approach here is to prove Euler's theorem by *only* considering the series generating function of $p_d(n)$ and an important (but under-utilized) *amalgamation property* of this series. We then convert the series generating function of $p_d(n)$ to the series generating function of $p_o(n)$ by a suitable *dissection* of the terms of the series. We use 2-modular Ferrers graphs to establish the equivalence. We then combine these ideas with the conjugation of the Ferrers graphs of partitions into distinct parts to improve a refinement of Euler's theorem due to Fine and to obtain a dual of a refinement due to Bessenrodt.

Sylvester improved upon many partition theorems of Euler by combinatorial methods. Using a graphical representation, he was led to the Theorem S which is a refinement of Theorem E. It is not easy to establish that his graphical representation yields a bijective proof of Theorem S; this was done by D. Kim and A. J. Yee in 1999.

Theorem S

The number of partitions of an integer n into odd parts of which exactly k are different is equal to the number of partitions of n into distinct parts which can be grouped into k (maximal) blocks of consecutive integers.

Yet another refinement of Theorem E was found by Fine, namely,

Theorem F

Let $p_d(n;k)$ denote the number of partitions of n into distinct parts with largest part k. Let $p_o(n;k)$ denote the number of partitions of n into odd parts such that the largest part plus twice the number of parts is $2k+1$. Then

$$p_d(n;k) = p_o(n;k).$$

Fine observed this in 1954 but published it only in his 1988 monograph. Fine's proof of Theorem F was not combinatorial, but q-theoretic. Andrews (1976) provided a q-theoretic proof, but prior to that in 1966, noted that Theorem F also falls out from Sylvester's graphical proof of Theorem E. Theorems S and F are *refinements* of Theorem E because by summing over k we get Theorem E. We will provide a simpler approach to Theorem E which yields a much simpler and very direct proof of Theorem F.

Bessenrodt obtained the following elegant reformulation of Theorem E from Sylvester's bijection for Theorem S. This refinement is also a limiting case of the deep lecture hall partition refinement of Theorem E due to Bousquet-Melou and Eriksson.

Theorem B

Let $p_{d,k}(n)$ denote the number of partitions of n into distinct parts such that the alternating sum starting with the largest part is k. Let $p_{o,k}(n)$ denote the number of partitions of n into odd parts with total number of parts equal to k. Then

$$p_{d,k}(n) = p_{o,k}(n).$$

We will provide a simple direct proof of Theorem B by considering conjugates of the Ferrers graphs of partitions into distinct parts and then using the amalgamation-dissection ideas to convert the generating function of $p_{d,k}(n)$ into that of $p_{o,k}(n)$. This also yields a dual of Theorem B and an improvement of Theorem F.

Examples of some recent works pertaining to Theorem E (including its analogs and Glaisher's generalization to all odd moduli) and Theorem B, emphasizing combinatorial arguments, are the papers of Berkovich-Uncu (2016), Straub (2016), and of Xiong and Keith (2019). Our approach is quite different.

Notation

We shall use the standard notation

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j), \tag{2}$$

and

$$(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a)_n = \prod_{j=0}^{\infty}(1 - aq^j), \quad \text{when} \quad |q| < 1. \tag{3}$$

When the base is q, we write $(a)_n$ as in (2) for simplicity, but when the base is anything other than q, it will be displayed.

For any partition π, we let

$$\lambda(\pi) = \text{largest part of}\,\pi,$$

$$\nu(\pi) = \text{number of parts of}\,\pi,$$

and

$$\sigma(\pi) = \text{the sum of the parts of}\,\pi.$$

Finally let D denote the set of partitions into distinct parts, and Ω, the set of partitions into odd parts.

2) New proof of Theorem E

The series generating function of $p_d(n)$ is

$$\sum_{n=0}^{\infty} p_d(n) q^n = \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q)_m}. \tag{4}$$

The term

$$\frac{q^{m(m+1)/2}}{(q)_m} \tag{5}$$

is the generating function of partitions into distinct parts with exactly m parts. These terms have an interesting *amalgamation property*:

$$\frac{q^{(2m-1)2m/2}}{(q)_{2m-1}} + \frac{q^{2m(2m+1)/2}}{(q)_{2m}} = \frac{q^{2m^2-m}}{(q)_{2m}}, \tag{6}$$

and its companion

$$\frac{q^{2m(2m+1)/2}}{(q)_{2m}} + \frac{q^{(2m+1)(2m+2)/2}}{(q)_{2m+1}} = \frac{q^{2m^2+m}}{(q)_{2m+1}}. \tag{7}$$

These amalgamation properties have not been fully exploited, and we shall use them here. We note from (4) and (6) that

$$\sum_{n=0}^{\infty} p_d(n) q^n = 1 + \sum_{m=1}^{\infty} \frac{q^{2m^2-m}}{(q)_{2m}}. \tag{8}$$

Next we dissect the denominator terms in (8) into its odd and even components. So we rewrite (8) as

$$1 + \sum_{m=1}^{\infty} \frac{q^{2m^2-m}}{(q)_{2m}} = 1 + \sum_{m=1}^{\infty} \frac{q^{2m^2-m}}{(q^2;q^2)_m (q;q^2)_m}. \tag{9}$$

We now show that the series on the right of (9) is the generating function of $p_o(n)$.

Represent a partition into odd parts as a 2-modular Ferrers graph, namely a Ferrers graph in which there is a 1 at the node on the extreme right of each row, and there is a 2 at every other node. Consider the Durfee square in this 2-modular Ferrers graph, namely the largest square of nodes starting from the upper left hand corner. Let the Durfee square be of dimension $m \times m$. Now the part below this Durfee square is a partition into odd parts the largest of which is $\leq 2m-1$. The generating function of such partitions is

$$\frac{1}{(q;q^2)_m}. \tag{10}$$

The portion consisting of the Durfee square and the nodes to its right forms a partition into exactly m odd parts each $\geq 2m-1$. If $2m-1$ is removed from each of the m rows of this part of the graph, we remove $2m^2-m$ in total. The remaining portion is a 2-modular Ferrer's graph with only twos in it, and which, if read columnwise, is a partition into even parts each $\leq 2m$. Thus the generating function of the Durfee square and portion to its right is

$$\frac{q^{2m^2-m}}{(q^2;q^2)_m}. \tag{11}$$

Thus we have shown that

$$1+\sum_{m=1}^{\infty}\frac{q^{2m^2-m}}{(q^2;q^2)_m(q;q^2)_m}=\sum_{n=0}^{\infty}p_o(n)q^n. \tag{12}$$

Theorem E follows from (12), (9) and (8).

3) Simple proof of Theorem F

By following the ideas of Euler that we described prior to (5), we get

$$\sum_{n=0}^{\infty} \frac{z^n t^n q^{n(n+1)/2}}{(tq)_n} = \sum_{\pi \in D} z^{\nu(\pi)} t^{\lambda(\pi)} q^{\sigma(\pi)}. \tag{13}$$

Even though this two parameter refined generating function of partitions into distinct parts is fundamental, it has not been given much attention because it does not have a product representation. However, if we set $t = 1$ and count only the number of parts, then we get a product representation for the expression in (13), namely

$$\prod_{k=0}^{\infty} (1 + zq^k),$$

and this product has been investigated in detail since the time of Euler. When we set $z = 1$ in (13) and keep track only of the largest part, we *do not* get a product representation for the series

$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n+1)/2}}{(tq)_n} = \sum_{\pi \in D} t^{\lambda(\pi)} q^{\sigma(\pi)}. \tag{14}$$

Our emphasis here is on series and not infinite products. The terms of the series on the left in (14) amalgamate as in (6) and (7). That is, we have,

$$\frac{t^{2m-1}q^{(2m-1)2m/2}}{(tq)_{2m-1}} + \frac{t^{2m}q^{2m(2m+1)/2}}{(tq)_{2m}} = \frac{t^{2m-1}q^{2m^2-m}}{(tq)_{2m}}. \tag{15}$$

and its companion

$$\frac{t^{2m}q^{2m(2m+1)/2}}{(tq)_{2m}} + \frac{t^{2m+1}q^{(2m+1)(2m+2)/2}}{(tq)_{2m+1}} = \frac{t^{2m}q^{2m^2+m}}{(tq)_{2m+1}}. \tag{16}$$

Thus using (15) we get

$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n+1)/2}}{(tq)_n} = 1 + \sum_{m=1}^{\infty} \frac{t^{2m-1}q^{2m^2-m}}{(tq)_{2m}}. \tag{17}$$

As in (9), we dissect the denominator on the right in (17) and rewrite it as

$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n+1)/2}}{(tq)_n} = 1 + \sum_{m=1}^{\infty} \frac{t^{2m-1} q^{2m^2-m}}{(tq^2;q^2)_m (tq;q^2)_m}. \quad (18)$$

We will now combinatorially interpret the coefficient of t^k in (18). We know already from (10) and (11) that the expression

$$\frac{t^{2m-1} q^{2m^2-m}}{(tq^2;q^2)_m (tq;q^2)_m} \quad (19)$$

without the parameter t, is the generating function of partitions π^* into odd parts whose 2-modular Ferrers graphs have an $m \times m$ Durfee square. How are the powers of t generated in the expression in (19) and what does the power of t represent in relation to the partition π^*?

To understand this, write the exponent k in the power of t, say t^k, in (19) as

$$k = (2m-1) + i + j, \tag{20}$$

where t^i is generated from the factor $(tq^2; q^2)_m$, t^j is generated by the factor $(tq; q^2)_m$, and t^{2m-1} comes from the numerator. Thus from the arguments underlying (11), we see that in the 2-modular graphs under consideration, $(2m-1)+2i = \lambda(\pi^*)$, the largest part. Also $m+j = \nu(\pi^*)$, the number of parts, because there are j parts below the Durfee square, and the Durfee square is of size $m \times m$. So (20) yields

$$\lambda(\pi^*) + 2\nu(\pi^*) = (2m-1) + 2i + 2(m+j) = 2k+1. \tag{21}$$

On the other hand, the coefficient of t^k in (14) is the generating function of the number of partitions into distinct parts with largest part k. So by comparing coefficients of t^k on both sides of (18), we get Theorem F from (14) and (20).

Remark:

Fine's proof of Theorem F was not combinatorial but involved transformations of q-series. Our proof given above is simpler and more direct and it leads to the following refinement of Theorem F.

A further refinement: In the amalgamation in (6) and (7) we are adding the generating functions of partitions π into distinct parts for which $\nu(\pi) = 2m-1$ or $2m$. After amalgamation and dissection, we interpreted the expression in (19) as the generating function of partitions into odd parts π^* whose 2-modular Ferrer's graphs have an $m \times m$ Durfee square. Thus we have the following refinement of Theorem F:

Theorem F*:

Let $p_d(n;k,m)$ denote the number of partitions π of n into distinct parts with $\lambda(\pi) = k$ and $\nu(\pi) = 2m-1$ or $2m$. Let $p_o(n;k,m)$ denote the number of partitions π^* of n into odd parts with $\lambda(\pi^*) + 2\nu(\pi^*) = 2k+1$ and such that the 2-modular Ferrers graph of π^* has a Durfee square of dimension $m \times m$. Then

$$p_d(n;k,m) = p_o(n;k,m).$$

4) Conjugation of partitions into distinct parts

If π is a partition into distinct parts, then its conjugate π^* is a partition whose set of parts is the set of consecutive integers from 1 up to $\nu(\pi)$. Thus letting ν denote $\nu(\pi)$, we have

$$\sigma(\pi) = \sigma(\pi^*) = \sum_{i=1}^{\nu} i f_i, \tag{22}$$

where f_i represents the frequency with which i occurs in π^*. So the sum on the right in (22) is

$$\sigma(\pi^*) = (f_1 + f_2 + \ldots + f_\nu) + (f_2 + f_3 + \ldots + f_\nu) + \ldots + (f_\nu), \tag{23}$$

where the quantities within the parenthesis represent the parts of π in decreasing order.

Let $s(\pi)$ denote the alternating sum of the parts of π starting with the largest part. Then

$$\begin{aligned} s(\pi) &= (f_1 + f_2 + \ldots + f_\nu) - (f_2 + f_3 + \ldots + f_\nu) \\ &\quad + (f_3 + f_4 + \ldots + f_\nu) - (f_4 + f_5 + \ldots + f_\nu) + \ldots \\ &= f_1 + f_3 + f_5 + \ldots =: \nu_o(\pi^*), \end{aligned} \tag{24}$$

where

$$\nu_o(\pi^*) = \text{the number of odd parts of } \pi^*.$$

So Theorem B can be reformulated as follows:

Theorem C:

Let C denote the set of partitions with the property that all integers up to the largest part occur as parts. Let $p^*_C(n;\ell)$ denote the number of partitions π^* of $n, \pi^* \in C$, with $\nu_o(\pi^*) = \ell$. Let $p^*_o(n;\ell)$ denote the number of partitions of n into odd parts such that the number of parts is ℓ. Then

$$p^*_C(n;\ell) = p^*_o(n;\ell).$$

A partition with the property that all integers up to the largest part occur as parts is known as a partition without gaps. We call such partitions as *chain partitions*, and C is the set of such partitions. We now give a simple proof of Theorem C.

Proof of Theorem C using series: For $\pi^* \in C$, consider $\lambda(\pi^*)$. If $\lambda(\pi^*) = 2j-1$, then the generating function of such chain partitions π^* is

$$\frac{q^{2j^2-j} z^j}{(zq;q^2)_j (q^2;q^2)_{j-1}}, \tag{25}$$

where the power of z in (25) is $\nu_o(\pi^*)$. If $\lambda(\pi^*) = 2j$, then the generating function is

$$\frac{q^{2j^2+j} z^j}{(zq;q^2)_j (q^2;q^2)_j}. \tag{26}$$

Now if we add the expressions in (25) and (26), they amalgamate to

$$\frac{q^{2j^2-j}z^j}{(zq;q^2)_j(q^2;q^2)_j}.$$

So we get

$$\sum p_C(n;\ell)z^\ell t^n = \sum_{j=0}^{\infty} \frac{q^{2j^2-j}z^j}{(zq;q^2)_j(q^2;q^2)_j}. \tag{27}$$

Just as we showed (12) via Durfee squares, it follows that the series on the right in (27) is

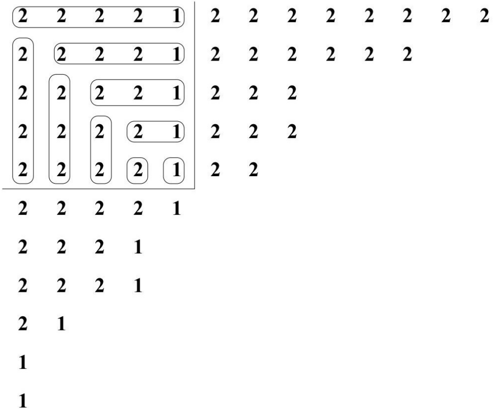
$$\sum_{j=0}^{\infty} \frac{q^{2j^2-j}z^j}{(zq;q^2)_j(q^2;q^2)_j} = \sum_n \sum_\ell p_o^*(n;\ell)z^\ell q^n. \tag{28}$$

Theorem C follows from (27) and (28) without any appeal to infinite products.

FIG 1: 5×5 DURFEE SQUARE

2	2	2	2	2	2	2	2	2	2	2	2	1
2	2	2	2	2	2	2	2	2	2	1		
2	2	2	2	2	2	2	1					
2	2	2	2	2	2	2	1					
2	2	2	2	2	2	1						
2	2	2	2	1								
2	2	2	1									
2	2	2	1									
2	1											
1												
1												

FIG 2: MODIFIED 5×5 DURFEE SQUARE



Graphical proof of Theorem C:

We now provide a bijective proof of Theorem C using 2-modular graphs.

Represent a partition π into odd parts as a 2-modular graph. We illustrate our bijective proof by considering the partition 25+21+15+15+13+9+7+7 +7+3+1+1. In this 2-modular graph, mark out the Durfee square. This is illustrated in Fig. 1.

Next delete the right most column of the Durfee square, fill them with ones, and move any twos that were in the right most column to the extreme right position on the same row. Thus the integer entries in the modified Durfee square (say of dimension j) add up to $2j^2 - j$. Group the integer entries in the modified Durfee square as indicated in Fig 2 to see that these represent the integers $1, 2, ..., 2j - 1$. The rows below the modified square represent odd parts $\leq 2j - 1$. The columns to the right of the modified square represent even parts $\leq 2j$. Thus if the modified graph is viewed in this fashion, we get a chain partition π^* with largest part either $2j - 1$ or $2j$. Notice that the number of parts of π equals then number of odd parts of π^* and this proves Theorem C.

5) A dual of Theorem C and an improvement of Theorem F

The graphical proof of Theorem C has interesting implications.

In the graphical proof given above, we focused on the number of parts.

We now see what happens if we consider the largest part.

Suppose the size of the largest part of the partition π represented in Fig 1 is $2k+1$. Let the size of the Durfee square and the modified Durfee square to be j. So the largest odd part $\lambda_o(\pi^*)$ of the chain partition π^* given by Fig. 2 is $2j-1$. Consequently, $2, 4, ..., 2j-2$ occur as even parts of π^* and these account for $j-1$ even parts. The number of even parts of π^* is given by $j-1$ plus the number of columns to the right of the modified Durfee square. Interpret $j-1$ as the number of twos in the first row of the modified Durfee square in Fig. 2. Thus the number of even parts of π^* is k. This leads to the following *dual* of Theorem C:

Theorem C*:

Let $p^*_{o,k}(n)$ denote the number of partitions of n into odd parts with largest part $2k+1$.
Let $p^*_{C,k}(n)$ denote the number of chain partitions of n with k even parts. Then

$$p^*_{o,k}(n) = p^*_{C,k}(n).$$

Remarks:

(i) Theorem C* is a dual of Theorem C because in Theorem C we counted the number of odd parts of π whereas in Theorem C* we count the size of the largest odd part of π; similarly in Theorem C we count the number of odd parts of π^* whereas in Theorem C* we count the number of even parts of π^*. Thus by reformulating Theorem B in terms of Theorem C, we have arrived at the dual Theorem C*.

(ii) By combining Theorems C and C* we get Fine's Theorem F. This is because with $\nu_e(\pi^*)$ representing the number of even parts of π^*, we can write Theorem C* in the form

$$\lambda(\pi) = 2\nu_e(\pi^*) + 1. \tag{29}$$

Similarly we may write Theorem C in the form

$$\nu(\pi) = \nu_o(\pi^*). \tag{30}$$

Thus (29) and (30) yield

$$\lambda(\pi) + 2\nu(\pi) = 2\nu_e(\pi^*) + 1 + \nu_o(\pi^*) = 2\nu(\pi^*) + 1. \tag{31}$$

By taking the conjugate of π^* the number of parts of π^* is converted to the largest part of the conjugate partition, which is a partition into distinct parts, and this is precisely Fine's theorem. Thus Theorems C and C* are improvements of Fine's theorem.

(iii) In section 4 we noted that the infinite series in (27) and (28) can be interpreted in two different ways to be realized as the generating function of the two partition functions $p_C^*(n;\ell)$ and $p_o^*(n;\ell)$ in Theorem C. Similarly, the analytic version of Theorem C* is

$$\sum_n \sum_k p_{o,k}^*(n) w^k q^n = 1 + \sum_{k=0}^{\infty} \frac{w^k q^{2k+1}}{(q;q^2)_{k+1}} = 1 + \sum_{j=1}^{\infty} \frac{w^{j-1} q^{2j^2-j}}{(q;q^2)_j (wq^2;q^2)_j} = \sum_n \sum_k p_{C,k}^*(n) w^k q^n. \tag{32}$$

The series on the right in (32) is the dual of the series in (27) and (28) because in (27) the power of z is counting the number of odd parts, whereas on the right in (32) the power of w is counting the number of even parts.

Identities (27), (28), and (32) can be combined into a single identity as follows:

$$1+\sum_{k=0}^{\infty}\frac{w^k zq^{2k+1}}{(zq;q^2)_{2k+1}}=1+\sum_{j=1}^{\infty}\frac{w^{j-1}z^j q^{2j^2-j}}{(zq;q^2)_j(wq^2;q^2)_j}. \tag{33}$$

In (27) and (28) we did not have the series on the left in (33) with $w=1$ because we did not need it. Instead we interpreted the series on the right in (33) with $w=1$ in two different ways one of which relied on an amalgamation.
The series on the right in (33) is a special case of a certain variant of the Rogers-Fine identity as we shall see in the next section.

6) Connection with the Rogers-Fine identity

In the previous section we studied the 2-modular graphs of partitions into odd parts by keeping track of the largest part and the number of parts. What if we also keep track of the number of different odd parts? This leads us to a variant of the Rogers-Fine identity as we show now.

The Rogers-Fine identity in the form obtained by Fine is

$$F(\alpha,\beta,\tau;q) =: \sum_{n=0}^{\infty} \frac{(\alpha q)_n \tau^n}{(\beta q)_n} = \sum_{n=0}^{\infty} \frac{(\alpha q)_n (\alpha\tau q/\beta)_n \beta^n \tau^n q^{n^2} (1-\alpha\tau q^{2n+1})}{(\beta q)_n (\tau)_{n+1}}. \tag{34}$$

Fine proved it by considering transformation properties of $F(\alpha,\beta,\tau;q)$ defined by the series on the left in (34).

Andrews has given a proof of the Rogers-Fine identity by considering a dilation $q \mapsto q^2$ in (34) and interpreting it combinatorially.

In 2009, I obtained* the following variant of the Rogers-Fine identity:

$$f(a,b,c;q) =: 1 + \sum_{k=1}^{\infty} \frac{(1-a)(abq)_{k-1} bc^k q^k}{(bq)_k}$$

$$= 1 + \sum_{j=1}^{\infty} \frac{b^j c^j q^{j^2}(1-a)(abq)_{j-1}(acq)_{j-1}(1-abcq^{2j})}{(bq)_j (cq)_j}. \tag{35}$$

Fine's function F and our function f are connected by the relation

$$\frac{(1-bq)}{(1-a)bcqI}\{f(a,b,c;q)-1\} = F(ab,bq,cq;q) \tag{36}$$

and so (35) and (36) are equivalent. The reason we investigated (36) was because it is combinatorially more interesting, and also can be established combinatorially in a very direct fashion.

*K. Alladi, *A new combinatorial study of the Rogers-Fine identity and a related theta series*, Int'l J. Num. Th. **5** (2009) 1311–1320 [KA-IJNT].

The function $f(a, b, c; q)$ defined by the series on the left in (35) is the generating function of unrestricted partitions in which the power of b keeps track of the number of parts, the power of c keeps track of the largest part, and the power of $1-a$ keeps track of the number of different parts. It is to be noted that for unrestricted partitions the generating function $f(a, b, c; q)$ has an infinite product representation only when b or c equals 1. When one keeps track of all three statistics $\lambda(\pi)$, $\nu(\pi)$, and $\nu_d(\pi)$ ($=$ the number of different parts of π), then one will NOT have a product representation but will have to deal only with a series representation. This is in line with the philosophy of this paper emphasizing series and removing dependence on infinite product representations. In order to pass from the defining series of f to the series on the right in (35), I studied in the Ferrers graphs of unrestricted partitions using Durfee squares and the fact that under conjugation $\lambda(\pi)$ and $\nu(\pi)$ get interchanged, and $\nu_d(\pi)$ remains invariant. We needed to use the invariance of $\nu_d(\pi)$ under conjugation only on the portion of the Ferrers graph to the right of the Durfee square. This aspect will be crucial in the remark below.

The ideas in [KA-IJNT] can be applied to the 2-modular Ferrers graphs of partitions into odd parts. Without getting into details, we simply point out that what this means is to replace

$$q \mapsto q^2, \quad \text{and} \quad b \mapsto bq^{-1} \tag{37}$$

in (35). This yields

$$\sum_{k=1}^{\infty} \frac{(1-a)(abq;q^2)_{k-1} bc^{k-1} q^{2k-1}}{(bq;q^2)_k}$$
$$= \sum_{j=1}^{\infty} \frac{b^j c^{j-1} q^{2j^2-j}(1-a)(abq;q^2)_{j-1}(acq;q^2)_{j-1}(1-abcq^{4j-1})}{(bq;q^2)_j (cq^2;q^2)_j}. \tag{38}$$

If we set $a=0$, then (38) reduces to (33) with the identifications $b=z$ and $c=w$.

We conclude by showing how Sylvester's theorem can be deduced from (38). For this purpose we state the dual of Sylvester's theorem by replacing partitions into distinct parts by chain partitions. Under conjugation, given a partition π into distinct parts having k maximal blocks of consecutive integers, its conjugate, namely the chain partition π^* will have $k-1$ repeating parts $<$ the largest part. So the dual of Sylvester's theorem is:

Theorem S*:

The number of partitions of an integer into odd parts of which exactly k are different equals the number of chain partitions of that integer having $k-1$ parts less than the largest part that repeat.

Theorem S* can be deduced from (38) as follows. First decompose

$$1 - abcq^{4j-1} = cq^{2j}(1 - abq^{2j-1}) + (1 - cq^{2j}). \tag{39}$$

When $1 - abcq^{4j-1}$ is first replaced by $cq^{2j}(1 - abq^{2j-1})$ on the right in (38), the resulting series can be interpreted as the generating function of chain partitions with largest part even. Similarly when $1-abcq^{4j-1}$ is next replaced by $(1 - cq^{2j})$ on the right in (38), the resulting series can be interpreted as the generating function of chain partitions with largest part odd. Thus Theorem S* will fall out of (38) and (39).

i) In the case of Ferrers graphs of unrestricted partitions, the number of different parts is the number of corners and this is invariant under conjugation. When one considers the 2-modular graphs of partitions into odd parts, the number of different parts is the number of corners, but under conjugation we do not have a 2-modular graph. This awkwardness is circumvented by replacing the graph in Fig 1 by the graph in Fig 2 so that the portion to the right of the Durfee square consists only of twos. The number of different odd parts that are at least as large as $2j-1$ where j is the dimension of the Durfee square is given by the number of corners to the right of the Durfee square and we can keep track of this by conjugation since that portion of the graph in Figure 2 has only twos in it.

ii) Zeng has studied combinatorially the original Rogers-Fine identity (34) under the dilation $q \mapsto q^2$, and with α, β chosen suitably so as to deal with partitions into odd parts. Our approach uses the variant (35) and so is combinatorially more direct. Also we have preferred to replace partitions into distinct parts by chain partitions. Hence there are essential differences between our approach and Zeng's.

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