Applying Asymptotics Methods to Kang and Park's Generalization of the Alder-Andrews Theorem

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Partitions

Definition (Partition)

A partition of a non-negative integer, n, is a sequence of non-increasing natural numbers, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, such that

$$\sum_{i=1}^{k} \lambda_i = n.$$

Each λ_i is called a part of the partition.
The partition function, $p(n)$, counts the number of partitions of n.

An Example

Example (Partitions of 5)

$$
\begin{aligned}
5 &= 5 \\
&= 4 + 1 \\
&= 3 + 2 \\
&= 3 + 1 + 1 \\
&= 2 + 2 + 1 \\
&= 2 + 1 + 1 + 1 \\
&= 1 + 1 + 1 + 1 + 1
\end{aligned}
$$

$$p(5) = 7$$

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$$\sum_{n=0}^{\infty} p(n|\text{distinct parts })q^n = (1+q)(1+q^2)(1+q^3)\cdots = (-q;q)_\infty$$

- Let $p(n|\text{odd parts})$ be the number of partitions of n which have only odd parts.

$$\sum_{n=0}^{\infty} p(n|\text{odd parts })q^n = \frac{1}{(q;q^2)_\infty}$$

Motivating Identities

- Euler: $p(n|\lambda_i \text{ distinct}) = p(n|\lambda_i \text{ odd})$

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- Rogers-Ramanujan 1 (1894, 1914):
$p(n|\lambda_i \text{ 2-distinct}) = p(n|\lambda_i \equiv \pm 1 \mod 5)$

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

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- Schur (1926): the number of partitions of n into parts differing by at least 3 with no consecutive multiples of 3 equals the number of partitions of n with parts congruent to $\pm 1 \mod 6$.

Generalizing the Motivating Identities

- $q_d^{(1)}(n) := p(n|\text{parts differing by at least } d)$ has the generating function:

$$\sum_{n=0}^{\infty} q_d^{(1)}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{d\binom{n}{2}+n}}{(q;q)_n}.$$

- $Q_d^{(1)}(n) := p(n|\text{parts } \equiv \pm 1 \mod d+3)$ has the generating function:

$$\sum_{n=0}^{\infty} Q_d^{(1)}(n)q^n = \frac{1}{(q;q^{d+3})_\infty (q^{d+2};q^{d+3})_\infty}.$$

- When $d = 2$, we recover the first Rogers-Ramanujan identity

$$\sum_{n=0}^{\infty} q_2^{(1)}(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} = \sum_{n=0}^{\infty} Q_2^{(1)}(n)q^n.$$

Alder's Conjecture

- To study the relative sizes of each object, Alder considered their difference: $\Delta_d^{(1)}(n) := q_d^{(1)}(n) - Q_d^{(1)}(n)$.
- By Euler, $\Delta_1^{(1)}(n) = 0$.
- By RR1, $\Delta_2^{(1)}(n) = 0$.
- By Schur, $\Delta_3^{(1)}(n) \geq 0$.
- Alder proved for $d \geq 3$ there exists $n \in \mathbb{N}$ such that $\Delta_d^{(1)}(n) \neq 0$.

Conjecture (Alder, 1956)

For all $d \geq 1, n \geq 0$, $\Delta_d^{(1)}(n) \geq 0$.

Proof of Alder's Conjecture

Theorem (Andrews, 1971)

$\Delta_d^{(1)}(n) \geq 0$ *for all* $n > 0$ *if* $d = 2^r - 1$ *and* $r \geq 4$.

Theorem (Yee, 2008)

$\Delta_d^{(1)}(n) \geq 0$ *for all* $n > 0$ *if* $d = 7$ *or* $d \geq 32$.

Theorem (Alfes, Jameson, Lemke Oliver, [1] 2010)

The Alder-Andrews Conjecture is true.

Asymptotic Expressions for $q_d^{(1)}(n)$, $Q_d^{(1)}(n)$

Theorem (Alfes, Jameson, Lemke Oliver, 2010)

If $d \geq 4$ and n is a positive integer, then

$$Q_d^{(1)}(n) = \frac{(3d+9)^{-\frac{1}{4}}}{4\sin\frac{\pi}{d+3}} n^{-\frac{3}{4}} \exp\left(n^{\frac{1}{2}} \frac{2\pi}{\sqrt{3(d+3)}} \right) + R(n),$$

where $R(n)$ is an explicitly bounded function.

Theorem (Alfes, Jameson, Lemke Oliver, 2010)

Let α be the unique real number in $[0,1]$ satisfying $\alpha^d + \alpha - 1 = 0$, and let $A_d := \frac{d}{2}\log^2\alpha + \sum_{r=1}^{\infty} \frac{\alpha^{rd}}{r^2}$. If n is a positive integer, then

$$q_d^{(1)}(n) = \frac{A_d^{\frac{1}{4}}}{2\sqrt{\pi\alpha^{d-1}(d\alpha^{d-1}+1)}} n^{-\frac{3}{4}} \exp(2\sqrt{nA_d}) + r_d(n),$$

where $|r_d(n)|$ can be explicitly bounded.

Method

- Since the error is explicit, we can compute $N(d)$, the value of n such that for all $n \geq N(d)$

$$\frac{A_d^{\frac{1}{4}}}{2\sqrt{\pi\alpha^{d-1}(d\alpha^{d-1}+1)}} n^{-\frac{3}{4}} \exp(2\sqrt{nA_d}) > |r_d(n)| + \frac{(3d+9)^{-\frac{1}{4}}}{4\sin\frac{\pi}{d+3}} n^{-\frac{3}{4}} \exp\left(n^{\frac{1}{2}}\frac{2\pi}{\sqrt{3(d+3)}}\right) + |R(n)|$$

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- Alfes, Jameson, and Lemke Oliver found that $N(d)$ is on the order of 10^6.
- This means the asymptotics guarantee that for all $n > 10^6$, Alder's Conjecture is true!
- What about $n < 10^6$? Check that $q_d(n) \geq Q_d(n)$ for all $4 \leq d \leq 31$ and $1 \leq n \leq N(d)$.

Generalization of RR2

Theorem (Rogers-Ramanujan 2)

For all integers $n \geq 0$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

From Kang and Park (2020):

- $q_d^{(a)}(n) := p(n|parts \geq a, \text{ and parts differ by at least } d)$
- $Q_d^{(b)}(n) := p(n|\text{parts} \equiv \pm b \pmod{d+3})$
- and $\Delta_d^{(a,b)}(n) = q_d^{(a)}(n) - Q_d^{(b)}(n)$.
- When $a = b = d = 2$, we recover the second Rogers-Ramanujan identity.
- When $a = b$ we write $\Delta_d^{(a)}(n) = \Delta_d^{(a,a)}(n)$.

Kang and Park's Conjecture

- $\Delta_3^{(2)}(4) = -1$.
- Let $Q_d^{(2,-)}(n)$ be the number of partitions of n into parts congruent to $\pm 2 \pmod{d+3}$ excluding parts equal to $d+1$.

Conjecture (Kang Park[4] 2020)

For all $n \geq 0, d \geq 1$,

$$\Delta_d^{(2,-)}(n) := q_d^{(2)}(n) - Q_d^{(2,-)}(n) \geq 0.$$

Theorem (Duncan Khunger Swisher Tamura [2], 2020)

For $d \geq 62$ and $n \geq 1$,

$$\Delta_d^{(2,-)}(n) \geq 0.$$

My Project

- Given the work by Duncan et al., there are finitely many values of d left to prove Kang and Park's conjecture for.
- $d = 2$ is done by the second Rogers-Ramanujan identity.
- Thus $d = 1$ and $3 \leq d \leq 61$ remain.
- Our goal is to use the methods used by Alfes et al. to prove Kang and Park's conjecture for as many of the remaining d values as possible.

Asymptotics for $q_d^{(2)}(n)$

- Can we find asymptotic expressions with explicit error bounds for $Q_d^{(2,-)}(n)$ and $q_d^{(2)}(n)$?

Theorem (Duncan et al. 2022)

Let α be the unique real number in $[0,1]$ satisfying $\alpha^d + \alpha - 1 = 0$, and let $A_d := \frac{d}{2}\log^2\alpha + \sum_{r=1}^{\infty} \frac{\alpha^{rd}}{r^2}$. If n is a positive integer, then

$$q_d^{(2)}(n) = \frac{A_d^{\frac{1}{4}}}{2\sqrt{\pi\alpha^{d-3}(d\alpha^{d-1}+1)}} n^{-\frac{3}{4}} \exp(2\sqrt{nA_d}) + r_d(n),$$

where $|r_d(n)|$ can be explicitly bounded.

Asymptotic Approaches

- Meinardus's results do not permit us to find an asymptotic expression for $Q_d^{(2,-)}(n)$ directly.
- By a theorem of Andrews
$$Q_d^{(1)}(n) \geq Q_d^{(2,-)}(n).$$
- Also, trivially
$$Q_d^{(2)}(n) \geq Q_d^{(2,-)}(n).$$
- We can either show $q_d^{(2)}(n) \geq Q_d^{(1)}(n)$ for large positive integers n or we can show $q_d^{(2)}(n) \geq Q_d^{(2)}(n)$ for large positive integers.
- So, we have two different approaches to try, and we will need both!

Asymptotics for $Q_d^{(2)}(n)$

- When d is odd, $d+3$ is even.
- In this case $Q_d^{(2)}(2n+1) = 0$.

Theorem (S- Swisher, 2022)

If $d \geq 4$ is even and n is a positive integer, then

$$Q_d^{(2)}(n) = \frac{(3(d+3))^{-\frac{1}{4}}}{4\sin\frac{2\pi}{d+3}} n^{-\frac{3}{4}} \exp\left(n^{\frac{1}{2}} \frac{2\pi}{\sqrt{3(d+3)}}\right) + R(n),$$

where $R(n)$ is an explicitly bounded function.

Finding $N(d)$

- For even $d \geq 4$, we use the asymptotic expression for $Q_d^{(2)}(n)$ to find the value, $N(d)$ such that for all $n \geq N(d)$

$$q_d^{(2)}(n) \geq Q_d^{(2)}(n) \geq Q_d^{(2,-)}(n).$$

- For odd $d \geq 5$ we instead use the asymptotic expression for $Q_d^{(1)}(n)$ to find $N(d)$ such that

$$q_d^{(2)}(n) \geq Q_d^{(1)}(n) \geq Q_d^{(2,-)}(n)$$

for all $n \geq N(d)$.

Finding $N(d)$

- Write
$$Q_d^{(a)}(n) = M_Q(n) + R(n)$$
and
$$q_d^{(2)}(n) = M_q(n) + r_d(n).$$
- $N(d)$ is the value of n such that for all $n \geq N(d)$
$$M_q(n) \geq M_Q(n) + |R(n)| + |r_d(n)|.$$

Finding $N(d)$

- We find $N(d)$ by taking each term in the sum, $M_Q + R(n) + r_d(n)$, and finding individual weighted bounds N_1, N_2, N_3 such that

$$K_1 M_q(n) \geq M_Q(n),$$

$$K_2 M_q(n) \geq |R(n)|,$$

  and

$$K_3 M_q(n) \geq |r_d(n)|$$

  for all $n \geq N_1, N_2$ and N_3 respectively.
- By choosing weights $0 \leq K_1, K_2, K_3 \leq 1$ such that $K_1 + K_2 + K_3 = 1$ we find $N(d) = \max\{N_1, N_2, N_3\}$.
- More specifically, since $R(n)$ and $r_d(n)$ are sums of other terms, we actually have K_i, N_i for $1 \leq i \leq 8$ because we treat each of the terms in $R(n)$ and $r_d(n)$ separately.

Computed Values of $N(d)$

- For even $6 \leq d \leq 60$ and odd $9 \leq d \leq 61$, $N(d) \leq 10^7$.
- $N(4) = 3.8 \times 10^7$, $N(5) = 1.5 \times 10^8$, and $N(7) = 1.7 \times 10^7$
- We are able to compute $Q_d^{(2)}(n)$, $Q_d^{(2,-)}(n)$, and $q_d^{(2)}(n)$ up to about $n = 10^7$.

Computing $\Delta_d^{(2,-)}(n)$

- We use a modified version of the code that Alfes, Jameson, and Lemke Oliver used.
- The code is written in C++ and utilizes recursive algorithms to generate values for $Q_d^{(2,-)}(n)$ and $q_d^{(2)}(n)$ individually then finds their difference and outputs values of n for which $\Delta_d^{(2,-)}(n)$ is negative.

Theorem (S-, Swisher 2023)

For all positive integers $n \geq 1$ and $d = 1$ and $6 \leq d \leq 61$

$$\Delta_d^{(2,-)}(n) \geq 0.$$

Remaining Cases

- Thus, Kang and Park's conjecture is proven for all d except $d = 3, 4$ and 5.
- For each $d \in \{3, 4, 5\}$ we compute $\Delta_d^{(2,-)}(n)$ for all $1 \leq n \leq 10^7$.
- For each d except 3, we would only need improve the computed $N(d)$ value to be $N(d) \leq 10^7$ to finish proving those cases.
- For $d = 3$, it is possible that one could extend the asymptotic results to include $d = 3$ and proceed from there.

Future Work

- For how many even values of d is the dash unnecessary? I.e. for what values of d is $\Delta_d^{(2)}(n) \geq 0$ for all n?
- What about $\Delta_d^{(a,-)}(n)$ for higher values of a?
- Inagaki and Tamura [3] have proved $\Delta_d^{(3,-)}(n) \geq 0$ for all but finitely many d.
- Armstrong, Ducasse, Meyer, and Swisher proved $\Delta_d^{(a,-)}(n) \geq 0$ for $a \geq 1$, $\lceil \frac{d}{a} \rceil \geq 105$ and $1 \leq n \leq d + 2 + a$ or $d + 2a \leq n$.

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