

Algebraic structures on sets of partitions

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(joint with Megan du Preez and Rachel Van Surksum)¹

Online Partitions and q -Series Seminar

September 24, 2026

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Partitions

Definition

A **partition** λ of a non-negative integer n is a non-increasing sequence of positive integers which sum to n :

$$\lambda = (\lambda_1, \dots, \lambda_k) \vdash n.$$

We denote the set of all partitions by $\mathcal{P}$.

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Example

The partitions of 5 are:

(5)

(4, 1)

(3, 2)

(3, 1, 1)

(2, 2, 1)

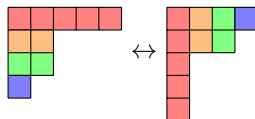
(2, 1, 1, 1)

(1, 1, 1, 1, 1)

Established Partition Operations

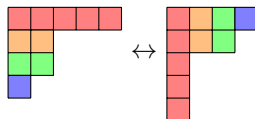
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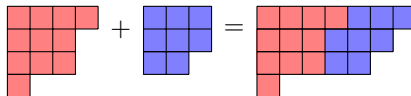


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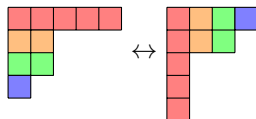


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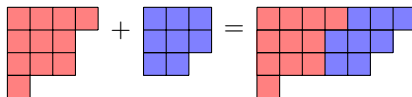


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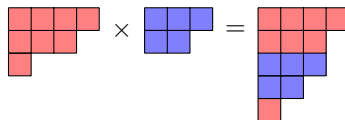
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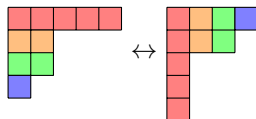


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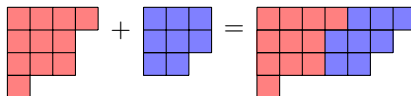


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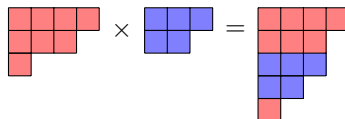
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- “Addition”



- “Multiplication” and “Division”



- “Scalar Multiplication” $3(4, 2, 1, 1) = (12, 6, 3, 3)$

Natural Questions About Operations

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We have a binary operation. Do partitions form a group?

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We also have scalar multiplication. Do partitions form a vector space?

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Interesting connections between various types of restricted partitions result from exploring these questions and related substructures.

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$$\begin{aligned} (7, 6, 3, 3, 1)(6^-, 4^-, 3^-, 1^-, 1^-) &= (7, 6, 6^-, 4^-, 3, 3, 3^-, 1, 1^-, 1^-) \\ &= (7, 4^-, 3, 1^-) \\ &= (7, 3)/(4, 1) \end{aligned}$$

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Theorem 1 (Schneider, 2018)

The set $\mathcal{Q}$ forms an abelian group under partition multiplication (and a ring with another operation similar to a tensor product).

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Goal

Identify algebraic structures on partitions without enlarging $\mathcal{P}$.

In other words, we want to find structures on partitions which follow the classical definition (positive parts) or with part restrictions.

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How would we fix this problem?

Two-Color Partitions

Definition

$\mathcal{P} \times \mathcal{P} :=$ the set of two-color partitions, where each part can be one of two colors.

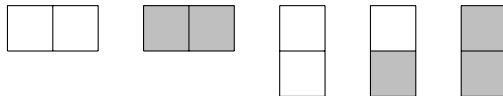


Figure: Two-color partitions of 2: (2_w) , (2_g) , $(1_w, 1_w)$, $(1_w, 1_g)$, $(1_g, 1_g)$

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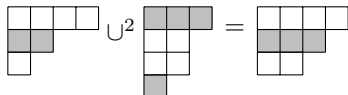
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Example: $(4_w, 2_g, 1_w) \cup^2 (3_g, 2_w, 2_w, 1_g) = (4_w, 3_g, 2_w)$



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- 5) We show in a few more cases that $\cup^2$ is commutative by counting the multiplicity of an arbitrary part in $\lambda \cup^2 \pi = \pi \cup^2 \lambda$.



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We discovered we need modular reduction to get inverses in $\mathcal{P}$.

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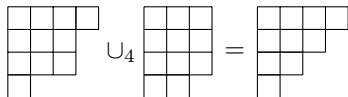
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Example: $(4, 3, 3, 1) \cup_4 (3, 3, 3, 2) = (4, 3, 2, 1)$



Concatenation with Reduction

$\mathcal{P}_{\cup_k} := \{\text{partitions where each part has multiplicity less than } k\}.$

Theorem 6

- 1) *For any integer $k \geq 2$, the set $\mathcal{P}_{\cup_k}$ forms an abelian group under concatenation and reduction modulo k .*
- 2) *For any prime p , the abelian group $(\mathcal{P}_{\cup_p}, \cup_p)$ forms a vector space over $\mathbb{Z}_p$.*

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Proof Idea.

For the abelian group:

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- 3) The inverse of $\lambda = \langle 1^{m_\lambda(1)}, \dots, r^{m_\lambda(r)} \rangle$ is

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- 4) For associativity and commutativity, we show that multiplicity reduction on both sides reduces an arbitrary part by the same number of occurrences.



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Proof Idea.

For the vector space:

- 1) Scalar multiplication $a\lambda$ is defined by $\underbrace{\lambda \cup_p \cdots \cup_p \lambda}_{a \text{ times}}$.
- 2) All required axioms follow from comparing the multiplicity modulo p of an arbitrary part on both sides of the equations.



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Question: Does viewing these restricted partitions as vectors have any advantages or useful applications?

Concatenation with Reduction: Sub-structures

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Theorem 7

Fix $k \in \mathbb{N}$ and $r \mid k$, $r > 0$. Let $\alpha \in \mathbb{N}$. The set $A_r(\alpha) := \{\lambda \in \mathcal{P}_{\cup_k} : r \mid m_\lambda(\alpha)\}$ forms a subgroup of $(\mathcal{P}_{\cup_k}, \cup_k)$.

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Remark

- 1) For a fixed part α contained in some partition in $\mathcal{P}_{\cup_k}$, the number of subgroups of the form $A_r(\alpha)$ in $\mathcal{P}_{\cup_k}$ is equal to the divisor function $d(k)$, the number of divisors of k .

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- 2) For a fixed part α and a fixed divisor $r \mid k$, the subgroup $A_r(\alpha)$ is the set of all partitions into parts with multiplicity less than k , where any part of size α occurs a multiple of r times.

Concatenation with Reduction: Sub-structures

Theorem 8

The set $\{\gamma \in \mathcal{P} : \ell(\gamma) = 1\}$ consisting of all partitions with only a single part forms a basis for the vector space $\mathcal{P}_{\cup_p}$ over $\mathbb{Z}_p$.

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Corollary 9

Let p be a prime and $m, a_1, a_2, \dots, a_m \in \mathbb{N}$. The subset of partitions generated by $\{(a_1), (a_2), \dots, (a_m)\}$ forms a subspace of the vector space $\mathcal{P}_{\cup_p}$ over $\mathbb{Z}_p$ and a subgroup of $\mathcal{P}_{\cup_p}$.

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Remark

The partitions generated by $\{(a_1), (a_2), \dots, (a_m)\}$ under $\cup_p$ are the partitions with parts from the set $\{a_1, a_2, \dots, a_m\}$ which all have multiplicity less than p .

Addition of Equivalence Classes

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Definition 10

*Component-wise congruence modulo k , $\sim_k$: $\lambda \sim_k \pi$ if $\lambda_j \equiv \pi_j \pmod{k}$ for all $1 \leq j \leq i$ and $\lambda_j \equiv 0 \pmod{k}$ for all $i < j \leq \ell$. If $\lambda \sim_k \pi$, then we call λ and π *component-wise congruent modulo k* .*

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Example 12

The equivalence class of $\lambda = (6, 1)$ modulo $k = 4$ contains the minimal-size partition $(2, 1)$ as well as the partitions $(6, 5)$, $(6, 5, 4)$, and $(22, 17, 16, 8, 4, 4)$.

Addition of Equivalence Classes

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Component-wise addition: Let $[\lambda] = [(\lambda_1, \lambda_2, \dots, \lambda_\ell)]$ and $[\pi] = [(\pi_1, \pi_2, \dots, \pi_i)]$ be equivalence classes of partitions with $\ell \geq i$. Then

$$[\lambda] + [\pi] := [(\lambda_1 + \pi_1, \lambda_2 + \pi_2, \dots, \lambda_i + \pi_i, \lambda_{i+1}, \dots, \lambda_\ell)].$$

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The set $\mathcal{P}/\sim_k$ forms an abelian group under component-wise addition and a vector space over $\mathbb{Z}_p$.

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The drawback is that we want a group structure on partitions, not on equivalence classes of partitions.

Addition with Reduction

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Component-wise reduction modulo k :

- 1) Reduce the smallest part $\lambda_{\ell(\lambda)}$ modulo k to $\lambda'_{\ell(\lambda)}$.
- 2) For each $1 \leq j < \ell(\lambda)$, replace λ_j by $\lambda'_j := \lambda_j - g_j k$, where g_j is defined backward recursively by

$$g_j := \left\lfloor \frac{\lambda_j - \lambda'_{j+1}}{k} \right\rfloor.$$

The new partition $\lambda' := (\lambda'_1, \lambda'_2, \dots, \lambda'_\ell)$ is the *component-wise reduction modulo k* of λ .

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The new partition $\lambda' := (\lambda'_1, \lambda'_2, \dots, \lambda'_\ell)$ is the *component-wise reduction modulo k* of λ . We call a partition $\lambda \in \mathcal{P}$ *component-wise reduced modulo k* if the difference between any two consecutive parts of λ is less than k and the smallest part $\lambda_{\ell(\lambda)}$ is less than k .

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- 2) For each $1 \leq j < \ell(\lambda)$, replace λ_j by $\lambda'_j := \lambda_j - g_j k$, where g_j is defined backward recursively by

$$g_j := \left\lfloor \frac{\lambda_j - \lambda'_{j+1}}{k} \right\rfloor.$$

The new partition $\lambda' := (\lambda'_1, \lambda'_2, \dots, \lambda'_\ell)$ is the *component-wise reduction modulo k* of λ . We call a partition $\lambda \in \mathcal{P}$ *component-wise reduced modulo k* if the difference between any two consecutive parts of λ is less than k and the smallest part $\lambda_{\ell(\lambda)}$ is less than k .

Remark: component-wise reduction modulo k reduces a partition λ to the partition λ' of minimal size such that each part of λ' is congruent modulo k to the corresponding part of λ .

Addition with Reduction

Example 16

Let $k = 4$ and $\lambda = (24, 18, 13, 9, 7, 2) \in \mathcal{P}$. We perform component-wise reduction modulo 4 on λ as follows:

$$\begin{aligned}
 (24, 18, 13, 9, 7, 2) &\rightarrow (24, 18, 13, 9, 7, 2 - 0 \cdot 4) \\
 &\rightarrow (24, 18, 13, 9, 7 - 1 \cdot 4, 2) \\
 &\rightarrow (24, 18, 13, 9 - 1 \cdot 4, 3, 2) \\
 &\rightarrow (24, 18, 13 - 2 \cdot 4, 5, 3, 2) \\
 &\rightarrow (24, 18 - 3 \cdot 4, 5, 5, 3, 2) \\
 &\rightarrow (24 - 4 \cdot 4, 6, 5, 5, 3, 2) = (8, 6, 5, 5, 3, 2).
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 &\rightarrow (24, 18, 13 - 2 \cdot 4, 5, 3, 2) \\
 &\rightarrow (24, 18 - 3 \cdot 4, 5, 5, 3, 2) \\
 &\rightarrow (24 - 4 \cdot 4, 6, 5, 5, 3, 2) = (8, 6, 5, 5, 3, 2).
 \end{aligned}$$

Observe that $\lambda' = (8, 6, 5, 5, 3, 2)$ is the partition of smallest size whose parts are congruent mod 4 to the parts of $\lambda = (24, 18, 13, 9, 7, 2)$.

Addition with Reduction

Example 17

Let $k = 3$ and $\lambda = (8, 7, 6, 3, 3, 3) \in \mathcal{P}$. Component-wise reduction modulo 3 on λ yields

$$(8, 7, 6, 3, 3, 3) \rightarrow (2, 1).$$

Addition with Reduction

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Note: the reduced partition $(2, 1)$ has length $\ell(\lambda) - 4 = 2$, since the number of smallest parts of λ which are all divisible by 3 is 4.

Addition with Reduction

Example 17

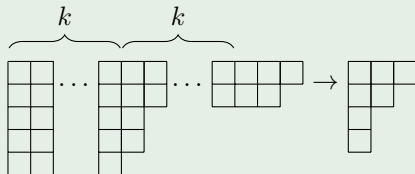
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Note: the reduced partition $(2, 1)$ has length $\ell(\lambda) - 4 = 2$, since the number of smallest parts of λ which are all divisible by 3 is 4.

Example 18

$$(2k + 3, 2k + 2, k + 1, k + 1, k) \rightarrow (3, 2, 1, 1).$$



Addition with Reduction

Definition 19

Component-wise addition and reduction modulo k :

- 1) component-wise addition,
- 2) then component-wise reduction modulo k .

Addition with Reduction

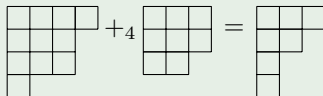
Definition 19

Component-wise addition and reduction modulo k :

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Example 20

$$(4, 3, 3, 1) +_4 (3, 3, 2) = (3, 2, 1, 1)$$



Addition with Reduction

Theorem 21

For any integer $k \geq 2$, the set $\mathcal{P}_k^+$ forms an abelian group under component-wise addition and reduction modulo k and a vector space over $\mathbb{Z}_p$.

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Proof Idea.

The map defined by taking any equivalence class $[\lambda] \in \mathcal{P}/\sim_k$ to the component-wise reduced modulo k partition $\lambda \rightarrow \gamma \in \mathcal{P}_k^+$ is an isomorphism. □

Group Isomorphism

Group Isomorphism

Theorem 22

$$\mathcal{P}_{\cup_k} \simeq \mathcal{P}_k^+.$$

Proof Idea.

The conjugation map is an isomorphism between partitions which are multiplicity reduced modulo k , under concatenation and reduction modulo k , and partitions which are component-wise reduced modulo k , under component-wise addition and reduction modulo k . $\square$

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Proof Idea.

The conjugation map is an isomorphism between partitions which are multiplicity reduced modulo k , under concatenation and reduction modulo k , and partitions which are component-wise reduced modulo k , under component-wise addition and reduction modulo k . $\square$

Remark

Conjugation gives a correspondence not only between multiplicity reduced and component-wise reduced partitions, but also between concatenation and component-wise addition.

Interesting Partition Bijection

Let $p_{\lambda \bmod k}(m)$ denote the number of partitions in $\mathcal{P}$ of size $|\lambda| + mk$ which component-wise reduce modulo k to the same partition λ .

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$p_{\lambda \bmod k}(m)$ counts the number of ways a component-wise reduced modulo k partition can have m additions of size k to any of its parts.

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Example: Let $k = 5$, $m = 4$, and $\lambda = (8, 6, 4, 3) \in \mathcal{P}_k^+$. Then $|\lambda| = 21$, and the $p(4) = 5$ possible partitions γ which component-wise reduce modulo 5 to $\lambda = (8, 6, 4, 3)$ and have size $21 + 4 \cdot 5 = 41$ are:

$$\gamma = (8 + 4 \cdot 5, 6, 4, 3) = (28, 6, 4, 3);$$

$$\gamma = (8 + 3 \cdot 5, 6 + 1 \cdot 5, 4, 3) = (23, 11, 4, 3);$$

$$\gamma = (8 + 2 \cdot 5, 6 + 2 \cdot 5, 4, 3) = (18, 16, 4, 3);$$

$$\gamma = (8 + 2 \cdot 5, 6 + 1 \cdot 5, 4 + 1 \cdot 5, 3) = (18, 11, 9, 3);$$

$$\gamma = (8 + 1 \cdot 5, 6 + 1 \cdot 5, 4 + 1 \cdot 5, 3 + 1 \cdot 5) = (13, 11, 9, 8).$$



Multiplication of Equivalence Classes

To find a partition ring structure, we need two compatible operations.

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Definition 24

Component-wise zero-congruence modulo k , $\sim'_k$: The same as $\sim_k$, with the added condition that any partition with some part divisible by k is automatically equivalent to $\emptyset$. Note: congruence preserves length.

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Theorem 25

$\sim'_k$ is an equivalence relation on $\mathcal{P}$.

Example 26

The equivalence class of $(6, 5, 1)$ modulo $k = 4$ contains the minimal-size partition $(2, 1, 1)$ as well as $(6, 1, 1)$ and $(14, 13, 5)$.

The equivalence class of $(6, 5, 4)$ modulo $k = 4$ contains the empty partition $\emptyset$ and all partitions of length 3 with any part divisible by 4.

Multiplication of Equivalence Classes

Denote the set of equivalence classes of length- r partitions under component-wise zero-congruence modulo k by $\mathcal{P}_r / \sim'_k$.

Definition 27

Component-wise multiplication: Let $[\lambda] = [(\lambda_1, \lambda_2, \dots, \lambda_r)]$ and $[\pi] = [(\pi_1, \pi_2, \dots, \pi_r)]$ be equivalence classes of partitions in $\mathcal{P}_r / \sim'_k$. Then

$$[\lambda] \cdot [\pi] := [(\lambda_1 \pi_1, \lambda_2 \pi_2, \dots, \lambda_r \pi_r)].$$

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$$[\lambda] \cdot [\pi] := [(\lambda_1 \pi_1, \lambda_2 \pi_2, \dots, \lambda_r \pi_r)].$$

Theorem 28

Let p be a prime, and let r be a non-negative integer. The set $\mathcal{P}_r / \sim'_p$ forms an abelian group under component-wise multiplication.

The proof of Theorem 28 is also straightforward. Again, the drawback is that we want a group structure on partitions, not equivalence classes.

Multiplication with Reduction

Definition 29

Component-wise zero-reduction modulo k :

- 1) If λ has some part divisible by k , then λ reduces to $\emptyset$.
- 2) If λ has no part divisible by k , then component-wise reduction modulo k .

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We call a partition $\lambda \in \mathcal{P}$ *component-wise zero-reduced modulo k* if λ is $\emptyset$, or if λ has no parts divisible by k and is component-wise reduced modulo k .

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We call a partition $\lambda \in \mathcal{P}$ *component-wise zero-reduced modulo k* if λ is $\emptyset$, or if λ has no parts divisible by k and is component-wise reduced modulo k .

Remark: a component-wise zero-reduced modulo k partition either is $\emptyset$ or has no parts divisible by k , difference between any two consecutive parts less than k , and smallest part less than k .

Multiplication with Reduction

Definition 30

Component-wise multiplication and zero-reduction modulo k , $\cdot_k^{(0)}$:

- 1) component-wise multiplication,
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Component-wise multiplication and zero-reduction modulo k , $\cdot_k^{(0)}$:

- 1) component-wise multiplication,
- 2) then component-wise zero-reduction modulo k .

Let $\mathcal{P}_{k,r}^{(0)}$, for any fixed non-negative integer r , denote the set of length- r partitions which are component-wise zero-reduced modulo k .

Theorem 31

$\mathcal{P}_{p,r}^{(0)}$ forms an abelian group under $\cdot_p^{(0)}$.

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$\mathcal{P}_{p,r}^{(0)}$ forms an abelian group under $\cdot_p^{(0)}$.

Proof Idea.

As before, we prove $\mathcal{P}_r / \sim'_p \simeq \mathcal{P}_{p,r}^{(0)}$ by component-wise zero-reduction modulo k . □

Commutative Ring Structure

Now, we have two compatible operations: $+_k$ and a simpler type of component-wise multiplication and reduction modulo k , $\cdot_k$.

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$\mathcal{P}_k^+$ forms a commutative ring under $+_k$ and $\cdot_k$.

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$\mathcal{P}_k^+$ forms a commutative ring under $+_k$ and $\cdot_k$.

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It is straightforward to show by congruence modulo k that $\cdot_k$ is associative and that the distributive law holds. □

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$\mathcal{P}_k^+$ forms a commutative ring under $+_k$ and $\cdot_k$.

Proof Idea.

It is straightforward to show by congruence modulo k that $\cdot_k$ is associative and that the distributive law holds. □

Remark: $\mathcal{P}_k^+$ is NOT a field, because we have zero-divisors. Using zero-congruence would remove zero-divisors, but then additive inverses would no longer be unique.

Ideals of the Partition Ring

We also prove a structure theorem for ideals of our partition ring.

Theorem 33

For any prime p , every finitely generated ideal of $\mathcal{P}_p^+$ is principal.

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Remark

Any finitely generated ideal of partitions with smallest part less than p and difference between consecutive parts less than p consists entirely of component-wise products of a single generating partition.

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For any prime p , every finitely generated ideal of $\mathcal{P}_p^+$ is principal.

Remark

Any finitely generated ideal of partitions with smallest part less than p and difference between consecutive parts less than p consists entirely of component-wise products of a single generating partition.

The unique generating partition is the minimal-size partition with parts congruent modulo p to the greatest common divisors of the parts of the partitions in the ideal.

Ideals of the Partition Ring

Example 34

Let $p = 3$, $\lambda = (3, 2)$, $\pi = (5, 4, 3, 1)$, and $\tau = (9, 8, 6, 4, 2)$.

Consider $I = \langle \lambda, \pi, \tau \rangle \subset \mathcal{P}_3^+$.

Ideals of the Partition Ring

Example 34

Let $p = 3$, $\lambda = (3, 2)$, $\pi = (5, 4, 3, 1)$, and $\tau = (9, 8, 6, 4, 2)$.

Consider $I = \langle \lambda, \pi, \tau \rangle \subset \mathcal{P}_3^+$.

The 5-tuple of component-wise greatest common divisors of λ, π, τ is $g = (1, 2, 3, 1, 2)$, so the unique generating partition of I is $\gamma = (10, 8, 6, 4, 2)$:

$$(3, 1) \cdot_p (10, 8, 6, 4, 2) = (3, 2);$$

$$(2, 2, 1, 1) \cdot_p (10, 8, 6, 4, 2) = (5, 4, 3, 1);$$

$$(3, 1, 1, 1, 1) \cdot_p (10, 8, 6, 4, 2) = (9, 8, 6, 4, 2).$$

Further Questions

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- 1) Now we have algebraic structures within the set of partitions. What can we do with them? Can we use this added structure to say anything about the specific types of restricted partitions involved?

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- 2) How do our operations interact with classical partition statistics like rank, crank, etc.

Thank you!