Properties of sequentially congruent partitions

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Online Partitions and q-Series Seminar

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Partitions

Definition

A **partition** λ of a nonnegative integer n is a nonincreasing sequence of positive integers which sum to n: $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$.

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Example

$p(5) = 7$: the partitions of 5 are

$$(5)$$
$$(4, 1)$$
$$(3, 2)$$
$$(3, 1, 1)$$
$$(2, 2, 1)$$
$$(2, 1, 1, 1)$$
$$(1, 1, 1, 1, 1)$$

Partition Bijections

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- Partitions into distinct parts,
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When $n = 5$: $p_{\text{odd}}(5) = 3$ and $p_{\text{distinct}}(5) = 3$

(5)	(5)
$(3,1,1)$	$(4,1)$
$(1,1,1,1,1)$	$(3,2)$

Sequentially Congruent Partitions

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A **sequentially congruent partition** of n is a partition $(\lambda_1, \ldots, \lambda_r)$ of n such that

1. $\lambda_i \equiv \lambda_{i+1} \pmod{i}$ for all $1 \leq i < r$, and
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Example

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (21, 16, 14, 8) \vdash 59$ is sequentially congruent:

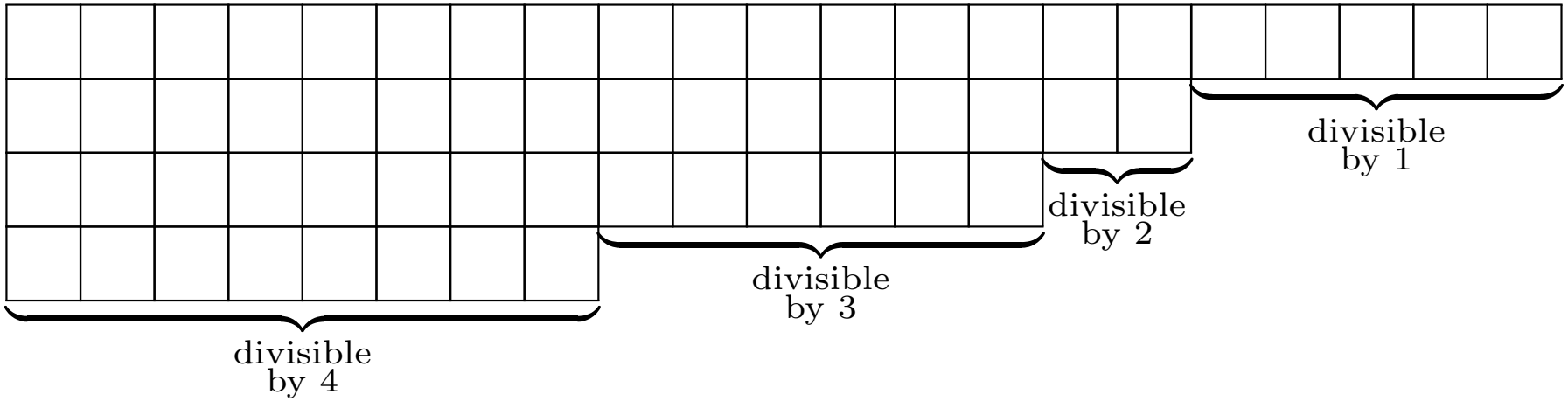
$$\begin{aligned} &1 \mid (21 - 16), \\ &2 \mid (16 - 14), \\ &3 \mid (14 - 8), \text{ and} \\ &4 \mid 8. \end{aligned}$$

Example

Definition

A **sequentially congruent partition** of n is a partition in which each part is congruent to the next modulo its index.

Sequentially congruent partition $(21, 16, 14, 8)$ has Young diagram:



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Definition

For any two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\gamma = (\gamma_1, \gamma_2, \dots)$, define

$$\lambda \oplus \gamma := \mu,$$

where $\mu = (\mu_1, \mu_2, \dots)$ is the partition such that $\mu_i = \lambda_i + \gamma_i$ for all i.

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In other words, $\oplus$ is componentwise addition from the left.

Example

$$\begin{aligned} &(5,3,2,2) \oplus (3,2,1) \\ &= (5+3, 3+2, 2+1, 2+0) \\ &= (8,5,3,2) \end{aligned}$$

Sequentially Congruent Partition Notation

Definition

For a nonnegative integer c, define $c(\lambda_1, \ldots, \lambda_r) := (c\lambda_1, \ldots, c\lambda_r)$, so

$$c\lambda = \underbrace{\lambda \oplus \lambda \oplus \cdots \oplus \lambda}_{c \text{ times}}.$$

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We now have a new way to write a sequentially congruent partition:

Theorem 1 (Cochran–D.–Harrell–Saunders, 2023)

A partition λ is sequentially congruent if and only if it can be written uniquely in the form

$$\lambda = c_1(1) \oplus c_2(2,2) \oplus c_3(3,3,3) \oplus \cdots$$

for nonnegative integers $c_1, c_2, c_3, \ldots,$ with finitely many c_i nonzero.

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The partition $(30, 26, 18, 15, 15) \vdash 104$ is sequentially congruent.

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\begin{aligned}
&(30, 26, 18, 15, 15) \\
&= (30, 26, 18, 15, 3 \cdot 5) \\
&= (30, 26, 18, 0 \cdot 4 + 3 \cdot 5, 3 \cdot 5) \\
&= (30, 26, 1 \cdot 3 + 0 \cdot 4 + 3 \cdot 5, 0 \cdot 4 + 3 \cdot 5) \\
&= (30, 4 \cdot 2 + 1 \cdot 3 + 0 \cdot 4 + 3 \cdot 5, 1 \cdot 3 + 0 \cdot 4 + 3 \cdot 5, 0 \cdot 4 + 3 \cdot 5, 3 \cdot 5) \\
&= (4 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 + 0 \cdot 4 + 3 \cdot 5, 4 \cdot 2 + 1 \cdot 3 + 0 \cdot 4 + 3 \cdot 5, \\
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&= (30, 26, 18, 0 \cdot 4 + 3 \cdot 5, 3 \cdot 5) \\
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&\qquad 1 \cdot 3 + 0 \cdot 4 + 3 \cdot 5, 0 \cdot 4 + 3 \cdot 5, 3 \cdot 5)
\end{aligned}
$$

Then

$$(30, 26, 18, 15, 15) = 4(1) \oplus 4(2, 2) \oplus 1(3, 3, 3) \oplus 0(4, 4, 4, 4) \oplus 3(5, 5, 5, 5, 5).$$

Sequentially Congruent Partition Notation

Note: a sequentially congruent partition

$$\lambda = c_1(1) \oplus c_2(2,2) \oplus c_3(3,3,3) \oplus \cdots \oplus c_r(\underbrace{r,\ldots,r}_{r \text{ times}})$$

can be written in standard notation as

$$\lambda = (c_1 + 2c_2 + 3c_3 + \cdots + rc_r, 2c_2 + 3c_3 + \cdots + rc_r, \ldots, rc_r).$$

Sequentially Congruent Partition Notation

Proof of Theorem 1.

($\Leftarrow$) For a partition λ of the form

$$\lambda = c_1(1) \oplus c_2(2,2) \oplus c_3(3,3,3) \oplus \cdots \oplus c_r(\underbrace{r,\ldots,r}_{r \text{ times}})$$
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we see that

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- $\lambda_k - \lambda_{k+1} = kc_k \equiv 0 \pmod{k}$ for all $1 \leq k < r$.

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we see that

- $rc_r \equiv 0 \pmod{r}$
- $\lambda_k - \lambda_{k+1} = kc_k \equiv 0 \pmod{k}$ for all $1 \leq k < r$.

Therefore λ is sequentially congruent.

Sequentially Congruent Partition Notation

Proof of Theorem 1.

($\Rightarrow$) For a sequentially congruent partition λ, we must have that

$$\lambda_k \equiv \lambda_{k+1} \pmod{k}$$

for each $1 \leq k < r$.

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for some nonnegative integer c_k. Therefore $\lambda_k = kc_k + \lambda_{k+1}$ for all k, so we can write

$$\begin{aligned}\lambda &= (c_1 + 2c_2 + \cdots + rc_r, 2c_2 + \cdots + rc_r, \ldots, rc_r) \\ &= c_1(1) \oplus c_2(2,2) \oplus c_3(3,3,3) \oplus \cdots \oplus c_r(\underbrace{r, \ldots, r}_{r \text{ times}}).\end{aligned}$$

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Uniqueness can be proved by induction. $\square$

c-Notation

Since the representation in Theorem 1 is unique, we can denote any sequentially congruent partition

$$\lambda = c_1(1) \oplus c_2(2,2) \oplus c_3(3,3,3) \oplus \cdots \oplus c_r(\underbrace{r,\ldots,r}_{r \text{ times}})$$

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Note that if the summand $(3,3,3)$ appears zero times in the partition, then $c_3 = 0$.

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Original Proof.

- The conjugate of a sequentially congruent partition is called a *frequency congruent partition*:

$$m_i(\lambda) \equiv 0 \pmod{i} \text{ for all } i \geq 1.$$

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$$\left(1^{1\cdot e_1}, 2^{2\cdot e_2}, \ldots, i^{i\cdot e_i}\right) \text{ for some } e_i \geq 0.$$
- Prove that the following is a size-preserving bijection:
$$\left(1^{1\cdot e_1}, \ldots, i^{i\cdot e_i}\right) \mapsto \left(\left(1^2\right)^{e_1}, \ldots, \left(i^2\right)^{e_i}\right)$$

$\square$

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Proof using c-notation.

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The Young diagram of a sequentially congruent partition is composed of squares:

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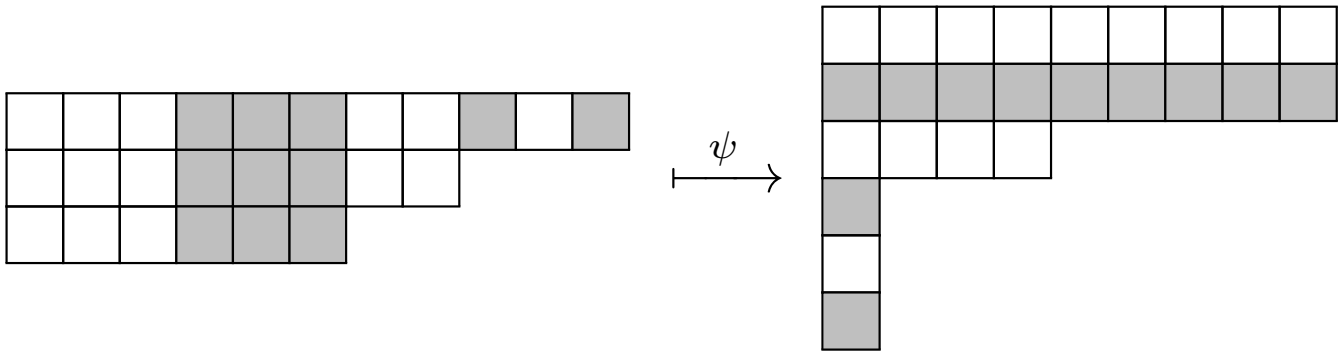
To map a sequentially congruent partition to a partition into squares, we transform each square into its own row:



Partitions Composed of Squares

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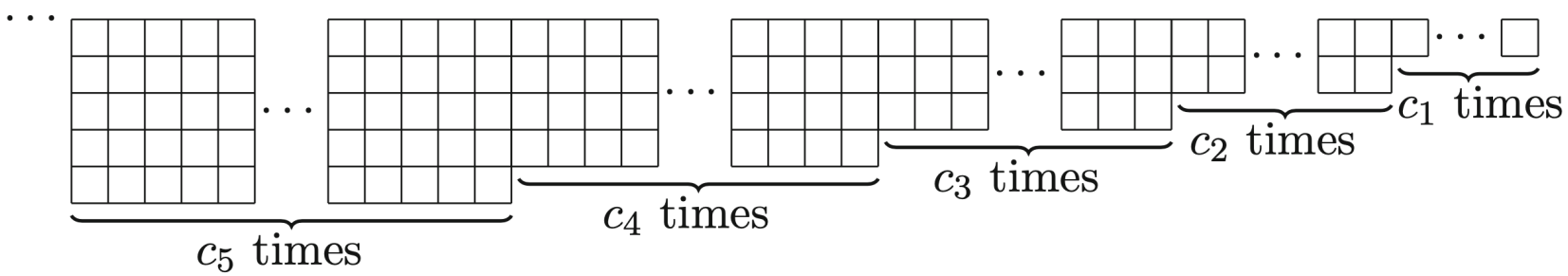

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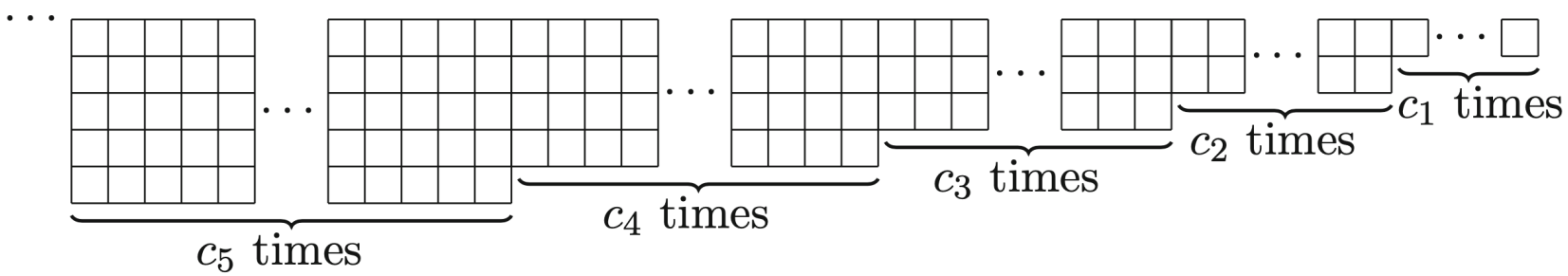

The number of $i \times i$ squares is equal to c_i for each $i \geq 1$.

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Theorem 4 (Schneider–Schneider, 2019)

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1. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, define $\pi : \lambda \mapsto \lambda'$ where $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ is the sequentially congruent partition with

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$$\lambda'_i = i\lambda_i + \sum_{j=i+1}^{r} \lambda_j.$$

2. For a sequentially congruent partition $\phi = (\phi_1, \ldots, \phi_r)$, define

$$\sigma : \phi \mapsto \left\langle 1^{\phi_1 - \phi_2}, 2^{(\phi_2 - \phi_3)/2}, 3^{(\phi_3 - \phi_4)/3}, \ldots, r^{\phi_r / r} \right\rangle.$$

Examples of Bijections

Example

Recall $\pi : (\lambda_1, \ldots, \lambda_r) \mapsto (\lambda'_1, \ldots, \lambda'_r)$ where $\lambda'_i = i\lambda_i + \sum_{j=i+1}^{r} \lambda_j$.

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Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (12, 8, 4, 3, 3)$.

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Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (12, 8, 4, 3, 3)$.

$$\lambda'_1 = 1\lambda_1 + \sum_{j=2}^{5} \lambda_j = 12 + 8 + 4 + 3 + 3 = 30,$$

$$\lambda'_2 = 2\lambda_2 + \sum_{j=3}^{5} \lambda_j = 16 + 4 + 3 + 3 = 26,$$

$$\lambda'_3 = 3\lambda_3 + \sum_{j=4}^{5} \lambda_j = 12 + 3 + 3 = 18,$$

$$\lambda'_4 = 4\lambda_4 + \sum_{j=5}^{5} \lambda_j = 12 + 3 = 15,$$

$$\lambda'_5 = 5\lambda_5 = 15.$$

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Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (12, 8, 4, 3, 3)$.

$$\lambda'_1 = 1\lambda_1 + \sum_{j=2}^{5} \lambda_j = 12 + 8 + 4 + 3 + 3 = 30,$$

$$\lambda'_2 = 2\lambda_2 + \sum_{j=3}^{5} \lambda_j = 16 + 4 + 3 + 3 = 26,$$

$$\lambda'_3 = 3\lambda_3 + \sum_{j=4}^{5} \lambda_j = 12 + 3 + 3 = 18,$$

$$\lambda'_4 = 4\lambda_4 + \sum_{j=5}^{5} \lambda_j = 12 + 3 = 15,$$

$$\lambda'_5 = 5\lambda_5 = 15.$$

Therefore

$$\pi((12, 8, 4, 3, 3)) = (30, 26, 18, 15, 15)$$

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Let $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (30, 26, 18, 15, 15)$.

$$\begin{aligned} 1^{\phi_1 - \phi_2} &= 1^{30-26} = 1^4, \\ 2^{\frac{\phi_2 - \phi_3}{2}} &= 2^{\frac{26-18}{2}} = 2^4, \\ 3^{\frac{\phi_3 - \phi_4}{3}} &= 3^{\frac{18-15}{3}} = 3^1, \\ 4^{\frac{\phi_4 - \phi_5}{4}} &= 4^{\frac{15-15}{4}} = 4^0, \\ 5^{\frac{\phi_5}{5}} &= 5^{\frac{15}{5}} = 5^3. \end{aligned}$$

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Recall $\sigma : (\phi_1, \ldots, \phi_r) \mapsto \left\langle 1^{\phi_1 - \phi_2}, 2^{(\phi_2 - \phi_3)/2}, 3^{(\phi_3 - \phi_4)/3}, \ldots, r^{\phi_r / r} \right\rangle$.

Let $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (30, 26, 18, 15, 15)$.

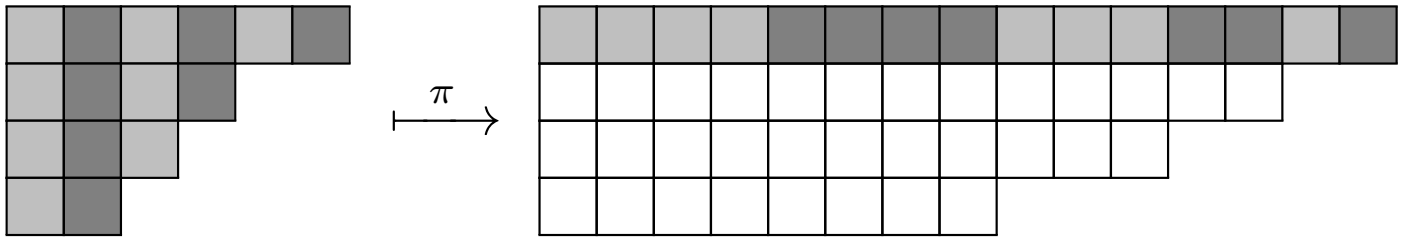
$$\begin{aligned} 1^{\phi_1 - \phi_2} &= 1^{30-26} = 1^4, \\ 2^{\frac{\phi_2 - \phi_3}{2}} &= 2^{\frac{26-18}{2}} = 2^4, \\ 3^{\frac{\phi_3 - \phi_4}{3}} &= 3^{\frac{18-15}{3}} = 3^1, \\ 4^{\frac{\phi_4 - \phi_5}{4}} &= 4^{\frac{15-15}{4}} = 4^0, \\ 5^{\frac{\phi_5}{5}} &= 5^{\frac{15}{5}} = 5^3. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma((30, 26, 18, 15, 15)) &= \left\langle 1^4, 2^4, 3^1, 5^3 \right\rangle \\ &= (5, 5, 5, 3, 2, 2, 2, 2, 1, 1, 1, 1) \end{aligned}$$

Young Diagram Transformations

Bijection 1 can be illustrated via Young diagrams as follows:



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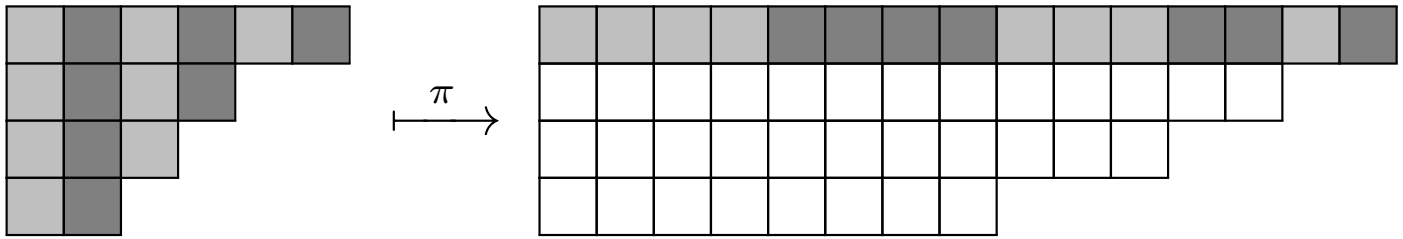
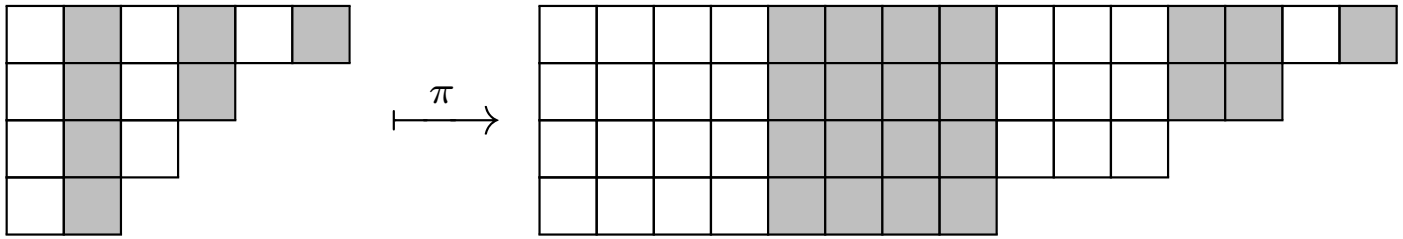


The map π "stretches" each $i \times 1$ column in the Young diagram into an $i \times i$ square.



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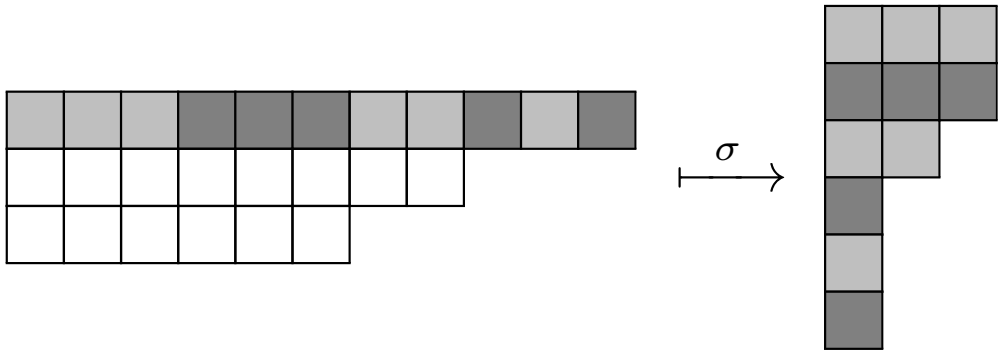

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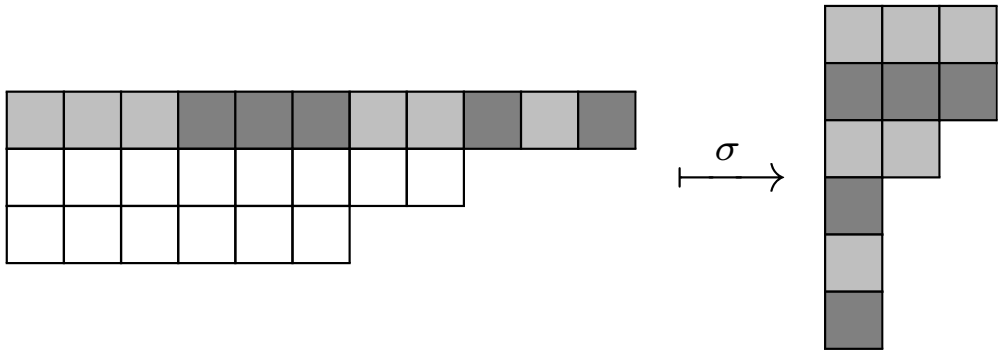
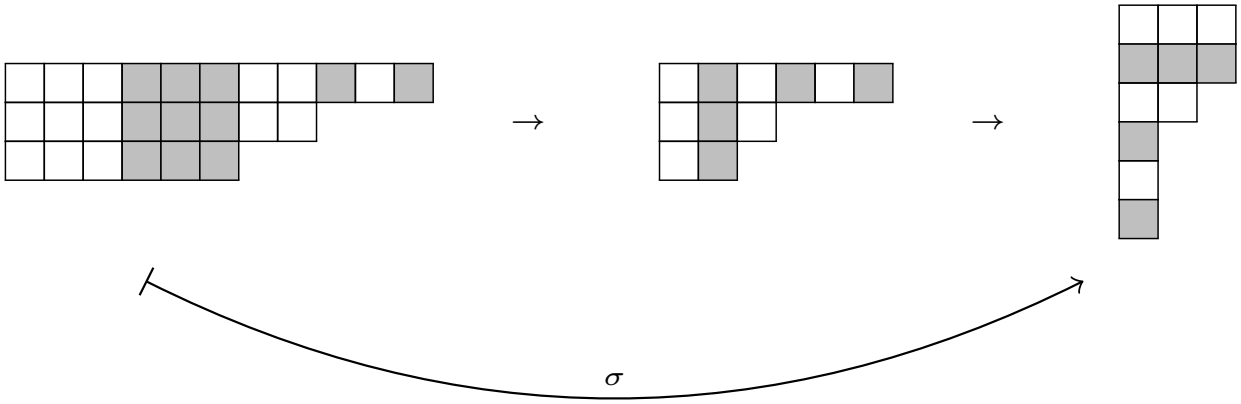


The map σ "squishes" each $i \times i$ square into an $i \times 1$ column, and then "flips" (conjugates) the diagram.



Natural Questions About Bijections

Observation

The bijections π and σ are not inverses of each other.

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The bijections π and σ are not inverses of each other.

- What is an explicit description of the composition $\sigma \circ \pi$?
- What is an explicit description of the composition $\pi \circ \sigma$?
- What is a combinatorial interpretation of both of these compositions?

Compositions of Bijections

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The conjugate of any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is the partition

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Moreover, a partition λ is self-conjugate if and only if

$$\lambda_i = \begin{cases} r & \text{when } i \leq \lambda_r, \\ r-1 & \text{when } \lambda_r < i \leq \lambda_{r-1}, \\ \vdots & \\ 2 & \text{when } \lambda_3 < i \leq \lambda_2, \\ 1 & \text{when } \lambda_2 < i \leq \lambda_1. \end{cases}$$

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Let $\phi = (\phi_1, \phi_2, \ldots, \phi_r)$ be a sequentially congruent partition. Then

$$(\pi \circ \sigma)(\phi) = \left\langle \left(\sum_{j=1}^{1} \sum_{i=j}^{r} \frac{\phi_i - \phi_{i+1}}{i} \right)^{\phi_1 - \phi_2}, \left(\sum_{j=1}^{2} \sum_{i=j}^{r} \frac{\phi_i - \phi_{i+1}}{i} \right)^{\frac{\phi_2 - \phi_3}{2}}, \ldots, \left(\sum_{j=1}^{r} \sum_{i=j}^{r} \frac{\phi_i - \phi_{i+1}}{i} \right)^{\frac{\phi_r}{r}} \right\rangle.$$

$$= [\underbrace{0, \ldots, 0, 1}_{c_r \ \text{terms}}, \underbrace{0, \ldots, 0, 1}_{c_{r-1} \ \text{terms}}, \ldots, \underbrace{0, \ldots, 0, 1}_{c_1 \ \text{terms}}].$$

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"Proofs" of Theorems 5 and 7.

- The composition $\sigma \circ \pi$ transforms the Young diagram of a partition by stretch-squish-flip.
- The composition $\pi \circ \sigma$ transforms the Young diagram of a sequentially congruent partition by squish-flip-stretch. □

Recap of Frequency Congruent Partitions

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Definition

Recall that **Frequency congruent partitions** are the conjugates of sequentially congruent partitions and are of the form

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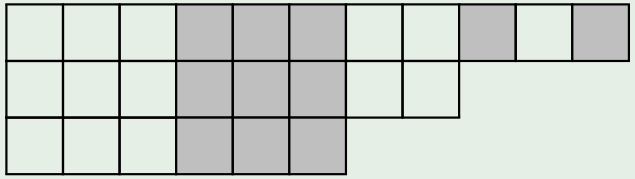
In other words, the part i occurs a multiple of i times.

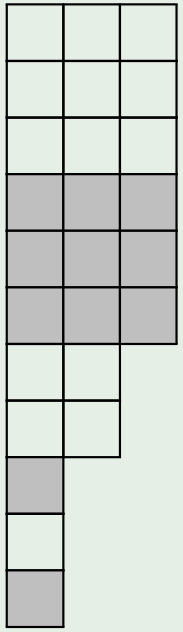
Recap of Frequency Congruent Partitions

Example

Sequentially congruent partition:



Corresponding frequency congruent partition:



Generalized Frequency Congruent Partitions

Let $B \subseteq \mathbb{N}$, and let A be a sequence of natural numbers.

Schneider–Schneider define the set of partitions with parts from B, where the part b_i occurs a multiple of a_i times:

$$\langle b_1^{n_1 a_1}, b_2^{n_2 a_2}, b_3^{n_3 a_3}, \dots \rangle,$$

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Example of a generalized frequency congruent partition for B and A:

$$\langle 2^4, 4^9, 6^5, 8^{14} \rangle$$

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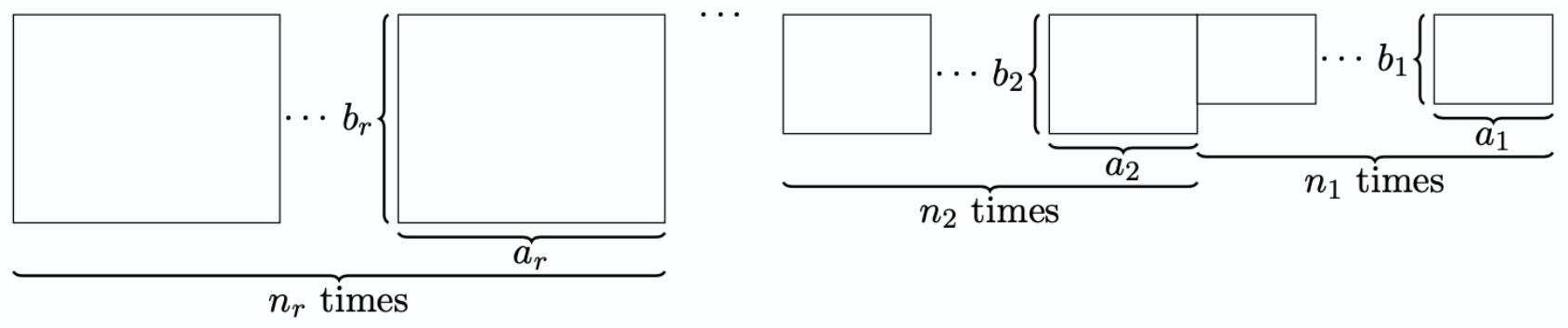
Example

If $B = \mathbb{N}, A = (1, 1, 1, \dots)$, these are sequentially congruent partitions.

Generalized Sequentially Congruent Partitions

Young diagram of a generalized sequentially congruent partition:



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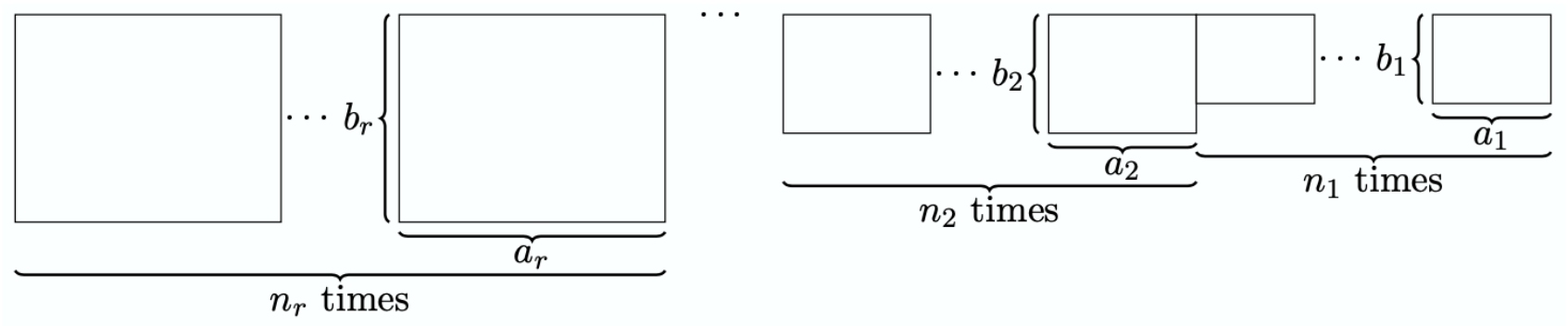

A generalized sequentially congruent partition can be uniquely written

$$\lambda = n_1(\underbrace{a_1, \ldots, a_1}_{b_1 \text{ times}}) \oplus n_2(\underbrace{a_2, \ldots, a_2}_{b_2 \text{ times}}) \oplus n_3(\underbrace{a_3, \ldots, a_3}_{b_3 \text{ times}}) \oplus \cdots$$

with finitely many nonzero n_i. We use the "n-notation"

$$\lambda = [n_1, n_2, \ldots, n_r]_{A,B}.$$

Generalized Bijections

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1. For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, define $\pi_{AB} : \lambda \mapsto \lambda'$ where $\lambda' = \left(\lambda'_1, \lambda'_2, \ldots, \lambda'_{b_r}\right)$ is the generalized sequentially congruent partition with

$$\lambda'_m = a_i \lambda_i + \sum_{j=i+1}^{r} (a_j - a_{j-1}) \lambda_j$$

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Examples of Generalized Bijections

Example

Recall $\pi_{AB} : (\lambda_1, \ldots, \lambda_r) \mapsto \left(\lambda'_1, \ldots, \lambda'_{b_r}\right)$, where for all $1 \leq i \leq r$, $\lambda'_m = a_i \lambda_i + \sum_{j=i+1}^{r} (a_j - a_{j-1}) \lambda_j$ for $b_{i-1} < m \leq b_i$.

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Let $B = \{2, 4, 6, 8, 10, \ldots\} \subseteq \mathbb{N}$ and $A = (1, 3, 5, 7, 9, \ldots)$, and let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (12, 8, 4, 3, 3)$.

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Therefore $\pi_{AB}((12, 8, 4, 3, 3)) = (48, 44, 32, 27, 27)$.

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$$\begin{array}{lll} 3 & b_1 \times a_1 = 2 \times 1 & \text{rectangles,} \\ 1 & b_2 \times a_2 = 4 \times 3 & \text{rectangle,} \\ 1 & b_3 \times a_3 = 6 \times 5 & \text{rectangle, and} \\ 2 & b_5 \times a_5 = 10 \times 9 & \text{rectangles.} \end{array}$$

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$$\begin{array}{lll} 3 & b_1 \times a_1 = 2 \times 1 & \text{rectangles,} \\ 1 & b_2 \times a_2 = 4 \times 3 & \text{rectangle,} \\ 1 & b_3 \times a_3 = 6 \times 5 & \text{rectangle, and} \\ 2 & b_5 \times a_5 = 10 \times 9 & \text{rectangles.} \end{array}$$

Therefore we found that

$$\sigma_{AB}((29, 29, 26, 26, 23, 23, 18, 18, 18, 18)) = (5, 5, 3, 2, 1, 1, 1).$$

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- Generalized bijections mapping between rectangles of different sizes or between different rectangles with the same area

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Sequentially congruent partitions enjoy many fascinating properties:

- Abbreviated c-notation
- Young diagrams composed of squares
- Bijections with various sets of partitions
- An analogue of conjugation within sequential congruence
- Generalizations with Young diagrams composed of rectangles
- Generalized bijections