Self-conjugate 6-cores and quadratic forms

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Definitions

Self-conjugate partitions

Each partition of a positive integer n can be represented by its Ferrers diagram.

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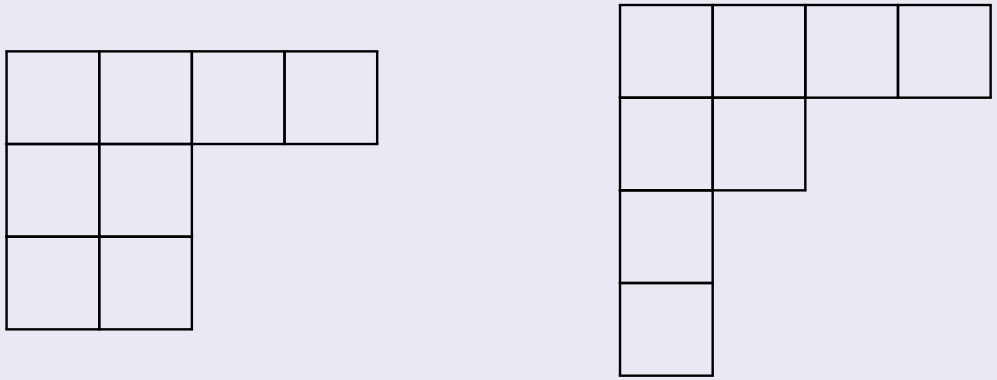

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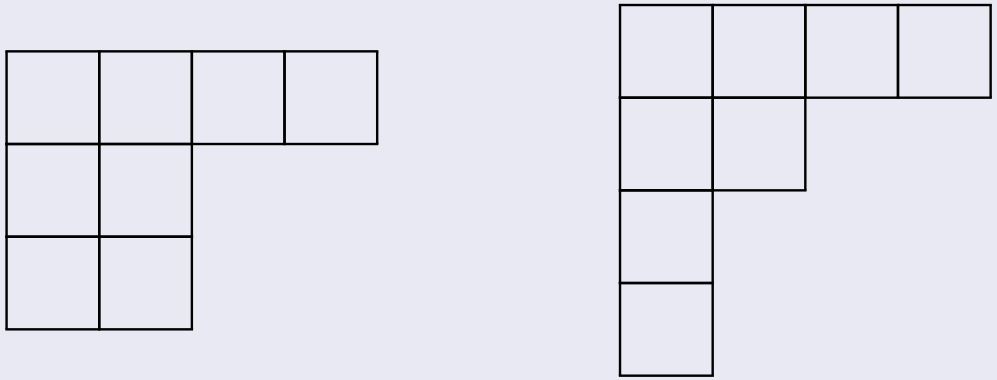

represent the partitions $4+2+2$ and $4+2+1+1$.
A partition is called self-conjugate if the Ferrers diagram does not change when its rows and columns are switched.

Definitions (continued)

t-core partitions

Each cell in the Ferrers diagram has a hook length, which is the number of cells to the right or below that cell (including itself).

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A partition is t-core if none of its hook lengths are multiples of t.

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Let $sc_t(n)$ be the number of self-conjugate t-core partitions of n.

Motivating result

Theorem (Ono–Raji 2019)

$sc_7(n)$ is essentially a Hurwitz class number. E.g., if $n \equiv 1 \pmod 4$ and $n \not\equiv 5 \pmod 7$ then

$$sc_7(n) = \frac{1}{4} H(-28n - 56).$$

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Sketch of proof: Write the generating function in terms of q-Pochhammer symbols, view it as a modular form, and then decompose it as the sum of well-understood Eisenstein series.

Background

- Hanusa–Nath (2013): The generating function for $sc_t(n)$ is an eta-quotient. For example,

$$\sum_{n\geq 0} sc_6(n)q^n = \prod_{n\geq 1} \frac{(1-q^{2n})^2(1-q^{12n})^3}{(1-q^n)(1-q^{4n})}.$$

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- Bringmann–Kane–Males (2021): $sc_7(n)$ is a linear combination of Hurwitz class numbers for all n, and is also related to $c_4(n)$.
- Males–Tripp (2020) and Dawsey–Sharp (2022): combinatorial considerations give insights related to hook lengths/parts of the partition, sums of squares, ...

Background (continued)

Motivating Question

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Conjecture (Hanusa–Nath 2013)

$sc_6(n) > 0$ for all positive integers except when $n \in \{2, 12, 13, 73\}$.

Main Theorem

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Theorem (Hanson-J)

Assuming the Generalized Riemann Hypothesis, the Hanusa–Nath conjecture is true, i.e., $sc_6(n) > 0$ *for all positive integers except when* $n \in \{2, 12, 13, 73\}$.

Alpoge's work

The first step is to connect $sc_6(n)$ to a quadratic form.

Theorem (Alpoge 2014*)

For all $n \geq 0$,

$$sc_6(n) = \frac{1}{12}\#\{(x,y,z) \in \mathbb{Z}^3 : 24n+35 = 3x^2+32y^2+32yz+32z^2\}.$$

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are positive (except when $n \in \{2, 12, 13, 73\}$).

Proof that $sc_6(n) = \frac{1}{12} r_Q(n)$

- Rewrite the generating function of Hanusa and Nath as

$$\sum_{n \geq 0} sc_6(n) q^{24n+35} = \left(\frac{\eta(48z)^2}{\eta(24z)} \right) \left(\frac{\eta(288z)^3}{\eta(96z)} \right),$$

where $q = e^{2\pi i z}$ and $\eta(z) := q^{1/24} \prod_{n \geq 1} (1 - q^n)$.

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$$\frac{\eta(48z)^2}{\eta(24z)} = \sum_{n \geq 0} q^{3(2n+1)^2} = \frac{1}{2} \sum_{n \in \mathbb{Z}} q^{3(2n+1)^2}$$

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and the second is

$$\frac{\eta(288z)^3}{\eta(96z)} = \sum_{n \geq 0} c_3(n) q^{32(3n+1)}$$

where $c_3(n)$ is the number of 3-core partitions of n.

Proof that $sc_6(n) = \frac{1}{12} r_Q(n)$ (continued)

- Work of Han and Ono give

$$c_3(n) = \frac{1}{6} \#\{(x, y) \in \mathbb{Z}^2 : 3n + 1 = x^2 + xy + y^2\}$$

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- Work of Han and Ono give

$$c_3(n) = \frac{1}{6}\#\{(x,y) \in \mathbb{Z}^2 : 3n+1 = x^2 + xy + y^2\}$$

and thus it follows that

$$sc_6(n) = \frac{1}{12}\#\{(x,y,z) \in \mathbb{Z}^3 : 24n+35 = 3x^2+32y^2+32yz+32z^2\}$$

as desired.

$\square$

So, our new goal is to prove the following:

Theorem (Hanson-J)

Assume the GRH and let n be a positive integer. Then $r_Q(24n+35) > 0$ except when $n \in \{2, 12, 13, 73\}$.

So, our new goal is to prove the following:

Theorem (Hanson-J)

Assume the GRH and let n be a positive integer. Then $r_Q(24n+35) > 0$ except when $n \in \{2, 12, 13, 73\}$.

For this, we follow the approach of Ono and Soundararajan (1997).

Understanding the theta function

- The theta function associated to Q

$$\theta_Q(z) := \sum_{\mathbf{x}\in\mathbb{Z}^3} q^{Q(\mathbf{x})} = \sum_{n\geq 0} r_Q(n)q^n = 1 + 2q^3 + 2q^{12} + \cdots$$

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- Write

$$\theta_Q(z) = E(z) + C(z)$$

  where E is an Eisenstein series and C is a cusp form.

Understanding the Eisenstein series

- First we turn to

$$E(z) = \sum_{n \geq 0} b(n) q^n = 1 + \frac{1}{2} q^3 + 3q^{11} + 2q^{12} + \frac{7}{2} q^{27} + 6q^{32} + 6q^{35} + \cdots .$$

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- For squarefree N in a fixed square class $\prod_{p|6} \mathbb{Q}_p^\times / \left(\mathbb{Q}_p^\times\right)^2$, there exist constants a and b such that that

$$b(N) = a \cdot h(-bN).$$

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  In fact, for $N = 24n + 35$, we have that $a = 3$ and $b = 1$.
- Dirichlet's class number formula:

$$h(-N) = \frac{1}{\pi} \sqrt{N} L(\chi_{-N}, 1).$$

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- Now we turn to

$$C(z) = \sum_{n \geq 0} a(n) q^n = \frac{3}{2} q^3 - 3q^{11} - \frac{3}{2} q^{27} + 6q^{35} + \cdots \in S_{3/2}(\Gamma_0(96)).$$

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- From Shimura correspondence, we get a multiple of the newform

$$F(z) = \sum_{n \geq 0} A(n)q^n = q - q^3 - 2q^5 + q^9 + 4q^{11} - 2q^{13} + \cdots \in S_2(\Gamma_0(24))$$

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which is the cusp form associated to the elliptic curve $E : y^2 = x^3 - x^2 + x$.

Understanding the cusp form

- Now a theorem of Waldspurger says that for square-free $N_1, N_2 \in \mathbb{N}$ with $N_1/N_2 \in \left(\mathbb{Q}_p^\times\right)^2$ for all $p \mid 6$, then

$$a(N_1)^2 L(F \otimes \chi_{-N_2}, 1) N_2^{1/2} = a(N_2)^2 L(F \otimes \chi_{-N_1}, 1) N_1^{1/2}.$$

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In fact, for squarefree $N = 24n + 35$, we have $d = 1.63384\ldots$.

Putting it all together

- Altogether for squarefree $N = 24n + 35$ we have

$$r_Q(N) = \frac{3}{\pi}\sqrt{N}L(\chi_{-N}, 1) \pm dN^{1/4}L(E \otimes \chi_{-N}, 1)^{1/2}.$$

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- So: if N is not represented by Q, then

$$\frac{L(E \otimes \chi_{-N}, 1)^{1/2}}{L(\chi_{-N}, 1)} = \frac{a\sqrt{b}}{d\pi}N^{1/4} \geq 0.5844N^{1/4}.$$

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- On the other hand, work of Chandee gives the upper bound

$$\frac{L(E \otimes \chi_{-N}, 1)^{1/2}}{L(\chi_{-N}, 1)} \leq 2.5889N^{0.14157}.$$

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- Combining gives $n \leq 916347.7794$, and check "by hand." $\square$

Recap

Theorem (Hanson-J)

Assuming GRH, the Hanusa–Nath conjecture is true, i.e., $sc_6(n) > 0$ *for all positive integers except when* $n \in \{2, 12, 13, 73\}$.

Thank you!