Classical problems and zeros of polynomials

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Basic properties of sequences

Basic properties of sequences

Let $\alpha : \mathbb{N}_0 \to \mathbb{R}$ be a sequence of positive numbers.

Definition

$$R_\alpha(n) = \sqrt[n]{\alpha(n)}, \qquad Q_\alpha(n) = \frac{\alpha(n)}{\alpha(n-1)}.$$

Complex analysis:

$$\sum_{n=0}^{\infty} \alpha(n) z^n$$

has radius of convergence

$$\rho = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\alpha(n)}} = \frac{1}{\limsup_{n\to\infty} R_\alpha(n)}$$

and in case it exists also

$$\rho = \lim_{n\to\infty} \frac{\alpha(n-1)}{\alpha(n)} = \lim_{n\to\infty} \frac{1}{Q_\alpha(n)}.$$

Basic properties of sequences

Note the following equivalences for $n \geq 1$

$$R_\alpha(n-1) > R_\alpha(n) \Leftrightarrow (\alpha(n-1))^n > (\alpha(n))^{n-1} \qquad (1)$$

$$\Leftrightarrow \quad \alpha(n-1) > \left(\frac{\alpha(n)}{\alpha(n-1)}\right)^{n-1} \Leftrightarrow R_\alpha(n-1) > Q_\alpha(n) \qquad (2)$$

$$\Leftrightarrow \quad \alpha(n) > \left(\frac{\alpha(n)}{\alpha(n-1)}\right)^{n} \Leftrightarrow R_\alpha(n) > Q_\alpha(n). \qquad (3)$$

Now additional assumptions:

$$Q_\alpha(n) \quad > \quad Q_\alpha(n+1), \qquad n \geq n_0, \qquad (4)$$

$$R_\alpha(n_0-1) \quad > \quad R_\alpha(n_0). \qquad (5)$$

Note that (4) is equivalent to log-concavity for $n \geq n_0$ and (5) is only a single condition.

Proposition

The conditions (4) and (5) ensure that the sequence $\left\{ \sqrt[n]{\alpha(n)} \right\}_{n \geq n_0 - 1}$ *is strictly decreasing.*

Proof

Mathematical induction: $n = n_0$ is assumption (5).
Now assume $n \geq n_0$. By induction hypothesis

$$R_\alpha(n-1) > R_\alpha(n)\,.$$

The equivalence (3) shows that in this case

$$R_\alpha(n) > Q_\alpha(n)\,.$$

Applying log-concavity (4):

$$R_\alpha(n) > Q_\alpha(n+1)\,.$$

By (2) this is equivalent to:

$$R_\alpha(n) > R_\alpha(n+1)$$

By mathematical induction this shows strict decreasing of $\{R_\alpha(n)\}_{n \geq n_0 - 1}$.

Applications

k-colored partitions

Nicolas' 1978 result that the number of partitions function $p(n)$ is log-concave for $n \geq 26$ implies:

Theorem (Wang–Zhu 2014)

The sequence $\left\{\sqrt[n]{p(n)}\right\}$ *is stricly decreasing for* $n \geq 6$.

n	1	2	3	4	5	6	7
$\sqrt[n]{p(n)} \approx$	1.00	1.41	1.44	1.50	1.48	1.49	1.47

Bringmann, Kane, Rolen, and Tripp's 2021 result that the number of k-colored partitions $p_k(n)$ are log-concave for almost all n implies:

Theorem (H. & N. 2026)

Let $k \geq 2$. *Then* $\sqrt[n]{p_k(n)} > \sqrt[n+1]{p_k(n+1)}$ *for all* k, n *except* $p_2(1) < \sqrt{p_2(2)}$ *and* $p_3(1) = \sqrt{p_3(2)}$.

Plane partitions and over-partitions

Ono, Pujahari, and Rolen's 2022 result that the number of plain partitions $pp(n)$ are log-concave for $n \geq n_0$ implies:

Theorem (H. & N. 2026)

The sequence $\left\{ \sqrt[n]{pp(n)} \right\}_{n \geq 1}$ *is strictly decreasing for* $n \geq 6$.

n	1	2	3	4	5	6	7
$\sqrt[n]{pp(n)} \approx$	1	1.732	1.817	1.899	1.888	1.906	1.890

Engel's 2017 result that the number of overpartitions function $\bar{p}(n)$ is log-concave for $n \geq 1$ implies:

Theorem (H. & N. 2026)

We have that $\bar{p}(1) = \sqrt{\bar{p}(2)} = \sqrt[3]{\bar{p}(3)}$ *and for* $n \geq 3$

$$\sqrt[n]{\bar{p}(n)} > \sqrt[n+1]{\bar{p}(n+1)}.$$

Polynomization

Polynomization

We consider arithmetic functions $g(n)$ with positive values and $g(1) = 1$ and sequences $\alpha(n) = P_n^g(1)$ as values of polynomials at $x = 1$ in the following way: $P_0(x) = 1$,

$$P_n^g(x) = \frac{x}{n} \sum_{k=1}^{n} g(k) P_{n-k}^g(x).$$

Example

$g(n) = \sigma(n) = \sum_{d|n} n$ sum-of-divisors function for partitions. Then the k-colored partition function $p_k(n) = P_n^\sigma(k)$.

Since the leading coefficient (and its parity) determine the behavior of a polynomial for $x \to \pm\infty$ we consider

$$\Delta_n^g(x) = \left(P_n^g(x)\right)^{n+1} - \left(P_{n+1}^g(x)\right)^n$$

and its zeros determine whether

$$\sqrt[n]{P_n^g(x)} > \sqrt[n+1]{P_{n+1}^g(x)}.$$

Polynomization

Lemma (H. & N. 2026)

Let the sequence $\left\{P_n^g(x)\right\}_{n\geq 0}$ *be given.*
Let x_n^g *be the largest real zero of* $\Delta_n^g(x)$*.*
Then $\Delta_n^g(x) > 0$ *for all* $x > x_n^g$*.*

Proof.

Leading coefficient of $\Delta_n^g(x)$:

$$\left(\frac{1}{n!}\right)^{n+1} - \left(\frac{1}{(n+1)!}\right)^n = \left(\frac{1}{n!}\right)^n \left(\frac{1}{n!} - \frac{1}{(n+1)^n}\right) > 0.$$

∎

The polynomials $\Delta_n^g(x)$

$$\begin{aligned}
\Delta_1^g(x) &= \frac{x}{2}\left(x - g(2)\right), \\
\Delta_2^g(x) &= \frac{x^2}{72}\Big(7x^4 + 15g(2)\,x^3 + \Big(9\left(g(2)\right)^2 - 8g(3)\Big)\,x^2 \\
&\quad + \Big(9\left(g(2)\right)^3 - 24g(3)\,g(2)\Big)\,x - 8\left(g(2)\right)^2\Big).
\end{aligned}$$

Lemma (H. & N. 2026)

We have that $\Delta_2^g(n) \geq 0$ *for all* $x \geq g(2)$ *if and only if* $g(3) \leq \left(g(2)\right)^2$.

Proof

Let $\frac{x^2}{72}E(x) = \Delta_2^g(x)$. Then

$$E(x+g(2)) = 7x^4 + 43g(2)x^3 + \left(96(g(2))^2 - 8g(3)\right)x^2 + \left(100(g(2))^3 - 40g(3)g(2)\right)x + 8\left((g(2))^2 - g(3)\right)\left(5(g(2))^2 + g(3)\right). \tag{6}$$

Suppose $g(3) \le (g(2))^2$. Then

$$\begin{aligned} 96(g(2))^2 - 8g(3) &\ge 88(g(2))^2 \\ 100(g(2))^3 - 40g(3)g(2) &\ge 60(g(2))^3. \end{aligned}$$

We have

$$E(x) \ge 7x^4 + 43g(2)x^3 + 88(g(2))^2x^2 + 60(g(2))^3x \ge 0$$

for $x \ge 0$. If $g(3) > (g(2))^2$ then $E(0) < 0$ by (6).

Approximate largest real zeros of $\Delta_n^{\sigma}(x)$

n	x_n^{σ}
1	3.000000000
2	1.257838489
3	1.566540996
4	0.491819667
5	1.240332594
6	0.025070743
7	0.983481535
8	0.369878066
9	0.791972724
10	0.004377003
11	0.781443228
12	0.000040172
13	0.656499115
14	0.346567890
15	0.579481233

Pochhammer and Laguerre polynomials

Pochhammer polynomials

Let $g(n) = 1$. Then $P_n^g(x) = \prod_{k=0}^{n-1} \frac{x+k}{1+k}$.

Proposition

Let $g(n) = 1$ *for all* n*. Then the largest zero of*

$$\Delta_n^g(x) = \left(P_n^g(x)\right)^{n+1} - \left(P_{n+1}^g(x)\right)^n$$

is $x_n^g = 1$ *for all* $n \geq 1$*.*

Laguerre polynomials

κ-associated Laguerre polynomials for $\quad \kappa = 1$:

$$L_n^{(\kappa)}(x) = \sum_{k=0}^{n} \binom{n+\kappa}{n-k} \frac{(-x)^k}{k!}.$$

$P_n^{\mathrm{id}}(x) = \frac{x}{n} L_{n-1}^{(1)}(-x) \quad$ for $\quad g(n) = \mathrm{id}(n) = n$.

Theorem (H., N., & Tröger 2022)

Turán inequality for $\quad x \geq 0$*:*

$$\left(P_n^{\mathrm{id}}(x)\right)^2 \geq P_{n-1}^{\mathrm{id}}(x)\, P_{n+1}^{\mathrm{id}}(x).$$

Laguerre polynomials

Theorem (H. & N. 2026)

Let $g = \mathrm{id}$. *The largest real zero of* $\Delta_n^{\mathrm{id}}(x)$ *satisfies* $x_n^{\mathrm{id}} \leq 2$ *for all* $n \in \mathbb{N}$. *Further for all* $n \geq 6$ *we have* $x_n^{\mathrm{id}} \leq 1$.

Remark

For all $x \geq 1$ and $n \geq 6$

$$\frac{x^{n+1}}{n^{n+1}} \left(L_{n-1}^{(1)}(-x) \right)^{n+1} > \frac{x^n}{(n+1)^n} \left(L_n^{(1)}(-x) \right)^n .$$

Approximate largest real zeros of $\Delta_n^{\mathrm{id}}(x)$

n	x_n^{id}
1	2.000000000
2	1.559199674
3	1.324963955
4	1.171508455
5	1.060347181
6	0.974804089
7	0.906236956
8	0.849633502
9	0.801850104
10	0.760797566
11	0.725023902
12	0.693483324
13	0.665400308
14	0.640185599
15	0.617382164