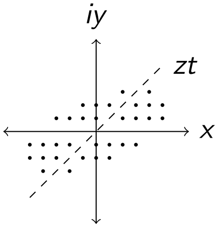
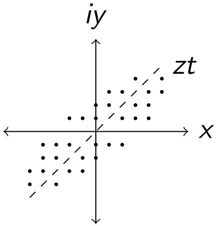
Partition Eisenstein series and semi-modular forms

Specialty Seminar in Partition Theory, q-Series and Related Topics

8 April 2021





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Department of Mathematics

Main idea

We define a new class of functions by embedding Ferrers diagrams of partitions into the complex plane.

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But first story time!

Overview

1. **Motivation**
   - The modular group and modular forms
   - Complex periodic functions
   - Semi-modular forms
2. **Construction of partition Eisenstein series**
   - Inspiration
   - Four-fold Ferrers-Young lattice
   - The partition Eisenstein series
3. **Questions and future directions**

$GL_2(\mathbb{Z})$ and some "special" subgroups

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$$SL_2(\mathbb{Z}) = \langle S, T \rangle$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Modular group

The automorphisms of the Riemann sphere are the Möbius transformations,

$$z \mapsto \frac{az+b}{cz+d}$$

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The modular group, Γ, is the set of Möbius transformations with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

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As groups, $\Gamma \cong PSL_2(\mathbb{Z})$.

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Two matrices $A, B \in SL_2(\mathbb{Z})$ define the same action iff $A = \pm B$. Therefore, we need to understand $S \cdot z$ and $T \cdot Z$, where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Modular forms

An (entire) modular form of weight k is an analytic function $f : \mathbb{H} \to \mathbb{C}$ such that

1. $f(S \cdot z) = z^k f(z)$ (weighted invariance under S)
2. $f(T \cdot z) = f(z)$ (invariance under T)
3. f remains bounded as $z \to i\infty$.

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Example: The Eisenstein series of weight $2k$ (k an integer greater than 1):

$$G_{2k}(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^{2k}}.$$

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Every entire modular form of weight k can be uniquely expressed as a $\mathbb{C}$-linear combination of $G_4^a G_6^b$ where $4a + 6b = k$ and $a, b \geq 0$.

Invariance under T

An analytic function $f : \mathbb{H} \to \mathbb{C}$ that is invariant under T satisfies

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The periodicity of f ensures that $\exp(2\pi iz) = \exp(2\pi iz')$ whenever $z - z' \in \mathbb{Z}$. Therefore, $z \mapsto q = \exp(2\pi iz)$ maps the upper half plane to the puncture unit disc, and we can expand the resulting function of q in a Laurent series

$$\sum_{n=-\infty}^{\infty} a_n q^n.$$

Example of invariance under T

The Eisenstein series can be expanded as follows:

$$G_{2k}(z) = 2\zeta(2k)\left(1 + \sum_{n\geq 1} \sigma_{2k-1}(n)q^n\right),$$

where $\sigma_a(n) = \sum_{d|n} d^a$.

What about functions that just satisfy $f(S \cdot z) = z^k f(z)$?

Semi-modular forms

An (entire) **semi-modular form** is an analytic function $f : \mathbb{H} \to \mathbb{C}$ for which $f(M \cdot z) = f(z)$ and $f(T \cdot z) = f(z)$ **or** $f(S \cdot z) = z^k f(z)$ for some k, where M is any matrix such that $GL_2(\mathbb{Z}) = \langle S, T, M \rangle$

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Note that we can take $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

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Question: Does there exist a semi-modular form that is invariant under R and S?

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$$= \tau^k \sum_{\substack{a,b\in\mathbb{Z} \\ (a,b)\neq(0,0)}} (b\tau - a)^{-k} = \tau^k G_k(\tau).$$

Partition bijections

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition (multiset of positive integers) of size $|\lambda| = \sum \lambda_i = n$, and $\mathcal{P}_n$ be the set of all partitions of size n.

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Then for a function $f : \mathcal{P}_n \to$ANYWHERE

$$\sum_{\lambda \in \mathcal{P}_n} f(\lambda) = \sum_{\lambda \in \mathcal{P}_n} f(\pi(\lambda)).$$

Embedding partitions into $\mathbb{C}$

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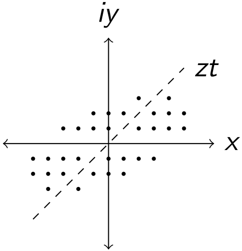
For $\lambda = (3, 2, 2, 1)$ and $z = 1 + i$:



Summing over the points in the lattice

For a partition λ, $z \in \mathbb{C} \setminus \mathbb{R}$, and $k \geq 0$, define

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4. Moreover, if $\overline{\lambda}$ is the conjugate of λ, we have

$$z^{2k} f_{2k}(\lambda, z) = f_{2k}\left(\overline{\lambda}, -\frac{1}{z}\right).$$

$z \mapsto -1/z$ induces partition conjugation!

Again let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, and $k > 0$. Then

$$f_{2k}(\lambda, z) = \frac{1}{2} \sum_{1 \leq a \leq r} \sum_{1 \leq b \leq \lambda_a} \left[(a + bz)^{-2k} + (a - bz)^{-2k}\right]$$

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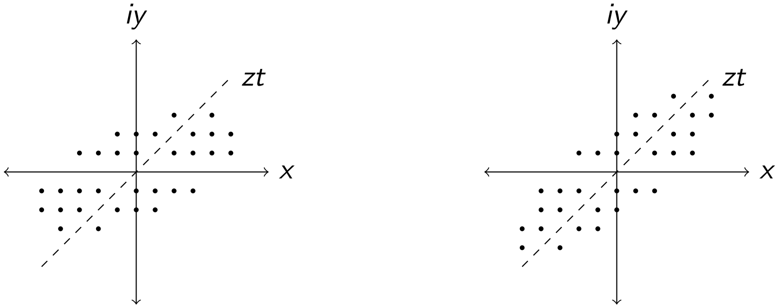
Conjugate partitions have dual lattices

Lattice for $\lambda = (3, 2, 2, 1)$ (left) and $\overline{\lambda} = (4, 3, 1)$ (right) with $z = 1 + i$.



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Hence $g_k(n, z)$ is a semi-modular form.

The q-series of a partition Eisenstein series

For $|q| < 1$

$$\mathscr{G}_k(z) \ = \ \mathscr{G}_k(z,q) \ := \sum_{n \geq 1} g_k(n,z) q^n$$

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$$= \sum_{\lambda\in\mathcal{P}} f_k(\lambda,z)q^{|\lambda|}.$$

Since $|1+z| \leq w$ for every $w \in \mathscr{F}(\lambda,z)$, we have

$$|g_k(n,z)| \leq np(n)|1+z|^{-k}$$

where $p(n) = |\mathcal{P}_n|$. Therefore, the convergence of the above series for $|q| < 1$ follows from the well known convergence of the series

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3. Connection to modular forms/quasimodular forms?
4. Other applications, particularly of Ferrers-Young lattices as combinatorial objects?

Thank You!

https://arxiv.org/pdf/2103.06239.pdf justmatt@uga.edu