Completing the A_2 Andrews–Schilling–Warnaar identities

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(Slides revised to fix typos)

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Preliminary notation

$$\begin{aligned}
(a;q)_n &= \prod_{0\le t<n}(1-aq^t),\, n\in\mathbb{Z}_{\ge 0}\cup\{\infty\}\\
(q)_n &= (q;q)_n\\
(a_1,a_2,\cdots,a_k;q)_n &= (a_1;q)_n(a_2;q)_n\cdots(a_k;q)_n\\
\theta(a;q) &= (a;q)_\infty(q/a;q)_\infty\\
\theta(a_1,a_2,\cdots,a_k;q) &= \theta(a_1;q)\theta(a_2;q)\cdots\theta(a_k;q)
\end{aligned}$$

Lepowsky and Wilson

Showed that principally specialized characters of the affine Kac–Moody Lie algebra $A_1^{(1)}$ are connected to the Andrews–Gordon and Andrews–Bressoud identities.

Andrews–Schilling–Warnaar 1999

- Invented a A_2 generalization of the (A_1) Bailey lemma.
- Found q-series multisums for many, but not all, principal characters of standard modules for $A_2^{(1)}$.
- Found identities for all moduli ≥ 5.

Sample:

$$(q)_\infty \sum_{r,s\geq 0} \frac{q^{r^2-rs+s^2+r+s}}{(q)_{r+s}(q)_{r+s+1}} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^3} = \frac{1}{\theta(q^2,q^3;\,q^6)} = \chi(\Omega(3\Lambda_0)).$$

ASW (mod 8) identities

$$\sum_{\substack{r_1\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} \frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2+r_2s_2+s_2^2+r_1+r_2+s_1+s_2}}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}} = \frac{1}{(q;q)_\infty\,(q^2,q^3,q^3,q^4,q^4,q^5,q^5,q^6;q^8)_\infty}$$

$$\sum_{\substack{r_1\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} \frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2+r_2s_2+s_2^2+r_2+s_1+s_2}}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}} = \frac{1}{(q;q)_\infty\,(q^1,q^2,q^3,q^4,q^4,q^5,q^6,q^7;q^8)_\infty}$$

$$\sum_{\substack{r_1\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} \frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2+r_2s_2+s_2^2+r_2+s_2}\left(1-q^{r_1+s_1+1}\right)}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}} = \frac{1}{(q;q)_\infty\,(q^1,q^1,q^3,q^3,q^5,q^5,q^7,q^7;q^8)_\infty}$$

$$\sum_{\substack{r_1\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} \frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2+r_2s_2+s_2^2+s_2}\left(1-q^{r_2+s_1+1}\right)}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}} = \frac{1}{(q;q)_\infty\,(q^1,q^1,q^2,q^4,q^4,q^6,q^7,q^7;q^8)_\infty}$$

$$\sum_{\substack{r_1\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} \frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2+r_2s_2+s_2^2+s_1+s_2}}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}} = \frac{1}{(q;q)_\infty\,(q^1,q^2,q^2,q^3,q^5,q^6,q^6,q^7;q^8)_\infty}$$

ASW (mod 8) identities

Define:

$$S_8(A,B \mid C,D) := \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 + r_2 s_2 + s_2^2 + Ar_1 + Br_2 + Cs_1 + Ds_2}}{(q)_{r_1 - r_2}(q)_{s_1 - s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}$$

ASW (mod 8) identities

$$S_8(1,1\mid 1,1)=\frac{\theta\left(q^1,q^1,q^2;q^8\right)_\infty\left(q^8;q^8\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_8(0,1\mid 1,1)=\frac{\theta\left(q^1,q^2,q^3;q^8\right)_\infty\left(q^8;q^8\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_8(0,0\mid 1,1)=\frac{\theta\left(q^2,q^2,q^4;q^8\right)_\infty\left(q^8;q^8\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_8(0,1\mid 0,1)-qS_8(1,1\mid 1,1)=\frac{\theta\left(q^2,q^3,q^3;q^8\right)_\infty\left(q^8;q^8\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_8(0,0\mid 0,1)-qS_8(0,1\mid 1,1)=\frac{\theta\left(q^1,q^3,q^4;q^8\right)_\infty\left(q^8;q^8\right)_\infty^2}{(q;q)_\infty^3}$$

What are the products?

$$\begin{aligned}\theta\left(q^1,q^3,q^4;q^8\right)_\infty\left(q^8;q^8\right)_\infty^2 &= \left(q^1,q^3,q^4,q^4,q^5,q^7,q^8,q^8;q^8\right)_\infty\\ &= \left(q^1,q^3,q^4,q^{1+3},q^{1+4},q^{3+4},q^8,q^8;q^8\right)_\infty\\ \theta\left(q^1,q^1,q^2;q^8\right)_\infty\left(q^8;q^8\right)_\infty^2 &= \left(q^1,q^1,q^2,q^6,q^7,q^7,q^8,q^8;q^8\right)_\infty\\ &= \left(q^1,q^1,q^6,q^{1+1},q^{1+6},q^{1+6},q^8,q^8;q^8\right)_\infty\end{aligned}$$

What are the products?

If $k = a + b + c + 3$, define

$$J_k(a,b,c) := \left(q^{a+1}, q^{b+1}, q^{c+1}, q^{(a+1)+(b+1)}, q^{(a+1)+(c+1)}, q^{(b+1)+(c+1)}, q^k, q^k; q^k\right)_\infty$$

Then:

$$\begin{aligned}
\theta\left(q^1, q^1, q^2; q^8\right)_\infty \left(q^8; q^8\right)_\infty^2 &= J_8(5,0,0)\\
\theta\left(q^1, q^2, q^3; q^8\right)_\infty \left(q^8; q^8\right)_\infty^2 &= J_8(4,1,0)\\
\theta\left(q^2, q^2, q^4; q^8\right)_\infty \left(q^8; q^8\right)_\infty^2 &= J_8(3,1,1)\\
\theta\left(q^2, q^3, q^3; q^8\right)_\infty \left(q^8; q^8\right)_\infty^2 &= J_8(2,2,1)\\
\theta\left(q^1, q^3, q^4; q^8\right)_\infty \left(q^8; q^8\right)_\infty^2 &= J_8(3,2,0)
\end{aligned}$$

One product for each partition of $8 - 3 = 5$ into ≤ 3 parts.

ASW (mod 8) identities

$$\begin{aligned}
S_8(1,1 \mid 1,1) &= \frac{J_8(5,0,0)}{(q;q)_\infty^3} \\
S_8(0,1 \mid 1,1) &= \frac{J_8(4,1,0)}{(q;q)_\infty^3} \\
S_8(0,0 \mid 1,1) &= \frac{J_8(3,1,1)}{(q;q)_\infty^3} \\
S_8(0,1 \mid 0,1) - qS_8(1,1 \mid 1,1) &= \frac{J_8(2,2,1)}{(q;q)_\infty^3} \\
S_8(0,0 \mid 0,1) - qS_8(0,1 \mid 1,1) &= \frac{J_8(3,2,0)}{(q;q)_\infty^3}
\end{aligned}$$

ASW (mod 10) identities

$$S_8(A,B \mid C,D) = \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 + r_2 s_2 + s_2^2 + Ar_1 + Br_2 + Cs_1 + Ds_2}}{(q)_{r_1 - r_2}(q)_{s_1 - s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}$$

$$S_{10}(A,B \mid C,D) := \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 - r_2 s_2 + s_2^2 + Ar_1 + Br_2 + Cs_1 + Ds_2}}{(q)_{r_1 - r_2}(q)_{s_1 - s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}$$

ASW (mod 10) identities

$$S_{10}(1,1 \mid 1,1) = \frac{\theta\left(q^1,q^1,q^2;q^{10}\right)_\infty \left(q^{10};q^{10}\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_{10}(0,1 \mid 1,1) = \frac{\theta\left(q^1,q^2,q^3;q^{10}\right)_\infty \left(q^{10};q^{10}\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_{10}(0,0 \mid 1,1) = \frac{\theta\left(q^1,q^3,q^4;q^{10}\right)_\infty \left(q^{10};q^{10}\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_{10}(0,1 \mid 0,1) - qS_{10}(1,1 \mid 1,1) = \frac{\theta\left(q^2,q^2,q^4;q^{10}\right)_\infty \left(q^{10};q^{10}\right)_\infty^2}{(q;q)_\infty^3}$$

$$S_{10}(0,1 \mid 0,1) - qS_{10}(1,1 \mid 1,1) = \frac{\theta\left(q^3,q^3,q^4;q^{10}\right)_\infty \left(q^{10};q^{10}\right)_\infty^2}{(q;q)_\infty^3}$$

ASW (mod 10) identities

$$S_{10}(1,1\mid 1,1)=\frac{J_{10}(7,0,0)}{(q;q)_\infty^3}$$

$$S_{10}(0,1\mid 1,1)=\frac{J_{10}(6,1,0)}{(q;q)_\infty^3}$$

$$S_{10}(0,0\mid 1,1)=\frac{J_{10}(5,2,0)}{(q;q)_\infty^3}$$

$$S_{10}(0,1\mid 0,1)-qS_{10}(1,1\mid 1,1)=\frac{J_{10}(5,1,1)}{(q;q)_\infty^3}$$

$$S_{10}(0,1\mid 0,1)-qS_{10}(1,1\mid 1,1)=\frac{J_{10}(3,2,2)}{(q;q)_\infty^3}$$

Missing (mod 10) identities

$$S_{10}(0,0 \mid 0,0) - qS_{10}(0,1 \mid 0,1) = \frac{J_{10}(4,2,1)}{(q;q)_\infty^3}$$

$$S_{10}(-1,0 \mid 1,1) - S_{10}(0,1 \mid 0,1) + qS_{10}(1,1 \mid 1,1) = \frac{J_{10}(4,3,0)}{(q;q)_\infty^3}$$

$$\begin{aligned} &S_{10}(-1,0 \mid 0,1) - S_{10}(0,1 \mid 0,0) + qS_{10}(1,1 \mid 0,1) \\ &\qquad -qS_{10}(-1,1 \mid 1,1) - S_{10}(0,0 \mid 1,1) = \frac{J_{10}(3,3,1)}{(q;q)_\infty^3} \end{aligned}$$

The triangle picture

The arrangement is as follows:

$J_{10}(7,0,0)$		
$J_{10}(6,1,0)$	$J_{10}(5,1,1)$	
$J_{10}(5,2,0)$	$J_{10}(4,2,1)$	$J_{10}(3,2,2)$
$J_{10}(4,3,0)$	$J_{10}(3,3,1)$	

The triangle picture

The arrangement is as follows:

$J_{10}(7,0,0) =$ $S_{10}(1,1 \mid 1,1)$		
$J_{10}(6,1,0) =$ $S_{10}(0,1 \mid 1,1)$	$J_{10}(5,1,1) =$ $S_{10}(0,1 \mid 0,1)$ $-qS_{10}(1,1 \mid 1,1)$	
$J_{10}(5,2,0) =$ $S_{10}(0,0 \mid 1,1)$	$J_{10}(4,2,1) =$ $S_{10}(0,0 \mid 0,1)$ $-qS_{10}(0,1 \mid 1,1)$	$J_{10}(3,2,2) =$ $S_{10}(0,0 \mid 0,0)$ $-qS_{10}(0,1 \mid 0,1)$
$J_{10}(4,3,0)$	$J_{10}(3,3,1)$	

ASW (mod 13) identities

$$S_{10}(A,B \mid C,D) = \sum_{\substack{r_1\ge r_2\ge 0\\ s_1\ge s_2\ge 0}} \frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2-r_2s_2+s_2^2+Ar_1+Br_2+Cs_1+Ds_2}}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}$$

$$S_{13}(A,B,C \mid D,E,F) :=$$

$$\sum_{\substack{r_1\ge r_2\ge r_3\ge 0\\ s_1\ge s_2\ge s_3\ge 0}} \frac{q^{\left(r_1^2-r_1s_1+s_1^2\right)+\left(r_2^2-r_2s_2+s_2^2\right)+\left(r_3^2-r_3s_3+s_3^2\right)+Ar_1+Br_2+Cr_3+Ds_1+Es_2+Fs_3}}{(q)_{r_1-r_2}(q)_{r_2-r_3}(q)_{s_1-s_2}(q)_{s_2-s_3}(q)_{r_3}(q)_{s_3}(q)_{r_3+s_3+1}}$$

The triangle picture for (mod 13)

For the products $J_{13}(a, b, c)/(q;q)_\infty^3$, index the rows by b and the columns by c:

	$c = 0$	$c = 1$	$c = 2$	$c = 3$
$b = 0$	$S(1,1,1 \mid 1,1,1)$			
$b = 1$	$S(0,1,1 \mid 1,1,1)$	$S(0,1,1 \mid 0,1,1)$ $-qS(1,1,1 \mid 1,1,1)$		
$b = 2$	$S(0,0,1 \mid 1,1,1)$	$S(0,0,1 \mid 0,1,1)$ $-qS(0,1,1 \mid 1,1,1)$	$S(0,0,1 \mid 0,0,1)$ $-qS(0,1,1 \mid 0,1,1)$	
$b = 3$	$S(0,0,0 \mid 1,1,1)$	$S(0,0,0 \mid 0,1,1)$ $-qS(0,0,1 \mid 1,1,1)$	$S(0,0,0 \mid 0,0,1)$ $-qS(0,0,1 \mid 0,1,1)$	$S(0,0,0 \mid 0,0,0)$ $-qS(0,0,1 \mid 0,0,1)$
$b = 4$	$J_{13}(6,4,0)$????????	$J_{13}(5,4,1)$????????		
$b = 5$	$J_{13}(5,5,0)$????????			

Recap so far.

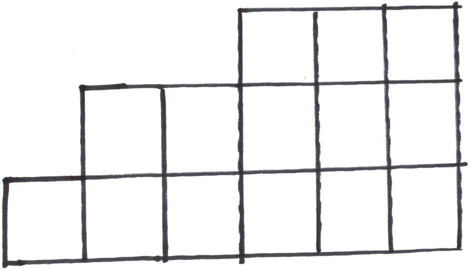
- We have conjectures for all "above-the-line" identities for (mod 3k+1).
- We also have (not shown) similar conjectures for the (mod 3k-1) and (mod 3k) cases.
- No real patterns for the "below-the-line" identities.
- Q: How to prove?
- A: Cylindric partitions.

Cylindric partitions

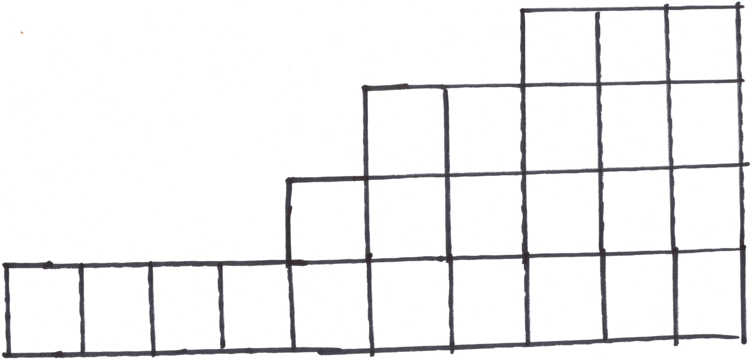
Introduced in 1997 by Gessel and Krattenthaler.
Idea: Take a composition $(c_0, \ldots, c_r)$ of ℓ, where $c_0, \ldots, c_r$ are non-negative integers and $\ell = c_0 + c_1 + \cdots + c_r$.
Consider a sequence $\Lambda = (\lambda^{(0)}, \ldots, \lambda^{(r)})$ where each $\lambda^{(j)}$ is a partition $\lambda^{(j)} = \lambda_1^{(j)} + \lambda_2^{(j)} + \cdots$ arranged in a weakly descending order.
Assume that each $\lambda^{(j)}$ continues indefinitely with only finitely many non-zero entries and ends with an infinite sequence of zeros.
We say that Λ is a cylindric partition with profile c if for all i and j,

$$\lambda_j^{(i)} \geq \lambda_{j+c_{(i+1)}}^{(i+1)} \quad \text{and} \quad \lambda_j^{(r)} \geq \lambda_{j+c_0}^{(0)}.$$

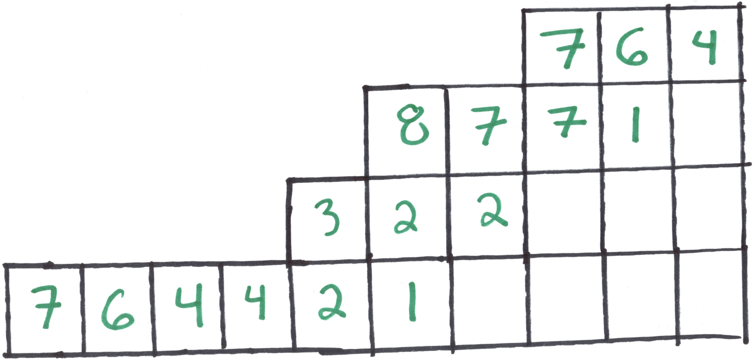
Cylindric partition example: profile $(4, 2, 1)$



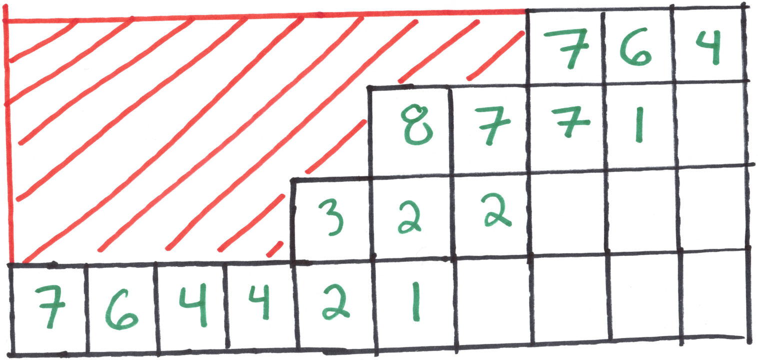
Cylindric partition example: profile $(4, 2, 1)$



Cylindric partition example: profile $(4, 2, 1)$



Cylindric partition example: profile $(4, 2, 1)$



Cylindric partitions

Let $F_c(z,q) = F_{(c_0,c_1,\ldots,c_r)}(z,q)$ be the generating function of cylindric partitions of profile c, where z corresponds to the maximum part statistic and q corresponds to the total weight of the cylindric partition.

$$F_c(z,q) = \sum_{\Lambda \in \mathcal{C}_c} z^{\max(\Lambda)} q^{\mathrm{wt}(\Lambda)},$$

where $\mathcal{C}_c$ is the set of cylindric partitions of profile c
Obvious cyclic symmetry:

$$F_{(c_0,c_1,\ldots,c_r)}(z,q) = F_{(c_1,c_2,\ldots,c_r,c_0)}(z,q).$$

Cylindric partitions and the product side

Theorem (Borodin 2007)

Let $c = (c_0, c_1, \ldots, c_r)$ be a composition of ℓ. Set $m = r + 1 + \ell$. Then:

$$F_c(1, q) = \frac{(q^m; q^m)_\infty^r}{(q)_\infty^{r+1}} \prod_{1 \le i < j \le r+1} \theta(q^{j-i+c_i+c_{i+1}+\cdots+c_{j-1}}; q^m)$$

Cylindric partitions and the product side

Set $r = 2$:

Theorem (Borodin 2007)

Let $c = (c_0, c_1, c_2)$ be a composition of ℓ. Set $m = \ell + 3$. Then:

$$F_c(1, q) = \frac{(q^m; q^m)_\infty^2}{(q)_\infty^3} \prod_{1 \le i < j \le 3} \theta(q^{j-i+c_i+c_{i+1}+\cdots+c_{j-1}}; \, q^m)$$

Cylindric partitions and the product side

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Let $c = (c_0, c_1, c_2)$ be a composition of ℓ. Set $m = \ell + 3$. Then:

$$F_c(1, q) = \frac{(q^m; q^m)_\infty^2}{(q)_\infty^3} \prod_{1 \leq i < j \leq 3} \theta(q^{j-i+c_i+c_{i+1}+\cdots+c_{j-1}}; q^m)$$

$$= \frac{J_m\left(c_0, c_1, c_2\right)}{(q)_\infty^3}$$

Cylindric partitions and the product side

If $r = 1$ (profile has length 2), this can be used to deduce Andrews-Gordon identities.
New goal: show that the series count cylindric partitions.

Corteel–Welsh recurrence

Define

$$\begin{aligned} G_c(z,q) &= (zq;\,q)_\infty F_c(z,q), \\ H_c(z,q) &= \frac{G_c(z,q)}{(q;\,q)_\infty} = \frac{(zq;\,q)_\infty}{(q;\,q)_\infty} F_c(z,q). \end{aligned}$$

Note that we have:

$$\begin{aligned} H_c(1,q) = F_c(1,q) &= \frac{(q^m;q^m)_\infty^r}{(q)_\infty^{r+1}} \prod_{1\le i<j\le r+1} \theta(q^{j-i+c_i+c_{i+1}+\cdots+c_{j-1}};\,q^m) \\ &= \frac{J_m\left(c_0,c_1,c_2\right)}{(q)_\infty^3} \end{aligned}$$

Corteel–Welsh recurrence for our case

Now, given a composition c, we define the following:

$$I_c = \{0 \leq i \leq r \,|\, c_i \neq 0\}.$$

With $c_{-1} = c_r$ and a subset $\phi \subsetneq J \subseteq I_c$, define a new composition $c(J) = (c_0(J), c_1(J), \ldots, c_r(J))$ by:

$$c_i(J) = \begin{cases} c_i - 1 & i \in J \text{ and } (i-1) \notin J, \\ c_i + 1 & i \notin J \text{ and } (i-1) \in J, \\ c_i & \text{otherwise.} \end{cases}$$

It is not hard to check that if c is a composition of ℓ, then so is $c(J)$ for any $\phi \subsetneq J \subseteq I_c$.

Corteel–Welsh recurrence for our case

We then have:

Theorem (Corteel–Welsh)

Fix r, ℓ. The functions G_c for all compositions c of ℓ with length $r+1$ are the unique solutions to the following (finite) system of functional equations and the initial conditions:

$$G_c(z,q) = \sum_{\phi \subsetneq J \subseteq I_c} (-1)^{|J|-1} (zq;\, q)_{|J|-1} G_{c(J)}(zq^{|J|};\, q)$$

$$G_c(0,q) = 1, \quad G_c(z,0) = 1.$$

Proof idea: inclusion/exclusion.

Corteel–Welsh recurrence

Corollary

Fix r, ℓ. The functions H_c for all compositions c of ℓ with length $r+1$ are the unique solutions to the following (finite) system of functional equations and the initial conditions:

$$H_c(z,q) = \sum_{\phi \subsetneq J \subseteq I_c} (-1)^{|J|-1} (zq;\, q)_{|J|-1} H_{c(J)}(zq^{|J|};\, q),$$

$$H_c(0,q) = \frac{1}{(q)_\infty}, \quad H_c(z,0) = 1.$$

Corteel–Welsh recurrence for (mod 10)

$$
\begin{aligned}
H_{7,0,0}(z,q) &= H_{6,1,0}(zq,q)\\
H_{6,1,0}(z,q) &= H_{5,2,0}(zq,q) + H_{6,0,1}(zq,q) - (1-zq)H_{5,1,1}\left(zq^2,q\right)\\
H_{6,0,1}(z,q) &= H_{5,1,1}(zq,q) + H_{7,0,0}(zq,q) - (1-zq)H_{6,1,0}\left(zq^2,q\right)\\
H_{5,2,0}(z,q) &= H_{4,3,0}(zq,q) + H_{5,1,1}(zq,q) - (1-zq)H_{4,2,1}\left(zq^2,q\right)\\
H_{5,0,2}(z,q) &= H_{4,1,2}(zq,q) + H_{6,0,1}(zq,q) - (1-zq)H_{5,1,1}\left(zq^2,q\right)\\
H_{5,1,1}(z,q) &= H_{4,2,1}(zq,q) + H_{5,0,2}(zq,q) + H_{6,1,0}(zq,q)\\
&\quad - (1-zq)H_{4,1,2}\left(zq^2,q\right) - (1-zq)H_{5,2,0}\left(zq^2,q\right)\\
&\quad - (1-zq)H_{6,0,1}\left(zq^2,q\right) + (1-zq)\left(1-zq^2\right)H_{5,1,1}\left(zq^3,q\right)\\
H_{4,3,0}(z,q) &= H_{4,0,3}(zq,q) + H_{4,2,1}(zq,q) - (1-zq)H_{3,3,1}\left(zq^2,q\right)\\
H_{4,0,3}(z,q) &= H_{3,3,1}(zq,q) + H_{5,0,2}(zq,q) - (1-zq)H_{4,1,2}\left(zq^2,q\right)
\end{aligned}
$$

Corteel–Welsh recurrence for (mod 10)

$$\begin{aligned}
H_{4,2,1}(z,q) &= H_{3,3,1}(zq,q) + H_{4,1,2}(zq,q) + H_{5,2,0}(zq,q) \\
&\quad - (1-zq)H_{3,2,2}\left(zq^2,q\right) - (1-zq)H_{4,3,0}\left(zq^2,q\right) \\
&\quad - (1-zq)H_{5,1,1}\left(zq^2,q\right) + (1-zq)\left(1-zq^2\right)H_{4,2,1}\left(zq^3,q\right) \\
H_{4,1,2}(z,q) &= H_{3,2,2}(zq,q) + H_{4,0,3}(zq,q) + H_{5,1,1}(zq,q) \\
&\quad - (1-zq)H_{3,3,1}\left(zq^2,q\right) - (1-zq)H_{4,2,1}\left(zq^2,q\right) \\
&\quad - (1-zq)H_{5,0,2}\left(zq^2,q\right) + (1-zq)\left(1-zq^2\right)H_{4,1,2}\left(zq^3,q\right) \\
H_{3,3,1}(z,q) &= H_{4,1,2}(zq,q) + H_{3,2,2}(zq,q) + H_{4,3,0}(zq,q) \\
&\quad - (1-zq)H_{3,2,2}\left(zq^2,q\right) - (1-zq)H_{4,0,3}\left(zq^2,q\right) \\
&\quad - (1-zq)H_{4,2,1}\left(zq^2,q\right) + (1-zq)\left(1-zq^2\right)H_{3,3,1}\left(zq^3,q\right) \\
H_{3,2,2}(z,q) &= H_{3,2,2}(zq,q) + H_{3,3,1}(zq,q) + H_{4,2,1}(zq,q) \\
&\quad - (1-zq)H_{3,2,2}\left(zq^2,q\right) - (1-zq)H_{3,3,1}\left(zq^2,q\right) \\
&\quad - (1-zq)H_{4,1,2}\left(zq^2,q\right) + (1-zq)\left(1-zq^2\right)H_{3,2,2}\left(zq^3,q\right)
\end{aligned}$$

How to insert in z?

$$S_{10}(A, B \mid C, D) := \sum_{\substack{r_1 \geq r_2 \geq 0 \\ s_1 \geq s_2 \geq 0}} z^{r_1} \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 - r_2 s_2 + s_2^2 + A r_1 + B r_2 + C s_1 + D s_2}}{(q)_{r_1 - r_2} (q)_{s_1 - s_2} (q)_{r_2} (q)_{s_2} (q)_{r_2 + s_2 + 1}}$$

ASW (mod 10) identities

$$
\begin{aligned}
H_{(7,0,0)} &= S_{10}(1,1 \mid 1,1) \\
H_{(6,1,0)} &= S_{10}(0,1 \mid 1,1) \\
H_{(5,2,0)} &= S_{10}(0,0 \mid 1,1) \\
H_{(5,1,1)} &= S_{10}(0,1 \mid 0,1) - qS_{10}(1,1 \mid 1,1) \\
H_{(4,2,1)} &= S_{10}(0,0 \mid 0,1) - qS_{10}(0,1 \mid 1,1) \\
H_{(3,2,2)} &= S_{10}(0,0 \mid 0,0) - qS_{10}(0,1 \mid 0,1)
\end{aligned}
$$

ASW (mod 10) identities

$$\begin{aligned}
H_{(6,1,0)} &= S_{10}(0,1 \mid 1,1) \\
H_{(6,0,1)} &= S_{10}(1,1 \mid 0,1) - q(1-z)S_{10}(2,1 \mid 1,1) \\
H_{(5,2,0)} &= S_{10}(0,0 \mid 1,1) \\
H_{(5,0,2)} &= S_{10}(1,1 \mid 0,0) - q(1-z)S_{10}(2,1 \mid 0,1) \\
H_{(4,2,1)} &= S_{10}(0,0 \mid 0,1) - qS_{10}(0,1 \mid 1,1) \\
H_{(4,1,2)} &= S_{10}(0,1 \mid 0,0) - qS_{10}(1,1 \mid 0,1)
\end{aligned}$$

ASW (mod 10) identities

To find identities for $H_{(4,3,0)}$, $H_{(4,0,3)}$, and $H_{(3,3,1)}$, use Corteel–Welsh recursions.
Solving the recurrence for $H_{(5,2,0)}$ gets us:

$$\begin{aligned} H_{(4,3,0)} = \quad & S_{10}(-1,0 \mid 1,1) - S_{10}(0,1 \mid 0,1) \\ & + (1-z)S_{10}(1,0 \mid 0,1) + zqS_{10}(1,1 \mid 1,1). \end{aligned}$$

Solving the recurrence for $H_{(4,2,1)}$ gives:

$$\begin{aligned} H_{(3,3,1)} = \quad & S_{10}(-1,0 \mid 0,1) - S_{10}(0,1 \mid 0,0) + (1-z)S_{10}(1,0 \mid 0,0) \\ & + zqS_{10}(1,1 \mid 0,1) - qS_{10}(-1,1 \mid 1,1) - zS_{10}(0,0 \mid 1,1). \end{aligned}$$

ASW (mod 10) identities

Solving the recurrence for $H_{(4,3,0)}$ gives

$$\begin{aligned} H_{(4,0,3)} = \quad & S_{10}(0,0 \mid 0,1) - qS_{10}(0,1 \mid 1,1) - zqS_{10}(1,0 \mid 1,1) \\ & - S_{10}(1,1 \mid 0,0) + (1-qz)S_{10}(2,0 \mid 0,0) \\ & + zq^2 S_{10}(2,1 \mid 0,1) + zqS_{10}(2,1 \mid 0,0) \end{aligned}$$

Relations

For $A, B, C, D \in \mathbb{Z}$, we have:

$$
\begin{aligned}
&S_{10}(A, B \mid C, D) - S_{10}(A+1, B-1 \mid C, D) - zq^{A+1} S_{10}(A+2, B \mid C-1, D) &&= 0 \quad (R_1)\\
&S_{10}(A, B \mid C, D) - S_{10}(A, B \mid C+1, D-1) - q^{C+1} S_{10}(A-1, B \mid C+2, D) &&= 0 \quad (R_2)\\
&S_{10}(A, B \mid C, D) - S_{10}(A, B \mid C, D+1) - qS_{10}(A, B+1 \mid C, D+1)\\
&\qquad + qS_{10}(A, B+1 \mid C, D+2) - q^{C+D+2} S_{10}(A-1, B-1 \mid C+2, D+2) &&= 0 \quad (R_3)\\
&S_{10}(A, B \mid C, D) - S_{10}(A, B+1 \mid C, D) - qS_{10}(A, B+1 \mid C, D+1)\\
&\qquad + qS_{10}(A, B+2 \mid C, D+1) - zq^{A+B+2} S_{10}(A+2, B+2 \mid C-1, D-1) &&= 0 \quad (R_4)
\end{aligned}
$$

Proof idea for relations

$$
\begin{aligned}
&S_{10}(A,B \mid C,D) - S_{10}(A+1,B-1 \mid C,D)\\
&\quad = \sum_{\substack{r_1\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} z^{r_1}\frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2-r_2s_2+s_2^2+Ar_1+Br_2+Cs_1+Ds_2}(1-q^{r_1-r_2})}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}\\
&\quad = \sum_{\substack{\mathbf{r_1}>r_2\geq 0\\ s_1\geq s_2\geq 0}} z^{r_1}\frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2-r_2s_2+s_2^2+Ar_1+Br_2+Cs_1+Ds_2}}{(q)_{\mathbf{r_1-r_2-1}}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}\\
&\quad = \sum_{\substack{\mathbf{r_1}\geq r_2\geq 0\\ s_1\geq s_2\geq 0}} z^{r_1+1}\frac{q^{r_1^2-r_1s_1+s_1^2+r_2^2-r_2s_2+s_2^2+(\mathbf{A+2})\mathbf{r_1}+Br_2+(\mathbf{C-1})\mathbf{s_1}+Ds_2+\mathbf{A+1}}}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}}\\
&\quad = zq^{A+1}S_{10}(A+2,B \mid C-1,D).
\end{aligned}
$$

Proof procedure

- Replace $H_c(z, q)$ with the conjectured multisums in the Corteel–Welsh recurrences.
- Show that the resulting equations follow from (R_1), (R_2), (R_3), (R_4).
- Once we've done that (and we check initial conditions), we're done.

Sample calculations

$H_{(6,1,0)}$ recurrence:

$$zqS_{10}(2,1 \mid 0,1) + S_{10}(1,0 \mid 1,1) - S_{10}(0,1 \mid 1,1) = 0$$

which is nothing but $(R_1)\,(0,1,1,1)$.

Sample calculations

$H_{(5,1,1)}$ recurrence:

$$
\begin{aligned}
& q^3 z^2 S_{10}(3,1 \mid 0,1) - q^2 z S_{10}(3,1 \mid 0,1) + qz S_{10}(2,1 \mid 0,0) \\
& + qz S_{10}(2,0 \mid 1,1) - S_{10}(2,0 \mid 1,1) + S_{10}(1,1 \mid 1,1) \\
& + S_{10}(1,0 \mid 0,1) - S_{10}(0,1 \mid 0,1) = 0
\end{aligned}
$$

which can be obtained as:

$$
\begin{aligned}
& q\,(R_1)\,(-1,1,2,1) + (R_1)\,(0,1,1,0) - (1-qz)\,(R_1)\,(1,1,1,1) \\
& + (R_2)\,(0,1,0,1) - (R_2)\,(1,0,0,1)
\end{aligned}
$$

Sample calculations

$H_{(3,3,1)}$ recurrence:

$$
\begin{aligned}
&q^7z^3S(4,1,0,1) - q^6z^3S(4,0,0,0) - q^6z^3S(3,0,1,1) - q^6z^2S(4,1,0,1)\\
&+ q^5z^2S(4,0,0,0) + q^5z^2S(3,0,1,1) + q^4z^2S(4,1,0,0)\\
&- q^4z^2S(2,1,1,1) + q^3z^2S(4,0,0,0) - q^3z^2S(3,1,0,0)\\
&+ q^3z^2S(2,0,0,1) - q^3zS(4,1,0,0) + q^3zS(2,1,1,1) - q^2zS(4,0,0,0)\\
&+ q^2zS(3,1,0,0) - q^2zS(2,1,0,1) - q^2zS(2,0,0,1) + qzS(2,0,0,0)\\
&- qzS(1,1,0,1) + qS(2,1,1,1) - qS(1,1,0,1) + zS(1,0,0,0)\\
&+ zS(0,0,1,1) + qS(-1,1,1,1) - S(2,0,0,0) - S(1,1,0,1)\\
&+ S(1,1,0,0) + S(0,1,0,0) + S(0,0,1,1) - S(-1,0,0,1) = 0
\end{aligned}
$$

which can be obtained as:
[217 pages of Maple output]

Summary of our work

- Conjectures for all three families: $\pmod{3k+1}$, $\pmod{3k-1}$, $\pmod{3k}$ for all "above-the-line" sum sides
- A proof that we can use the Corteel–Welsh recurrences to determine all "below-the-line" sum sides, given the "above-the-line" sum sides
- Proofs of the identities in (mod 5), (mod 6), (mod 7), (mod 8), (mod 10)
  - New cases: (mod 6), (mod 10)

Avenues for future work

- Prove (mod 9) identities
- Prove identities in general
- A_3? A_r?
- Cylindric partitions with profile of length > 3?

Avenues for future work

Experimentally, we have the following conjectural identity.

Conjecture

We have:

$$F_{(k,k,k-3)}(z,q) = F_{(k+1,k-2,k-2)}(z,q) - zq^{k-2}F_{(3k-3,0,0)}(z,q).$$

If we set $z = 1$, this follows from an identity of Weierstraß, and gives us a few new (proved!) identities in the (mod $3k$) case.

Avenues for future work

Set $k = 3$:

Conjecture

$$F_{(3,3,0)}(z,q) = F_{(4,1,1)}(z,q) - zqF_{(6,0,0)}(z,q)$$

It would be interesting to find a combinatorial proof of this.