Companions to the Andrews-Gordon and Andrews-Bressoud identities, and recent conjectures of Capparelli, Meurman, Primc, and Primc

Matthew C. Russell

Texas A&M University

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Outline

- **Andrews-Gordon-Bressoud identities**
- CMPP admissible partitions
- Functional equations for admissible partitions: 2ℓ rows
- Functional equations for admissible partitions: $2\ell - 1$ rows
- Completing the bivariate generating function proof
- Bijection with two-rowed cylindric partitions

Preliminary notation

$$(a;q)_n = \prod_{0\le t<n} (1-aq^t), n \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$$

$$(a_1, a_2, \cdots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_k;q)_n$$

$$N_i = n_i + n_{i+1} + \cdots + n_\ell$$

Gordon-Andrews identities

Theorem (Gordon, Andrews)

Fix $\ell \geq 1$, and let $0 \leq i \leq \ell$. Let $B_{\ell,i}(n)$ denote the number of partitions of n satisfying $\lambda_j - \lambda_{j+\ell} \geq 2$, and there are at most i occurrences of 1 as parts. Let $A_{\ell,i}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm(i+1) \pmod{2\ell+3}$. Then $A_{\ell,i}(n) = B_{\ell,i}(n)$ for all n.

$$\sum_{n\geq 0} B_{\ell,i}(n) q^n = \sum_{n_1,n_2,\ldots,n_\ell \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}} = \frac{\left(q^{i+1}, q^{2\ell+2-i}, q^{2\ell+3}; q^{2\ell+3}\right)_\infty}{(q;q)_\infty}.$$

Refining the multisum

Let $B_{\ell,i,m}(n)$ denote the number of partitions of n with exactly m parts satisfying $\lambda_j - \lambda_{j+\ell} \geq 2$, and there are at most i occurrences of 1 as parts. Then

$$\sum_{m,n\geq 0} B_{\ell,i,m}(n) x^m q^n$$

$$= \sum_{n_1,n_2,\ldots,n_\ell \geq 0} \frac{x^{N_1+N_2+\cdots+N_\ell} q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}}.$$

Andrews-Bressoud identities

Theorem

Fix $\ell \geq 1$, and let $0 \leq i \leq \ell$. Let $B^{\star}_{\ell,i}(n)$ denote the number of partitions of n satisfying $\lambda_j - \lambda_{j+\ell} \geq 2$, and, if $\lambda_j - \lambda_{j+\ell-1} < 2$, then $\lambda_j + \lambda_{j+1} + \cdots + \lambda_{j+\ell-1} \equiv i \pmod 2$, and there are at most i occurrences of 1 as parts. Then

$$\sum_{n \geq 0} B^{\star}_{\ell,i}(n) q^n = \sum_{n_1, n_2, \ldots, n_\ell \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_\ell^2 + N_{i+1} + N_{i+2} + \cdots + N_\ell}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_{\ell-1}} \left(q^2;q^2\right)_{n_\ell}}$$

$$= \frac{\left(q^{i+1}, q^{2\ell+1-i}, q^{2\ell+2}; q^{2\ell+2}\right)_\infty}{(q;q)_\infty}.$$

Refining the multisum

Let $B^{\star}_{\ell,i,m}(n)$ denote the number of partitions of n counted by $B^{\star}_{\ell,i}(n)$ with exactly m parts. Then

$$\sum_{m,n\geq 0} B^{\star}_{\ell,i,m}(n) x^m q^n$$

$$= \sum_{n_1,n_2,\ldots,n_\ell \geq 0} \frac{x^{N_1+N_2+\cdots+N_\ell} q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{\ell-1}}(q^2;q^2)_{n_\ell}}.$$

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Capparelli, Meurman, Primc, Primc partitions: 2ℓ rows

Consider the following array of colored integers:

$$
\begin{array}{cccccccccccccc}
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & & \\
& 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & \\
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & & \\
& 2 & & 4 & & 6 & & 8 & & 10 & & 12 & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\
1 & & 3 & & 5 & & 7 & & 9 & & 11 & & & \\
& 2 & & 4 & & 6 & & 8 & & 10 & & 12 & &
\end{array}
$$

Parts with the same value (but in different rows) are distinct.

CMPP partitions: 2ℓ rows

Associate an array of frequencies:

$$
\begin{array}{ccccccccccccccc}
k_1 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & \\
k_3 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & \ldots, \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
k_2 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & \\
 & k_0 & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} &
\end{array}
$$

where each f_j indicates how many times the corresponding part occurs in the original array.

The nonnegative integers $k_0, \ldots, k_\ell$ provide initial conditions.

Downward paths

$$
\begin{array}{cccccccccccccc}
k_1 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} \\
k_3 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} \\
k_2 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & \\
 & k_0 & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12}
\end{array}
\quad \ldots,
$$

Admissible partitions

We say that an array of frequencies is $[k_0, \ldots, k_\ell]$-admissible if, for all downward paths $\mathcal{D}$ in the frequency array,

$$\sum_{m \in \mathcal{D}} f_m \leq k,$$

where $k = \sum_{i=0}^{\ell} k_i$.

CMPP conjecture: Generating function for $[k_0, \ldots, k_\ell]$-admissible partitions is an infinite product.

Base case: $\ell = 1$

$$\begin{array}{cccccccccccccc} k_1 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & \\ & k_0 & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & \ldots, \end{array}$$

$[k_0, k_1]$-admissible partitions: those that satisfy mod $2(k_0 + k_1) + 3$ Gordon-Andrews difference conditions.

Bivariate generating function for $k = 1$

Fix $k = 1$. For $0 \leq i \leq \ell$, let $F(i, j, n)$ be the number of $[k_0, k_1, \ldots, k_\ell] = [0, \ldots, 0, 1, 0, \ldots, 0]$-admissible colored partitions of n with exactly j parts, where $k_i = 1$ (and all others are 0). Then, define $P_i(z, q)$ to be the bivariate generating function

$$P_i(z, q) = \sum_{n, j \geq 0} F(i, j, n) z^j q^n.$$

Theorem

For $0 \leq i \leq \ell$,

$$P_i(z, q) = \sum_{n_1, n_2, \ldots, n_\ell \geq 0} \frac{z^{N_1} q^{N_1^2 + N_2^2 + \cdots + N_\ell^2 + N_{i+1} + N_{i+2} + \cdots + N_\ell}}{(q; q)_{n_1} (q; q)_{n_2} \cdots (q; q)_{n_\ell}}.$$

Bivariate generating function for $k = 1$

$$P_i(z,q) = \sum_{n_1,n_2,\dots,n_\ell \geq 0} \frac{z^{N_1} q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}}.$$

Nearly the same as in the Andrews-Gordon identities, except that the factor $x^{N_1+N_2+\cdots+N_\ell}$ has changed to z^{N_1}.

As an immediate corollary,

$$\begin{aligned} P_i(1,q) &= \sum_{n_1,n_2,\dots,n_\ell \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}} \\ &= \frac{\left(q^{i+1}, q^{2\ell+2-i}, q^{2\ell+3}; q^{2\ell+3}\right)_\infty}{(q;q)_\infty}. \end{aligned}$$

CMPP admissible partitions: $2\ell - 1$ rows

We can define $[k_0, k_1, \ldots, k_\ell]$-admissible partitions in an analogous way; for the sake of simplicity, we will jump straight to the case where $k_0 + k_1 + \cdots + k_\ell = 1$.

- Take i such that $0 \leq i \leq \ell$.
- If i is even, create an array with $2\ell - 1$ rows, where the top and bottom rows consist of even integers.
- If i is odd, create an array with $2\ell - 1$ rows, where the top and bottom rows consist of odd integers.
- Set $k_i = 1$, and all other $k_j = 0$.

CMPP conjecture: Generating function for $[k_0, \ldots, k_\ell]$-admissible partitions is an infinite product.

CMPP partitions: $2\ell - 1$ rows, $\ell = 6$ example

Locations of k_i for i even:

	k_0		2		4		6		8		10		12	
k_2		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	
k_4		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	
k_6		1		3		5		7		9		11		$\dots$
	$\cdot$		2		4		6		8		10		12	
$\cdot$		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	
$\cdot$		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	

CMPP partitions: $2\ell - 1$ rows, $\ell = 6$ example

Locations of k_i for i odd:

k_1		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	
k_3		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	
k_5		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	$\ldots$
$\cdot$		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	
$\cdot$		1		3		5		7		9		11		
	$\cdot$		2		4		6		8		10		12	
$\cdot$		1		3		5		7		9		11		

Bivariate generating function for $k = 1$

Fix $k = 1$. For $0 \leq i \leq \ell$, let $F^\star(i, j, n)$ be the number of $[k_0, k_1, \dots, k_\ell] = [0, \dots, 0, 1, 0, \dots, 0]$-admissible partitions of n (on $2\ell - 1$ rows) with exactly j parts, where $k_i = 1$ (and all others are 0).

Define

$$P_i^\star(z, q) = \sum_{n,j \geq 0} F^\star(i, j, n) z^j q^n.$$

Bivariate generating function for $k = 1$

We then have the corresponding result here for $P_i^\star(z,q)$:

Theorem

For $0 \leq i \leq \ell$,

$$P_i^\star(z,q) = \sum_{n_1,n_2,\ldots,n_\ell \geq 0} \frac{z^{N_1} q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{\ell-1}}(q^2;q^2)_{n_\ell}},$$

giving us (by the Andrews-Bressoud identity)

$$\begin{aligned} P_i^\star(1,q) &= \sum_{n_1,n_2,\ldots,n_\ell \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{\ell-1}}(q^2;q^2)_{n_\ell}} \\ &= \frac{\left(q^{i+1}, q^{2\ell+1-i}, q^{2\ell+2}; q^{2\ell+2}\right)_\infty}{(q;q)_\infty}. \end{aligned}$$

Past work

- M. Primc and Trupčević proved 2ℓ-rowed conjectures in the special case of initial conditions $[k, 0, \ldots, 0]$.
- Dousse and Konan used perfect crystals to prove the $k = 1$ case of the $2\ell - 1$-row conjecture (in the case of odd first and last rows).
- $k = 1$ cases of all families of conjectures turn out to be equivalent to a 2001 theorem of Jing, Misra, and Savage.

In all of the cases, the bivariate multisums are new.

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Functional equations for $P_i(z,q)$

Suppose we have 2ℓ rows. Recall $F(i,j,n)$ is the number of $[k_0, k_1, \ldots, k_\ell] = [0, \ldots, 0, 1, 0, \ldots, 0]$-admissible colored partitions of n with exactly j parts, and

$$P_i(z,q) = \sum_{n,j \geq 0} F(i,j,n) z^j q^n.$$

Then:

Theorem

$$P_i(z,q) - P_{i+1}(zq,q) = \sum_{j=1}^{i} zq^j P_{i-j+1}\left(zq^{j+1}, q\right), \quad 0 \leq i \leq \ell - 1$$

$$P_\ell(z,q) - P_\ell(zq,q) = \sum_{j=1}^{\ell} zq^j P_{\ell-j+1}\left(zq^{j+1}, q\right)$$

Functional equations: $\ell = 4$

$$
\begin{aligned}
P_0(z,q) &= P_1(zq,q)\\
P_1(z,q) &= P_2(zq,q) + zq^1 P_1\left(zq^2,q\right)\\
P_2(z,q) &= P_3(zq,q) + zq^1 P_2\left(zq^2,q\right) + zq^2 P_1\left(zq^3,q\right)\\
P_3(z,q) &= P_4(zq,q) + zq^1 P_3\left(zq^2,q\right) + zq^2 P_2\left(zq^3,q\right)\\
&\quad + zq^3 P_1\left(zq^4,q\right)\\
P_4(z,q) &= P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right)\\
&\quad + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)
\end{aligned}
$$

Illustration for $\ell = 4$

Here, we have the following initial picture, where exactly one of the k_i equals 1, and the rest are zero:

$$\begin{array}{cccccccccccccccc}
k_1 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
k_3 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
k_4 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
k_2 & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & k_0 & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14}
\end{array} \quad \ldots,$$

Here are frequency arrays for the five generating functions that we care about, with their initial conditions shown as $\widehat{1}$, and $\cdot$ representing parts that are forbidden from appearing:

$P_0(z,q)$

$$
\begin{array}{cccccccccccccccc c}
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & \cdot & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \widehat{1} & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_1(z, q)$

$$
\begin{array}{ccccccccccccccccc}
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
 & \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & \cdot & & \cdot & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_2(z, q)$

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_3(z,q)$

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_4(z, q)$

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

Functional equations

$$
\begin{aligned}
P_0(z,q) &= P_1(zq,q)\\
P_1(z,q) &= P_2(zq,q) + zq^1 P_1\left(zq^2,q\right)\\
P_2(z,q) &= P_3(zq,q) + zq^1 P_2\left(zq^2,q\right) + zq^2 P_1\left(zq^3,q\right)\\
P_3(z,q) &= P_4(zq,q) + zq^1 P_3\left(zq^2,q\right) + zq^2 P_2\left(zq^3,q\right)\\
&\quad + zq^3 P_1\left(zq^4,q\right)\\
P_4(z,q) &= P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right)\\
&\quad + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)
\end{aligned}
$$

$P_0(z,q) = P_1(zq,q)$: This is clear by inspection: simply "flip" the picture for $P_1(z,q)$ upside down and shift.

$P_1(z, q)$

$$
\begin{array}{ccccccccccccccccc}
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & \cdot & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_0(z, q)$

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & \cdot & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & \cdot & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \widehat{1} & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_1(z,q) = P_2(zq,q) + zq^1 P_1\left(zq^2,q\right)$

Consider a partition counted by $P_1(z,q)$.

$$
\begin{array}{ccccccccccccccccc}
\hat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & \cdot & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

Either there is a 1 in this partition, or there is not.

$P_1(z,q) = P_2(zq,q) + zq^1 P_1\left(zq^2,q\right)$

If there is a 1, we have the following picture:

$$
\begin{array}{cccccccccccccccc}
\cdot & & 1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \star & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \star & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \cdot & & \star & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \cdot & & \star & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \cdot & & \cdot & & \star & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \cdot & & \cdot & & \star & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \cdot & & \cdot & & \cdot & & \star & & f_{10} & & f_{12} & & f_{14}
\end{array}
\quad \ldots,
$$

We can see that this corresponds to $zqP_1\left(zq^2,q\right)$.

$P_1(z,q) = P_2(zq,q) + zq^1 P_1\left(zq^2, q\right)$

The other possibility is that there is no 1:

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \star & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \ldots, \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & \cdot & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

which corresponds to $P_2(zq,q)$.

$$
P_2(z, q)
$$

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_2(z,q) = P_3(zq,q) + zq^1 P_2\left(zq^2,q\right) + zq^2 P_1\left(zq^3,q\right)$

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

Either there is a 1, a 2 in the final row, or neither of these occur.

$P_2(z,q) = P_3(zq,q) + zq^1 P_2\left(zq^2,q\right) + zq^2 P_1\left(zq^3,q\right)$

1 occurs:

$$
\begin{array}{cccccccccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & \star & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \cdot & & \cdot & & \star & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \cdot & & \star & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \cdot & & \star & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \star & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \star & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & 1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \star & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14}
\end{array}
\quad \ldots,
$$

Corresponds to $zq^1 P_2\left(zq^2,q\right)$.

$P_2(z,q) = P_3(zq,q) + zq^1 P_2\left(zq^2,q\right) + zq^2 P_1\left(zq^3,q\right)$

A 2 in the final row:

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & \star & & \star & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & \star & & \star & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \star & & \star & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \star & & \star & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \star & & \star & & f_7 & & f_9 & & f_{11} & & f_{13} & & \ldots, \\
& \cdot & & \star & & \star & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \star & & \star & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & 2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

which corresponds to $zq^2 P_1\left(zq^3,q\right)$.

$P_2(z,q) = P_3(zq,q) + zq^1 P_2\left(zq^2,q\right) + zq^2 P_1\left(zq^3,q\right)$

Neither of those occur:

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \dots, \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \star & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \star & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

which corresponds to $P_3(zq,q)$.

Functional equations: $\ell = 4$

$$
\begin{aligned}
P_0(z,q) &= P_1(zq,q)\\
P_1(z,q) &= P_2(zq,q) + zq^1 P_1\left(zq^2,q\right)\\
P_2(z,q) &= P_3(zq,q) + zq^1 P_2\left(zq^2,q\right) + zq^2 P_1\left(zq^3,q\right)\\
P_3(z,q) &= P_4(zq,q) + zq^1 P_3\left(zq^2,q\right) + zq^2 P_2\left(zq^3,q\right)\\
&\quad + zq^3 P_1\left(zq^4,q\right)\\
P_4(z,q) &= P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right)\\
&\quad + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)
\end{aligned}
$$

$P_4(z,q) = P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right) + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)$

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \cdots \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

Exactly one of the following must happen: a 1 occurs, a 2 in the sixth row occurs, a 3 in the seventh row occurs, a 4 in the eighth row occurs, or none of these occur.

$P_4(z,q) = P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right) + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)$

A 1 occurs in the partition:

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \color{red}{\star} & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & \color{red}{\star} & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \color{red}{\star} & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \color{red}{\star} & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \color{red}{1} & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \dots, \\
 & \cdot & & \color{red}{\star} & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \color{red}{\star} & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & \color{red}{\star} & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

corresponding to $zq^1 P_4\left(zq^2,q\right)$.

$P_4(z,q) = P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right) + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)$

A 2 occurs in the sixth row:

$$
\begin{array}{cccccccccccccccc}
\cdot & & \cdot & & \cdot & & \color{red}{\star} & & \color{red}{\star} & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \cdot & & \color{red}{\star} & & \color{red}{\star} & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \color{red}{\star} & & \color{red}{\star} & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \color{red}{\star} & & \color{red}{\star} & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \color{red}{\star} & & \color{red}{\star} & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \color{red}{2} & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \color{red}{\star} & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
& \cdot & & \cdot & & \color{red}{\star} & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14}
\end{array}
\quad \ldots,
$$

corresponding to $zq^2 P_3\left(zq^3,q\right)$.

$P_4(z,q) = P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right) + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)$

A 3 occurs in the seventh row:

$$
\begin{array}{ccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \star & & \star & & \star & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \star & & \star & & \star & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \star & & \star & & \star & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \star & & \star & & \star & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \star & & \star & & \star & & f_7 & & f_9 & & f_{11} & & f_{13} & & \ldots, \\
& \cdot & & \star & & \star & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & 3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \star & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

corresponding to $zq^3 P_2\left(zq^4,q\right)$.

$P_4(z,q) = P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right) + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)$

A 4 occurs in the eighth row:

$$\begin{array}{cccccccccccccccccc}
\cdot & & \cdot & & \cdot & & \star & & \star & & \star & & \star & & f_{13} & & & \\
& \cdot & & \cdot & & \star & & \star & & \star & & \star & & f_{12} & & f_{14} & & \\
\cdot & & \cdot & & \star & & \star & & \star & & \star & & f_{11} & & f_{13} & & & \\
& \cdot & & \star & & \star & & \star & & \star & & f_{10} & & f_{12} & & f_{14} & & \\
\cdot & & \star & & \star & & \star & & \star & & f_9 & & f_{11} & & f_{13} & & & \ldots, \\
& \cdot & & \star & & \star & & \star & & f_8 & & f_{10} & & f_{12} & & f_{14} & & \\
\cdot & & \cdot & & \star & & \star & & f_7 & & f_9 & & f_{11} & & f_{13} & & & \\
& \cdot & & \cdot & & 4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & &
\end{array}$$

corresponding to $zq^4 P_1\left(zq^5,q\right)$.

$P_4(z,q) = P_4(zq,q) + zq^1 P_4\left(zq^2,q\right) + zq^2 P_3\left(zq^3,q\right) + zq^3 P_2\left(zq^4,q\right) + zq^4 P_1\left(zq^5,q\right)$

Or none of those occur:

$$
\begin{array}{cccccccccccccccc}
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \star & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & \cdot & & \star & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} \\
\cdot & & \cdot & & \star & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & \\
 & \cdot & & \cdot & & \star & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14}
\end{array}
\quad \ldots,
$$

corresponding to $P_4(zq,q)$.

Rewritten functional equations

$$P_i(z,q) - P_{i+1}(zq,q) = \sum_{j=1}^{i} zq^j P_{i-j+1}\left(zq^{j+1},q\right), \quad 0 \le i \le \ell - 1$$

$$P_\ell(z,q) - P_\ell(zq,q) = \sum_{j=1}^{\ell} zq^j P_{\ell-j+1}\left(zq^{j+1},q\right)$$

Rewrite (suppressing the argument of q) as:

$$P_0(z) - P_1(zq) = 0$$

$$P_i(z) - P_{i+1}(zq) - P_{i-1}(zq) + (1-zq)P_i\left(zq^2\right) = 0, 1 \le i \le \ell - 1$$

$$P_\ell(z,q) - P_\ell(zq) - P_{\ell-1}(zq) + (1-zq)P_\ell\left(zq^2\right) = 0$$

Proof

Let $1 \leq i \leq \ell$. We now take the $(i-1)$ instance of the functional equation, dilate $z \mapsto zq$, and reïndex j:

$$P_{i-1}(z) - P_i(zq) = \sum_{j=1}^{i-1} zq^j P_{i-j}\left(zq^{j+1}\right)$$

$$P_{i-1}(zq) - P_i\left(zq^2\right) = \sum_{j=1}^{i-1} zq^{j+1} P_{i-j}\left(zq^{j+2}\right)$$

$$P_{i-1}(zq) - P_i\left(zq^2\right) = \sum_{j=2}^{i} zq^j P_{i-j+1}\left(zq^{j+1}\right), \qquad 1 \leq i \leq \ell$$

Proof

If $i \leq \ell - 1$, subtracting this from the i instance of the functional equation gives

$$\begin{aligned} &(P_i(z) - P_{i+1}(zq)) - \left(P_{i-1}(zq) - P_i\left(zq^2\right)\right) \\ &= \sum_{j=1}^{i} zq^j P_{i-j+1}\left(zq^{j+1}\right) - \sum_{j=2}^{i} zq^j P_{i-j+1}\left(zq^{j+1}\right) \\ &= zqP_i\left(zq^2\right) \end{aligned}$$

or

$$P_i(z) - P_{i+1}(zq) - P_{i-1}(zq) + (1 - zq)P_i\left(zq^2\right) = 0, \qquad 1 \leq i \leq \ell - 1.$$

Proof

If $i = \ell$, subtracting this from the $i = \ell$ instance of the functional equation gives, similarly,

$$\begin{aligned}
&(P_\ell(z) - P_\ell(zq)) - \left(P_{\ell-1}(zq) - P_\ell\left(zq^2\right)\right) \\
&= \sum_{j=1}^{\ell} zq^j P_{\ell-j+1}\left(zq^{j+1}\right) - \sum_{j=2}^{\ell} zq^j P_{\ell-j+1}\left(zq^{j+1}\right) \\
&= zqP_\ell\left(zq^2\right)
\end{aligned}$$

or

$$P_\ell(z) - P_\ell(zq) - P_{\ell-1}(zq) + (1 - zq)P_\ell\left(zq^2\right) = 0.$$

Outline

- Andrews-Gordon-Bressoud identities
- CMPP admissible partitions
- Functional equations for admissible partitions: 2ℓ rows
- **Functional equations for admissible partitions: $2\ell - 1$ rows**
- Completing the bivariate generating function proof
- Bijection with two-rowed cylindric partitions

Functional equations for $P_i^\star(z,q)$

Suppose we have $2\ell-1$ rows. Recall $F^\star(i,j,n)$ is the number of $[k_0,k_1,\ldots,k_\ell]=[0,\ldots,0,1,0,\ldots,0]$-admissible colored partitions of n with exactly j parts, and

$$P_i^\star(z,q)=\sum_{n,j\geq 0} F^\star(i,j,n)z^j q^n.$$

Then:

Theorem

$$P_i^\star(z,q)-P_{i+1}^\star(zq,q)=\sum_{j=1}^{i} zq^j P_{i-j+1}^\star\left(zq^{j+1},q\right), \quad 0\leq i\leq \ell-1$$

$$P_\ell^\star(z,q)-P_{\ell-1}^\star(zq,q)=\sum_{j=1}^{\ell} zq^j P_{\ell-j+1}^\star\left(zq^{j+1},q\right)$$

Rewritten functional equations

Theorem

$$P_0^\star(z) - P_1^\star(zq) = 0$$
$$P_i^\star(z) - P_{i+1}^\star(zq) - P_{i-1}^\star(zq) + (1-zq)P_i^\star\big(zq^2\big) = 0,\ 1 \le i \le \ell - 1$$
$$P_\ell^\star(z) - 2P_{\ell-1}^\star(zq) + (1-zq)P_\ell^\star\big(zq^2\big) = 0$$

$P_0^\star(z,q)$

$$
\begin{array}{ccccccccccccccccc}
 & \widehat{1} & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \ldots \\
 & \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & \cdot & & \cdot & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

$P_1^{\star}(z, q)$

$$
\begin{array}{ccccccccccccccccc}
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \dots \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
& \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & \cdot & & f_7 & & f_9 & & f_{11} & & f_{13} & &
\end{array}
$$

$P_2^{\star}(z,q)$

$$
\begin{array}{ccccccccccccccccc}
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\widehat{1} & & f_1 & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & f_2 & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & f_3 & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \ldots, \\
 & \cdot & & \cdot & & f_4 & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} & \\
\cdot & & \cdot & & \cdot & & f_5 & & f_7 & & f_9 & & f_{11} & & f_{13} & & \\
 & \cdot & & \cdot & & \cdot & & f_6 & & f_8 & & f_{10} & & f_{12} & & f_{14} &
\end{array}
$$

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- Andrews-Gordon-Bressoud identities
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- Bijection with two-rowed cylindric partitions

Bivariate generating functions proof

Recall:

Theorem

For $0 \leq i \leq \ell$,

$$P_i(z,q) = \sum_{n_1,n_2,\ldots,n_\ell \geq 0} \frac{z^{N_1} q^{N_1^2+N_2^2+\cdots+N_\ell^2+N_{i+1}+N_{i+2}+\cdots+N_\ell}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_\ell}}.$$

We actually haven't shown this yet.

Bivariate generating functions proof

Let us now define $T_i(z)$ $(0 \leq i \leq \ell)$ to be the sum side here:

$$T_i(z) = \sum_{n_1, n_2, \ldots, n_\ell \geq 0} \frac{z^{N_1} q^{N_1^2 + N_2^2 + \cdots + N_\ell^2 + N_{i+1} + N_{i+2} + \cdots + N_\ell}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_\ell}}.$$

Goal: Show that $T_i(z)$ satisfies the following system of functional equations:

$$\begin{aligned} T_0(z) - T_1(zq) &= 0 \\ T_i(z) - T_{i+1}(zq) - T_{i-1}(zq) + (1 - zq) T_i\left(zq^2\right) &= 0, \quad 1 \leq i \leq \ell - 1 \\ T_\ell(z) - T_\ell(zq) - T_{\ell-1}(zq) + (1 - zq) T_\ell\left(zq^2\right) &= 0 \end{aligned}$$

First approach

As observed by Ole Warnaar, this set of functional equations exactly matches (normalized) Corteel-Welsh functional equations for two-line cylindric partitions with profile (c_1, c_2), where $c_1 + c_2$ is odd.

As such, we can simply complete the proof of the theorem by observing that we can recover the formulas on the previous slide by taking the limit as $L \to \infty$ in a pair of Warnaar's equations.

Second approach

Let $\mathbf{v} = \langle v_1, \ldots, v_\ell \rangle$ and $\mathbf{n} = \langle n_1, \ldots, n_\ell \rangle$ be vectors in $\mathbb{R}^\ell$. Define

$$
\begin{aligned}
S_{\mathbf{v}}(z) &= \sum_{n_1, n_2, \ldots, n_\ell \geq 0} \frac{z^{N_1} q^{N_1^2 + N_2^2 + \cdots + N_\ell^2 + \mathbf{v}\cdot\mathbf{n}}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_\ell}} \\
\vec{\mathbf{0}} &= \langle 0, 0, \ldots, 0 \rangle \\
\vec{\mathbf{1}} &= \langle 1, 1, \ldots, 1 \rangle \\
\mathbf{e}_i &= \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle & 1 \leq i \leq \ell \\
\mathbf{t}_i &= \langle 0, \ldots, 0, 1, 2, 3, \ldots, \ell - i - 1, \ell - i \rangle, & 1 \leq i \leq \ell
\end{aligned}
$$

where, in $\mathbf{e}_i$ and $\mathbf{t}_i$, the 1 is located in position i.

Note that $T_i(z) = S_{\mathbf{t}_{i+1}}(z)$.

Atomic relations

Lemma

For all $\mathbf{v}$ *and* $1 \le i \le \ell$,

$$S_{\mathbf{v}} - S_{\mathbf{v}+\mathbf{e}_i} = zq^{\mathbf{v}\cdot\mathbf{e}_i + i} S_{\mathbf{v}+2\mathbf{t}_1 - 2\mathbf{t}_{i+1}}.$$

Proof.

$$\begin{aligned} S_{\mathbf{v}} - S_{\mathbf{v}+\mathbf{e}_i} &= \sum_{n_1, n_2, \ldots, n_\ell \ge 0} \frac{z^{N_1} q^{N_1^2 + N_2^2 + \cdots + N_\ell^2 + \mathbf{v}\cdot\mathbf{n}} (1 - q^{n_i})}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_\ell}} \\ &= \sum_{n_1, n_2, \ldots, n_\ell \ge 0} \frac{z^{n_1 + n_2 + \cdots + n_\ell} q^{N_1^2 + N_2^2 + \cdots + N_\ell^2 + \mathbf{v}\cdot\mathbf{n}}}{(q;q)_{n_1} (q;q)_{n_2} \cdots (q;q)_{n_i - 1} \cdots (q;q)_{n_\ell}} \end{aligned}$$

□

Atomic relations

Proof.

$$
\begin{aligned}
&S_{\mathbf{v}} - S_{\mathbf{v}+\mathbf{e}_i} \\
&= \sum_{n_1,n_2,\ldots,n_\ell \geq 0} \frac{z^{n_1+n_2+\cdots+n_\ell} q^{N_1^2+N_2^2+\cdots+N_\ell^2+\mathbf{v}\cdot\mathbf{n}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_i-1}\cdots(q;q)_{n_\ell}} \\
&= \sum_{n_1,n_2,\ldots,\widehat{n}_i,\ldots,n_\ell \geq 0} \frac{z^{n_1+\cdots+\widehat{n}_i+1+\cdots+n_\ell} q^{(N_1+1)^2+\cdots+(N_i+1)^2+N_{i+1}^2+\cdots+N_\ell^2+\mathbf{v}\cdot\mathbf{n}+\mathbf{v}\cdot\mathbf{e}_i}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{\widehat{n}_i}\cdots(q;q)_{n_\ell}} \\
&= zq^{\mathbf{v}\cdot\mathbf{e}_i+i} \sum_{n_1,n_2,\ldots,\widehat{n}_i,\ldots,n_\ell \geq 0} \frac{z^{n_1+n_2+\cdots+\widehat{n}_i+\cdots+n_\ell} q^{N_1^2+\cdots+N_\ell^2+2(N_1+\cdots+N_i)+\mathbf{v}\cdot\mathbf{n}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{\widehat{n}_i}\cdots(q;q)_{n_\ell}} \\
&= zq^{\mathbf{v}\cdot\mathbf{e}_i+i} S_{\mathbf{v}+2\mathbf{t}_1-2\mathbf{t}_{i+1}}
\end{aligned}
$$

Slight abuse of notation: N_j and $\mathbf{n}$ have $\widehat{n}_i$ replacing n_i. □

Atomic relations

So now define, for $1 \le i \le \ell$,

$$\mathrm{rel}^{i}_{\mathbf{v}} := S_{\mathbf{v}} - S_{\mathbf{v}+\mathbf{e}_i} - zq^{\mathbf{v}\cdot\mathbf{e}_i + i} S_{\mathbf{v}+2\mathbf{t}_1-2\mathbf{t}_{i+1}}.$$

Each $\mathrm{rel}^{i}_{\mathbf{v}}$ is, of course, equal to zero.

If we can obtain a certain expression as a linear combination of $\mathrm{rel}^{i}_{\mathbf{v}}$s, then that expression will also be equal to zero.

Atomic relations

Recall $T_i(z) = S_{\mathbf{t}_{i+1}}(z)$. Goal:

$$T_i(z) - T_{i+1}(zq) - T_{i-1}(zq) + (1 - zq)\,T_i\left(zq^2\right) = 0, \quad 1 \leq i \leq \ell - 1$$
$$S_{\mathbf{t}_{i+1}}(z) - S_{\mathbf{t}_{i+2}}(zq) - S_{\mathbf{t}_i}(zq) + (1 - zq)S_{\mathbf{t}_{i+1}}\left(zq^2\right) = 0$$

Note that $S_{\mathbf{v}}(zq) = S_{\mathbf{v}+\vec{\mathbf{1}}}(z)$ and $S_{\mathbf{v}}\left(zq^2\right) = S_{\mathbf{v}+2\vec{\mathbf{1}}}(z)$. This allows us to rewrite these as

$$S_{\mathbf{t}_{i+1}}(z) - S_{\vec{\mathbf{1}}+\mathbf{t}_{i+2}}(z) - S_{\vec{\mathbf{1}}+\mathbf{t}_i}(z) + (1 - zq)S_{2\vec{\mathbf{1}}+\mathbf{t}_{i+1}}(z) = 0$$

Atomic relations proof

For $1 \leq j \leq \ell - 1$,

$$\sum_{i=1}^{j} \mathrm{rel}^{i}_{\mathbf{e}_1+\cdots+\mathbf{e}_{i-1}+\mathbf{t}_{j+1}} - \sum_{i=1}^{j-1} \mathrm{rel}^{i}_{2\vec{\mathbf{1}}-\mathbf{e}_1-\cdots-\mathbf{e}_i+\mathbf{t}_{j+1}}$$

$$= \cdots$$

$$= S_{\mathbf{t}_{j+1}} - S_{\mathbf{e}_1+\cdots+\mathbf{e}_j+\mathbf{t}_{j+1}} - S_{2\vec{\mathbf{1}}-\mathbf{e}_1-\cdots-\mathbf{e}_{j-1}+\mathbf{t}_{j+1}} + (1-zq)S_{2\vec{\mathbf{1}}+\mathbf{t}_{j+1}}$$

$$= S_{\mathbf{t}_{i+1}}(z,q) - S_{\vec{\mathbf{1}}+\mathbf{t}_{i+2}}(z,q) - S_{\vec{\mathbf{1}}+\mathbf{t}_i}(z,q) + (1-zq)S_{2\vec{\mathbf{1}}+\mathbf{t}_{i+1}}(z,q)$$

$$= 0.$$

Repeat this for the other functional equations.
Finally, checking the initial conditions ($P_i(0,q) = T_i(0,q) = 1$, $P_i(z,0) = T_i(z,0) = 1$) allows us to complete our proof.

Outline

- Andrews-Gordon-Bressoud identities
- CMPP admissible partitions
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- Functional equations for admissible partitions: $2\ell - 1$ rows
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- **Bijection with two-rowed cylindric partitions**

Cylindric partitions

Introduced in 1997 by Gessel and Krattenthaler.
Idea: Take a composition $(c_1, \ldots, c_r)$ of ℓ, where $c_1, \ldots, c_r$ are non-negative integers and $\ell = c_1 + \cdots + c_r$.
Consider a sequence $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ where each $\lambda^{(j)}$ is a partition $\lambda^{(j)} = \lambda_1^{(j)} + \lambda_2^{(j)} + \cdots$ arranged in a weakly descending order.
Assume that each $\lambda^{(j)}$ continues indefinitely with only finitely many non-zero entries and ends with an infinite sequence of zeros.
We say that Λ is a cylindric partition with profile c if for all i and j,

$$\lambda_j^{(i)} \geq \lambda_{j+c_{(i+1)}}^{(i+1)} \quad \text{and} \quad \lambda_j^{(r)} \geq \lambda_{j+c_1}^{(1)}.$$

Example

Profile $(2,1)$:

$$\lambda^{(1)} \rightarrow$$
$$\lambda^{(2)} \rightarrow$$

Example

Profile $(2,1)$:

$$\begin{array}{lccccc} \lambda^{(1)} \rightarrow & & 6 & 3 & 2 & 2 \\ \lambda^{(2)} \rightarrow & 5 & 4 & 1 & & \end{array}$$

Example

Profile $(2,1)$:

$$\begin{array}{lcccccc} \lambda^{(2)} \rightarrow & & & & 5 & 4 & 1 \\ \lambda^{(1)} \rightarrow & & 6 & 3 & 2 & 2 & \\ \lambda^{(2)} \rightarrow & 5 & 4 & 1 & & & \end{array}$$

Generating functions for two-rowed cylindric partitions

For a profile $c = (b, \ell - b)$, let

$$F_{(b,\ell-b)}(z, q) = \sum_{\Lambda \in \mathcal{C}_{(b,\ell-b)}} z^{\max(\Lambda)} q^{\mathrm{wt}(\Lambda)}.$$

Theorem

$$F_{(b,\ell-b)}(1, q) = \frac{\left(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2}\right)_\infty}{(q;q)_\infty^2}$$
$$= \frac{1}{(q;q)_\infty} \cdot \frac{\left(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2}\right)_\infty}{(q;q)_\infty}$$

Suggests a bijection between two-rowed cylindric partitions and pairs of (ordinary partitions) and (Gordon-Andrews or Andrews-Bressoud partitions).

Generating functions for two-rowed cylindric partitions

For a profile $c = (b, \ell - b)$, let

$$F_{(b,\ell-b)}(z,q) = \sum_{\Lambda \in \mathcal{C}_{(b,\ell-b)}} z^{\max(\Lambda)} q^{\mathrm{wt}(\Lambda)}.$$

Theorem

$$\begin{aligned} F_{(b,\ell-b)}(1,q) &= \frac{\left(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2}\right)_\infty}{(q;q)_\infty^2} \\ &= \frac{1}{(q;q)_\infty} \cdot \frac{\left(q^{b+1}, q^{\ell-b+1}, q^{\ell+2}; q^{\ell+2}\right)_\infty}{(q;q)_\infty} \end{aligned}$$

Suggests a bijection between two-rowed cylindric partitions and pairs of (ordinary partitions) and (CMPP admissible partitions on 2ℓ rows).

Bijection idea

Corteel: gave a bijective proof of

$$\frac{1}{(q;q)_\infty (q^2,q^3;q^5)_\infty} = \frac{1}{(q;q)_\infty} \sum_{n=0} \frac{q^{n^2+n}}{(q;q)_n}$$

using two-line cylindric partitions of profile $(3,0)$. (Note RR2 partitions are the same as CMPP $[1,0]$-admissible partitions.)
Modify to give:

cylindric partitions of profile $(2\ell+1,0) \leftrightarrow$

(ordinary partitions) $\times$ ($[1,0,\ldots,0]$-admissible partitions on 2ℓ rows)

$$\frac{\left(q^1, q^{2\ell+2}, q^{2\ell+3}; q^{2\ell+3}\right)_\infty}{(q;q)_\infty^2} = \frac{1}{(q;q)_\infty} \times \frac{\left(q^1, q^{2\ell+2}, q^{2\ell+3}; q^{2\ell+3}\right)_\infty}{(q;q)_\infty}$$

.

Bijection

Consider a cylindric partition $\Lambda = \left(\lambda^{(1)}, \lambda^{(2)}\right)$ of profile $(2\ell + 1, 0)$. We construct from Λ a colored partition $\widehat{\Lambda}$, where the allowable colors are $0, 1, \cdots, \ell$. We will use subscripts for colors: i_A is a part of size i with color A.

- a, b: number of nonzero parts of the partitions in each of the two rows, respectively, at a given step of the algorithm.
- Note that we must always have $0 \leq a - b \leq 2\ell + 1$.
- We delete 1 from each nonzero part, and insert into $\widehat{\Lambda}$ a part with size $(a + b)$ with color $\lfloor \frac{a-b}{2} \rfloor$.
- This essentially is taking the "conjugate" of the cylindric partition (but keeping track of colors in some way).

Bijection

The resulting colored partition $\widehat{\Lambda}$ has the following properties:

- Any two parts of size i that appear must have the same color.
- Parts of size 1 can only occur with color 0.
- For $1 \leq i < \ell$, parts of size $2i$ and $2i+1$ can only occur with colors $0, 1, \ldots, i$.
- Suppose $i < j$. i_A and j_B are forbidden from appearing together in $\widehat{\Lambda}$ if any of the following conditions are met.
  - C_1. i is even, $B \geq A$, and $j - i < 2(B - A)$.
  - C_2. i is odd, $B > A$, and $j - i < 2(B - A) - 1$.
  - C_3. i is even, $B < A$, and $j - i < 2(A - B) - 1$.
  - C_4. i is odd, $B \leq A$, and $j - i < 2(A - B)$.

Example

Suppose that $\ell = 3$, and our original cylindric partition $\Lambda = \left(\lambda^{(1)}, \lambda^{(2)}\right)$ with profile $(7, 0)$ is

$$\begin{array}{lcccccccccccccc}
\lambda^{(2)} \rightarrow & & & & & & & & 9 & 4 & 4 & 4 & 2 \\
\lambda^{(1)} \rightarrow & 9 & 8 & 8 & 5 & 5 & 3 & 3 & 3 & 3 & 3 & & \\
\lambda^{(2)} \rightarrow & 9 & 4 & 4 & 4 & 2 & & & & & & &
\end{array}$$

We will find $\widehat{\Lambda} = 15_2 + 15_2 + 14_3 + 9_0 + 6_2 + 4_1 + 4_1 + 4_1 + 2_0$.

Let's walk through this:

Example

$$\begin{array}{llllllllllll} \lambda^{(1)} \to & 9 & 8 & 8 & 5 & 5 & 3 & 3 & 3 & 3 & 3 \\ \lambda^{(2)} \to & 9 & 4 & 4 & 4 & 2 \end{array}$$

$a = 10,\ b = 5$: $a + b = 15$, $\lfloor \frac{10-5}{2} \rfloor = 2$. Insert in 15_2.

$$\begin{array}{llllllllllll} \lambda^{(1)} \to & 8 & 7 & 7 & 4 & 4 & 2 & 2 & 2 & 2 & 2 \\ \lambda^{(2)} \to & 8 & 3 & 3 & 3 & 1 \end{array}$$

$a = 10,\ b = 5$: $a + b = 15$, $\lfloor \frac{10-5}{2} \rfloor = 2$. Insert in 15_2.

$$\begin{array}{llllllllllll} \lambda^{(1)} \to & 7 & 6 & 6 & 3 & 3 & 1 & 1 & 1 & 1 & 1 \\ \lambda^{(2)} \to & 7 & 2 & 2 & 2 \end{array}$$

$a = 10,\ b = 4$: $a + b = 14$, $\lfloor \frac{10-4}{2} \rfloor = 3$. Insert in 14_3.

Example

$$\begin{array}{lccccc} \lambda^{(1)} \rightarrow & 6 & 5 & 5 & 2 & 2 \\ \lambda^{(2)} \rightarrow & 6 & 1 & 1 & 1 & \end{array}$$

$a = 5,\ b = 4$: $a + b = 9$, $\lfloor \frac{5-4}{2} \rfloor = 0$. Insert in 9_0.

$$\begin{array}{lccccc} \lambda^{(1)} \rightarrow & 5 & 4 & 4 & 1 & 1 \\ \lambda^{(2)} \rightarrow & 5 & & & & \end{array}$$

$a = 5,\ b = 1$: $a + b = 6$, $\lfloor \frac{5-1}{2} \rfloor = 2$. Insert in 6_2.

$$\begin{array}{lccc} \lambda^{(1)} \rightarrow & 4 & 3 & 3 \\ \lambda^{(2)} \rightarrow & 4 & & \end{array}$$

$a = 3,\ b = 1$: $a + b = 4$, $\lfloor \frac{3-1}{2} \rfloor = 1$. Insert in 4_1.

Example

$$\begin{array}{ll} \lambda^{(1)} \to & 3 \quad 2 \quad 2 \\ \lambda^{(2)} \to & 3 \end{array}$$

$a = 3,\ b = 1$: $a + b = 4$, $\lfloor \frac{3-1}{2} \rfloor = 1$. Insert in 4_1.

$$\begin{array}{ll} \lambda^{(1)} \to & 2 \quad 1 \quad 1 \\ \lambda^{(2)} \to & 2 \end{array}$$

$a = 3,\ b = 1$: $a + b = 4$, $\lfloor \frac{3-1}{2} \rfloor = 1$. Insert in 4_1.

$$\begin{array}{ll} \lambda^{(1)} \to & 1 \\ \lambda^{(2)} \to & 1 \end{array}$$

$a = 1,\ b = 1$: $a + b = 2$, $\lfloor \frac{1-1}{2} \rfloor = 0$. Insert in 2_0.

Example

Thus, $\widehat{\Lambda} = 15_2 + 15_2 + 14_3 + 9_0 + 6_2 + 4_1 + 4_1 + 4_1 + 2_0$. Observe that $\widehat{\Lambda}$ satisfies:

- Any two parts of size i that appear must have the same color.
- Parts of size 1 can only occur with color 0.
- For $1 \leq i < \ell$, parts of size $2i$ and $2i+1$ can only occur with colors $0, 1, \ldots, i$.
- Suppose $i < j$. i_A and j_B are forbidden from appearing together in $\widehat{\Lambda}$ if any of the following conditions are met.
  - C_1. i is even, $B \geq A$, and $j - i < 2\,(B - A)$.
  - C_2. i is odd, $B > A$, and $j - i < 2\,(B - A) - 1$.
  - C_3. i is even, $B < A$, and $j - i < 2\,(A - B) - 1$.
  - C_4. i is odd, $B \leq A$, and $j - i < 2\,(A - B)$.

Check C_3: $i = 14$, $j = 15$, $A = 3$, $B = 2$, $j - i = 1 = 2(3 - 2) - 1$.

Continuing with the bijection

Use $\widehat{\Lambda}$ to construct:

- ordinary partition μ
- ℓ-colored partition ν

When we add parts to μ, we "forget" their colors.

ν will be a $[1, 0, \ldots, 0]$-admissible partition of 2ℓ rows.

Continuing with the bijection

Take each part of $\widehat{\Lambda}$ that occurs $t > 1$ times, and add $t - 1$ occurrences of that part to μ. Add all parts colored 0 to μ. Then:

R_1. If i_A and j_B both appear as parts in $\widehat{\Lambda}$, where $i < j$ and i is even, $B \geq A > 0$, and $2(B-A) \leq j - i \leq 2(B-A)+1$, then insert i into μ.

R_2. If i_A and j_B both appear as parts in $\widehat{\Lambda}$, where $i < j$ and i is odd, $B > A > 0$, and $2(B-A)-1 \leq j - i \leq 2(B-A)$, then insert i into μ.

R_3. If i_A and j_B both appear as parts in $\widehat{\Lambda}$, where $i < j$ and i is even, $0 < B < A$, and $2(A-B)-1 \leq j - i \leq 2(A-B)$, then insert j into μ.

R_4. If i_A and j_B both appear as parts in $\widehat{\Lambda}$, where $i < j$ and i is odd, $0 < B \leq A$, and $2(A-B) \leq j - i \leq 2(A-B)+1$, then insert j into μ.

At the end, we form ν by taking all of the leftover parts from $\widehat{\Lambda}$ not inserted into μ, keeping their colors intact.

Continuing with the bijection

μ is clearly an (ordinary) partition. We claim that ν is a $[1, 0, \ldots, 0]$-admissible partition on 2ℓ rows, where the parts are colored in the following manner (shown for $\ell = 3$):

$$\begin{array}{cccccccccccc}
 & & & & & 7_3 & & 9_3 & & 11_3 & & \\
 & & & & 6_3 & & 8_3 & & 10_3 & & 12_3 & \\
 & & & 5_2 & & 7_2 & & 9_2 & & 11_2 & & \\
 & & 4_2 & & 6_2 & & 8_2 & & 10_2 & & 12_2 & \cdots. \\
 & 3_1 & & 5_1 & & 7_1 & & 9_1 & & 11_1 & & \\
2_1 & & 4_1 & & 6_1 & & 8_1 & & 10_1 & & 12_1 &
\end{array}$$

Continuing with the bijection

An alternative way of expressing the necessary conditions on ν for it to be so is the following:

- All parts i_A are distinct.
- Parts i_A and i_B $(A \neq B)$ cannot both appear.
- No parts have size 1.
- For $1 \leq i < \ell$, parts of size $2i$ and $2i+1$ can only occur with colors $1, \ldots, i$.
- Suppose $i < j$. i_A and j_B $(A, B > 0)$ are forbidden from appearing together if:

  D_1. i is even, $B \geq A$, and $j - i \leq 2(B-A)+1$.
  D_2. i is odd, $B > A$, and $j - i \leq 2(B-A)$.
  D_3. i is even, $B < A$, and $j - i \leq 2(A-B)$.
  D_4. i is odd, $B \leq A$, and $j - i \leq 2(A-B)+1$.

Continuing with the bijection

The first condition above is met by the fact that duplicated parts were sent to μ, while the second, third, and fourth are met through the properties of $\widehat{\Lambda}$. Conditions D_1, D_2, D_3, and D_4 are nearly the same as those in C_1, C_2, C_3, and C_4 in the construction of $\widehat{\Lambda}$, except for the following cases:

- i is even, $B \geq A$, and $2\,(B - A) \leq j - i \leq 2\,(B - A) + 1$.
- i is odd, $B > A$, and $2\,(B - A) - 1 \leq j - i \leq 2\,(B - A)$.
- i is even, $B < A$, and $2\,(A - B) - 1 \leq j - i \leq 2\,(A - B)$.
- i is odd, $B \leq A$, and $2\,(A - B) \leq j - i \leq 2\,(A - B) + 1$.

But, those four cases are exactly the ones that were dealt with in the "rules" above, and so we conclude that ν is, in fact, a $[1, 0, \ldots, 0]$-admissible partition on 2ℓ rows (with bivariate generating function $P_0(z, q)$).

Continuing with the example

Recall $\widehat{\Lambda} = 15_2 + 15_2 + 14_3 + 9_0 + 6_2 + 4_1 + 4_1 + 4_1 + 2_0$.
There are two copies of 15_2 and three copies of 4_1, along with 9_0 and 2_0, so start off μ as

$$\mu = 15 + 9 + 4 + 4 + 2.$$

R_3. If i_A and j_B both appear as parts in $\widehat{\Lambda}$, where $i < j$ and i is even, $0 < B < A$, and $2(A - B) - 1 \leq j - i \leq 2(A - B)$, then insert j into μ.

This applies to $15_2 + 14_3$, so insert the last copy of 15 into μ.

R_1. If i_A and j_B both appear as parts in $\widehat{\Lambda}$, where $i < j$ and i is even, $B \geq A > 0$, and $2(B - A) \leq j - i \leq 2(B - A) + 1$, then insert i into μ.

This applies to $6_2 + 4_1$, so also insert the final copy of 4 into μ.

Continuing with the example

None of the other rules apply, so the remaining parts (with their colors intact) make up ν:

$$\mu = 15 + 15 + 9 + 4 + 4 + 4 + 2$$
$$\nu = 14_3 + 6_2$$

μ is an ordinary partition and ν is a $[1, 0, 0, 0]$-admissible partition.

Reversing the map

Start with a partition μ and a $[1, 0, \ldots, 0]$-admissible colored partition ν with colors $1, \ldots, \ell$.
Construct $\widehat{\Lambda}$ by performing the following operations:

- If i is in μ, j_B is in ν, $i < j$, i is even, and $j - i < 2B$, then insert i_C into $\widehat{\Lambda}$, where C satisfies $2(B - C) \leq j - i \leq 2(B - C) + 1$.
- If i is in μ, j_B is in ν, $i < j$, i is odd, and $j - i < 2B - 1$, then insert i_C into $\widehat{\Lambda}$, where C satisfies $2(B - C) - 1 \leq j - i \leq 2(B - C)$.
- If j is in μ, i_A is in ν, $i < j$, i is even, and $j - i < 2A - 1$, then insert j_C into $\widehat{\Lambda}$, where C satisfies $2(A - C) - 1 \leq j - i \leq 2(A - C)$.
- If j is in μ, i_A is in ν, $i < j$, i is odd, and $j - i < 2A$, then insert j_C into $\widehat{\Lambda}$, where C satisfies $2(A - C) \leq j - i \leq 2(A - C) + 1$.

Reversing the map

- All parts of ν are then inserted into $\widehat{\Lambda}$ with their colors intact.
- All remaining parts i of μ are inserted into $\widehat{\Lambda}$.
- If there already is a part i_A in $\widehat{\Lambda}$, then those parts i are also given color A; otherwise, we insert in i_0 into $\widehat{\Lambda}$.
- To complete the bijection, we need to turn $\widehat{\Lambda}$ into a cylindric partition of profile $(2\ell + 1, 0)$.
- For each part j_B of $\widehat{\Lambda}$, successively increment $\lambda_i^{(1)}$ by 1 for $1 \leq i \leq \lceil j/2 \rceil + B$, and increment $\lambda_i^{(2)}$ by 1 for $1 \leq i \leq \lfloor j/2 \rfloor - B$.

$2\ell - 1$ rows

Play the same game for the case of admissible partitions on $2\ell - 1$ rows: can construct a bijection between cylindric partitions of profile $(2\ell, 0)$ and pairs of ordinary partitions and $[1, 0, \ldots, 0]$-admissible partitions on $2\ell - 1$ rows.
The only change in the proof is that, in $\widehat{\Lambda}$, parts with color ℓ cannot have odd size, which parallels exactly how parts with color ℓ in the admissible partition must have even size (again shown for $\ell = 3$):

$$
\begin{array}{ccccccccccccc}
 & & & & 6_3 & & 8_3 & & 10_3 & & 12_3 & & \\
 & & & 5_2 & & 7_2 & & 9_2 & & 11_2 & & & \\
 & & 4_2 & & 6_2 & & 8_2 & & 10_2 & & 12_2 & & \ldots. \\
 & 3_1 & & 5_1 & & 7_1 & & 9_1 & & 11_1 & & & \\
2_1 & & 4_1 & & 6_1 & & 8_1 & & 10_1 & & 12_1 & &
\end{array}
$$