s-Modular, s-congruent and s-duplicate partitions

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$$(a;q)_n := \begin{cases} \prod_{i=0}^{n-1}(1-aq^i) & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

$$(a_1, a_2, \ldots, a_j; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_j; q)_\infty.$$

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For example, the following figure illustrates the 3-modular Ferrers diagram of the partition $\lambda = (14^2, 12^2, 7, 6, 5, 4, 3)$ of $n = 77$ with the diagram of $(5^2, 4^2, 3, 2^3, 1)$.

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3	3	3	3	2
3	3	3	3	2
3	3	3	3	
3	3	3	3	
3	3	1		
3	3			
3	2			
3	1			
3				

Figure: The 3-modular Ferrers diagram of $\lambda = (14^2, 12^2, 7, 6, 5, 4, 3)$.

History

Throughout the history of integer partitions, most studies have investigated partitions with various constraints on their parts. Examples include:

- Partitions into odd (or even) parts,
- ℓ-regular partitions (parts indivisible by $\ell > 1$),
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Nevertheless, the multiplicities (i.e. number of occurrences) of parts have not received significant attention in the literature on integer partitions, as reflected in the limited number of related papers. Many of these few papers are dedicated to the partitions where parts appear fewer than m times (the number of such partitions is equal to the number of m-regular partitions).

Motivation

[7] A. Bazniar, M. Ahmia, J. L. Ramírez et al., New Modular Symmetric Function and its Applications: Modular s-Stirling Numbers. *Bull. Malays. Math. Sci. Soc.* **45** (2022), 1093–1109.

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Throughout the remainder of the presentation, s is always considered to be an even positive integer greater than or equal to 4.

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Example

For $(s,n)=(4,8)$, we have $M_4(8)=10$ and the corresponding set is

$$\mathbb{M}_4(8)=\{(8),(7,1),(6,2),(5,3),(5,2,1),(4,3,1),(3,1^5),(4,1^4),(2^4),(1^8)\}.$$

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A partition λ is said to be s-congruent if the parts are not congruent to $2, 4, 6, \ldots, (s-2)$ modulo s. We denote by $C_s(n)$ the number of s-congruent partitions of n.

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For $(s, n) = (6, 8)$, we have $C_6(8) = 7$ such that

$$\mathbb{C}_6(8) = \{(7,1), (6,1^2), (5,3), (5,1^3), (3^2,1^2), (3,1^5), (1^8)\}.$$

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We note that the s-congruent partitions were introduced by Ballantine and Welch [4] as partitions in which all even parts are congruent to 0 modulo s.

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Example

For $(s,n)=(6,6)$, we have $D_6(6)=5$ and the set of partitions is given by

$$\mathbb{D}_6(6) = \{(6),(5,1),(4,2),(3,2,1),(3^2)\}.$$

s-Duplicate partitions and the $\mathrm{pod}(n)$ function

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This class of partitions appears frequently in the literature, such as in the works of Andrews [2, 3] and Berkovich and Garvan [5],

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This intriguing connection suggests that this function and its combinatorial interpretations play a substantial role in representation theory and raise bigger questions for the general case of s-sduplicate partitions and their impact on these topics.

Combinatorial properties of s-modular, s-congruent and s-duplicate partitions

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&= \prod_{n\geq 1} \frac{1}{(1-q^{2n-1})(1-q^{sn})} = \sum_{n\geq 0} C_s(n)q^n \\
&= \prod_{n\geq 1} \frac{(1+q^n)}{(1+q^{sn/2})(1-q^{sn/2})} = \sum_{n\geq 0} D_s(n)q^n.
\end{aligned}$$

Combinatorial properties: Bijections

We establish a bijective function $f : \mathbb{A}(n) \to \mathbb{B}(n)$ between two sets of partitions $\mathbb{A}(n)$ and $\mathbb{B}(n)$, creating a one-to-one correspondence where $\mathbb{A}(n)$ and $\mathbb{B}(n)$ are finite sets of partitions of fixed size.

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For $\lambda = (\lambda_1^{u_1}, \lambda_2^{u_2}, \ldots, \lambda_k^{u_k}) \in \mathbb{A}(n)$ and $\beta = (\beta_1^{w_1}, \beta_2^{w_2}, \ldots, \beta_k^{w_k}) \in \mathbb{B}(n)$, we have

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$$\lambda = f^{-1}(\beta) = f^{-1}\big((\beta_1^{w_1}, \ldots, \beta_k^{w_k})\big) := \big(f^{-1}(\beta_1^{w_1}), \ldots, f^{-1}(\beta_k^{w_k})\big).$$

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$$f(\lambda_i^{u_i}) = \begin{cases} u_i^{\lambda_i} & \text{if } u_i \equiv \ 0 \pmod{s}, \\ (u_i - 1)^{\lambda_i}, \ell_i^{2^{r_i}} & \text{if } u_i \equiv \ 1 \pmod{s}, \end{cases}$$

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Else, if $\lambda_i \not\equiv 2, 4, 6, \ldots, (s-2) \pmod{s}$, then

$$f(\lambda_i^{u_i}) = \lambda_i^{u_i}.$$

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Let $\lambda = (\lambda_1^{u_1}, \lambda_2^{u_2}, \ldots, \lambda_k^{u_k}) \in \mathbb{C}_s(n)$ and define the inverse map $f^{-1} : \mathbb{C}_s(n) \to \mathbb{M}_s(n)$ by $\lambda \to f^{-1}(\lambda) = \bigcup_{i=1}^{k} f^{-1}(\lambda_i^{u_i})$.

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Else, if $\lambda_i \not\equiv 0 \pmod{s}$, define the vector W_{p-1} as

$$W_{p-1} = (2a_1, 2^2 a_2, \ldots, 2^{p-1} a_{p-1}),$$

where $a_i \in \{0, 1\}$ and $i \in \{1, 2, \ldots (p-1)\}$.

The bijection $\mathbb{M}_s(n) \Leftrightarrow \mathbb{C}_s(n)$ when $s = 2^p$ for $p \geq 2$.

The values of a_i are chosen in such a way that W_{p-1} attains a unique magnitude $\mid W_{p-1} \mid$ for which

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where m is the maximum possible value less than u_i. Consequently, we obtain the expansion

$$\lambda_i^{u_i} = \lambda_i^{m} + \lambda_i^{2a_1} + \cdots + \lambda_i^{2^{p-1}a_{p-1}},$$

where if $a_i = 0$ or $m = 0$, we exclude the corresponding part λ_i from the expansion.

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Recurrence Relations

Let $M_s(n, k)$ denote the number of s-modular partitions of n into k parts.

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Theorem

For every positive integers $n, k \geq 1$,

$$M_s(n,k) = \sum_{\substack{\ell=0 \\ \ell \equiv 0,1 \pmod{s}}}^{k} M_s(n-k, k-\ell).$$

Recurrence Relations

Let $C_s(n,k)$ denote the number of s-congruent partitions of n into k parts. Let $N(s)$ be the set consisting of s and all the odd positive integers less than s, and let $N'(\ell)$ be the set consisting of all the odd positive integers less than $\ell \in N(s)$.

Theorem

For every positive integers $n, k \geq 1$, we have

$$C_s(n,k) = \sum_{\ell \in N(s)} C_s^{\ell}(n,k) + C_s^{s+1}(n,k).$$

Moreover, for $\ell \in N(s)$, we have

$$C_s^{\ell}(n,k) = C_s(n-\ell, k-1) - \sum_{i \in N'(\ell)} C_s^{i}(n-\ell, k-1) \text{ and } C_s^{s+1}(n,k) = C_s(n-sk, k).$$

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We have the following dissection of $\mathbb{C}_s(n,k)$ into $\mid N(s) \mid +1$ disjoint subsets

$$\mathbb{C}_s(n,k) = \mathbb{C}_s^1(n,k) \cup \mathbb{C}_s^3(n,k) \cup \cdots \cup \mathbb{C}_s^s(n,k) \cup \mathbb{C}_s^{s+1}(n,k).$$

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- Consider the subset

$$\mathbb{C}_s(n-\ell, k-1) \setminus \cup_{i \in N'(\ell)} \mathbb{C}_s^i(n-\ell, k-1)$$

which contains all the partitions of $n-\ell$ into $k-1$ parts of sizes $\geq \ell$, where $\ell \in N(s)$.

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• To obtain partitions of $n-\ell$ into $k-1$ parts of sizes $\geq \ell$, we exclude those with at least one part of size $i \in N'(\ell)$ as the smallest part from $\mathbb{C}_s(n-\ell, k-1)$.

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• Conversely, by adding one part of size $\ell \in N(s)$ to each partition $\lambda \in \mathbb{C}_s(n-\ell, k-1) \setminus \cup_{i \in N'(\ell)} \mathbb{C}_s^i(n-\ell, k-1)$, we arrive at $\mathbb{C}_s^\ell(n, k)$.

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• Therefore, for all $\ell \in N(s)$ we deduce that

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• Similarly, by adding s to each part of $\lambda \in \mathbb{C}_s(n-sk, k)$, we get $\mathbb{C}_s^{s+1}(n, k)$. Therefore, we find that

$$\mid \mathbb{C}_s^{s+1}(n, k) \mid = \mid \mathbb{C}_s(n-sk, k) \mid .$$

Recurrence Relations

Let $\mathbb{D}_s(n,k)$ denote the set of s-duplicate partitions of n into k parts.

Theorem

For every positive integers $n, k \geq 1$,

$$D_s(n,k) = D_s\left(n - \frac{s}{2}, k-1\right) + \sum_{\alpha^j \in \mathbb{A}(s)} D_s\left(n - \frac{s(k-\ell(\alpha^j))}{2} - \mid \alpha^j \mid, k - \ell(\alpha^j)\right).$$

Where $\mathbb{O}(s)$ denote the set of all partitions into distinct parts less than or equal to $s/2-1$, $\mathbb{A}(s) = \mathbb{O}(s) \cup \{\emptyset\}$, and α^j is a partition of $\mathbb{A}(s)$.

Recurrence Relations

Example

For $s = 4$, we have the set $\mathbb{O}(4) = \{(1)\}$. Therefore, $\mathbb{A}(4) = \{\emptyset, (1)\}$ and the recurrence relation is given by

$$D_4(n,k) = D_4(n-2,k-1) + D_4(n-2k,k) + D_4(n-2(k-1)-1,k-1).$$

Series Expansion

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$$\frac{(-bq;q^2)_\infty}{(cq^2;q^2)_\infty} = 1+\sum_{k\geq 1}\frac{c^k q^{2k^2-1}(-bq;q^2)_{k-1}(-bc^{-1}q;q^2)_{k-1}(bc^{-1}+q)(1+bq^{4k-1})}{(cq^2;q^2)_k(q^2;q^2)_k},$$

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where the powers of b and c keep track of the number of odd and even parts respectively.

Series Expansion

Lebesgue's Identity:

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}(1+bq)(1+bq^2)\cdots(1+bq^n)}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m\geq 1} \frac{(1+bq^{2m})}{(1-q^{2m-1})}.$$

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Rogers-Fine identity:

$$F(\alpha,\beta,\tau;q) = \sum_{n\geq 0} \frac{(\alpha q;q)_n(\alpha\tau q/\beta;q)_n(1-\alpha\tau q^{2n+1})}{(\beta q;q)_n(\tau;q)_{n+1}} \beta^n\tau^n q^{n^2}.$$

Series Expansion

Given that $\mathtt{pod}(n)$ is a special case of s-duplicate partitions, we present a generalized series expansion for s-duplicate partitions, of which Alladi's series expansion emerges as a special case.

Series Expansion

Theorem

$$\sum_{n,r,l\geq 0} D_s(n,r,l)z^r b^l q^n = 1 + \sum_{k\geq 1} A_k(q)\times$$
$$\Big\{ z^k q^{sk^2/2}(-zbq^{s(k-1)/2+1};q)_{s/2-1}(-bq^{s(k-1)/2+1};q)_{s/2-1}$$
$$+ (1-zq^{sk/2})(1-q^{sk/2}) \sum_{i=1}^{s/2-1} bz^k q^{sk^2/2-i} \Big\},$$

where

$$A_k(q) = \frac{(-zbq,\dots,-zbq^{s/2-1};q^{s/2})_{k-1}(-bq,\dots,-bq^{s/2-1};q^{s/2})_{k-1}}{(zq^{s/2};q^{s/2})_k(q^{s/2};q^{s/2})_k}.$$

Partitions into parts simultaneously s-congruent and t-distinct

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A *t-distinct partition* λ of a positive integer n

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A *t-distinct partition* λ of a positive integer n is a finite sequence of positive integers such that $\lambda_1^{u_1} + \lambda_2^{u_2} + \cdots + \lambda_k^{u_k} = n$,

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A *t-distinct partition* λ of a positive integer n is a finite sequence of positive integers such that $\lambda_1^{u_1} + \lambda_2^{u_2} + \cdots + \lambda_k^{u_k} = n$, where $1 \leq u_i < t$ and $t \geq 2$. We shall impose an additional restriction on the set of the s-congruent partitions to obtain a new set of partitions into parts simultaneously s-congruent and t-distinct.

Parts simultaneously s-congruent and t-distinct

Partitions into parts simultaneously s-congruent and t-distinct

A partition into parts simultaneously s-congruent and t-distinct is a partition into parts not congruent to $2, 4, 6, \ldots, (s-2)$ modulo s and appearing fewer than t times. We denote by $C_s^t(n)$ the number of partitions into parts simultaneously s-congruent and t-distinct of n.

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The generating function for $C_s^t(n)$ is given by

$$\sum_{n\geq 0} C_s^t(n)q^n = \prod_{n\geq 1} \frac{(1-q^{t(2n-1)})(1-q^{tsn})}{(1-q^{2n-1})(1-q^{sn})}.$$

Parts simultaneously s-congruent and t-distinct

Setting $t = 2$ and $s = 4, 6$, the sequences match A261734, A261736, respectively, which seem to be related to the Generating functions for fixed points of the Mullineux map [8].

Parts simultaneously s-congruent and t-distinct

Among the most celebrated identities in the theory of partitions and q-series are those of Göllnitz and Gordon. These identities were initially discovered by Göllnitz in 1961, but remained unknown until Gordon independently rediscovered them in 1965.

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Theorem (Göllnitz-Gordon identities)

Fix a to be either 1 or 3. Given an integer n, the number of partitions of n in which parts are congruent to 4 or $\pm a$ modulo 8, is equal to the number of partitions of n in which parts are non-repeating and non-consecutive, with any two even parts differing by at least 4, and with all parts $\geq a$.

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In 1967, Andrews generalized the Göllnitz-Gordon identities in the following theorem.

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Theorem (Andrews-Göllnitz-Gordon)

Let i and k be integers with $0 < i \leq k$. Let $V_{k,i}(n)$ denote the number of partitions of n into parts not congruent to 2 modulo 4 and not congruent to $0, \pm(2i-1)$ modulo $4k$. Let $W_{k,i}(n)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ of n in which no odd part is repeated, $\lambda_j \geq \lambda_{j+1}$, $\lambda_j - \lambda_{j+k-1} \geq 2$ if λ_j odd, $\lambda_j - \lambda_{j+k-1} > 2$ if λ_j even, and at most $i-1$ parts are ≤ 2. Then

$$V_{k,i}(n) = W_{k,i}(n).$$

Parts simultaneously s-congruent and t-distinct

In further exploration of Andrews' general theorem, we enlarge its scope by expanding the class of partitions enumerated by $V_{k,i}(n)$ for certain values of k and i.

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Definition

Let $t \geq 3$ be a positive integer with $t \not\equiv 2, 4, 6, \ldots, (s-2) \pmod{s}$. Let $E_s^t(n)$ denote the number of partitions of n into parts not congruent to $2, 4, 6, \ldots, (s-2)$ modulo s and not congruent to $0, t(2r+1)$ modulo ts, where $r = 0, 1, 2, 3, \ldots, s/2 - 1$.

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In further exploration of Andrews' general theorem, we enlarge its scope by expanding the class of partitions enumerated by $V_{k,i}(n)$ for certain values of k and i.

Definition

Let $t \geq 3$ be a positive integer with $t \not\equiv 2, 4, 6, \ldots, (s-2) \pmod{s}$. Let $E_s^t(n)$ denote the number of partitions of n into parts not congruent to $2, 4, 6, \ldots, (s-2)$ modulo s and not congruent to $0, t(2r+1)$ modulo ts, where $r = 0, 1, 2, 3, \ldots, s/2 - 1$.

The generating function for $E_s^t(n)$ is given by

$$\sum_{n\geq 0} E_s^t(n) q^n = \frac{(q^2;q^2)_\infty (q^t;q^t)_\infty (q^{ts};q^{ts})_\infty}{(q;q)_\infty (q^s;q^s)_\infty (q^{2t};q^{2t})_\infty}.$$

Parts simultaneously s-congruent and t-distinct

Theorem

Let $t \geq 3$ be a positive integer with $t \not\equiv 2, 4, 6, \ldots, (s-2)$ modulo s. Then for every positive integer $n \geq 0$,

$$C_s^t(n) = E_s^t(n).$$

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The relation between the number $E_s^t(n)$ and Andrews' theorem becomes apparent by setting $s = 4$ in the definition of $E_s^t(n)$.

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Corollary

Let $t \geq 3$ be an odd integer. Then for every natural number $n \geq 0$,

$$C_4^t(n) = V_{t,(t+1)/2}(n) = W_{t,(t+1)/2}(n).$$

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$W_{3,2}(n)$: The number of partitions of n in which

- No odd part is repeated,
- $\lambda_j + \lambda_{j+2} \geq 2$ if λ_j is odd.
- $\lambda_j + \lambda_{j+2} > 2$ if λ_j is even.
- At most 1 part ≤ 2.

References I

- K. Alladi, Partitions with non-repeating odd parts and q-hypergeometric identities. *The legacy of Alladi Ramakrishnan in the mathematical sciences*, Springer, New York, NY. (2010), 169–182.
- G. E. Andrews, A generalization of the Göllnitz-Gordon partition theorems. *Proc. Am. Math. Soc.* 18 (1967), 945–952.
- G. E. Andrews, Two theorems of Gauss and allied identities proved arithmetically. *Pac. J. Math.* 41 (1972), 563–578.
- Cristina Ballantine, Amanda Welch, Generalizations of POD and PED partitions. Discrete Math. 347 (2024), 114150.
- A. Berkovich, F. G. Garvan, Some observations on Dyson's new symmetries of partitions. *J. Comb. Theory, Ser. A* 100 (2002), 61–93.

References II

- Mohammed L. Nadji and Moussa Ahmia, s-Modular, s-congruent and s-duplicate partitions. *Bol. Soc. Mat. Mex.* 30 (2024), Article 98.
- A. Bazniar, M. Ahmia, J. L. Ramírez et al., New Modular Symmetric Function and its Applications: Modular s-Stirling Numbers. *Bull. Malays. Math. Sci. Soc.* 45 (2022), 1093–1109.
- David J. Hemmer, Generating functions for fixed points of the Mullineux map, arXiv:2402.03643 [math.CO], 2024.

Thank You!