Parity Results for the Coefficients of the Reciprocals of False Theta Functions

Nayandeep Deka Baruah

Department of Mathematical Sciences
Tezpur University

Joint with Rituparna Nath

11/20/2025

Contents

Introduction

Theta Functions

Definition

Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}}, \ |ab| < 1,$$

whereas the false theta function $\Psi(a,b)$ is defined by

$$\Psi(a,b) := \sum_{n=0}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}} - \sum_{n=-\infty}^{-1} a^{\binom{n+1}{2}} b^{\binom{n}{2}}.$$

In a recent paper, Keith[1] investigated the coefficients of the reciprocals of false theta functions and proved several arithmetic identities.

Let $c_t(n)$ be defined by

$$\frac{1}{\Psi(-q^t, q)} := \sum_{n=0}^{\infty} c_t(n) q^n. \tag{1}$$

1

Keith, W.J.: *Reciprocals of false theta functions.* Ramanujan J. **68**, 27 (2025)

Keith found several congruences, an exponential asymptotic, and some side results, including an interesting connection to the truncated pentagonal number theorem studied by Andrews and Merca[2].

2

Andrews, G.E., Merca, M.: *The truncated pentagonal number theorem*. J. Comb. Theory Ser. A 119, 1639–1643 (2012)

Theorem 1.1

For $n \geq 0$,

$$c_5(8n+5) \equiv 0 \pmod{2},$$

$$c_5(2(pn+j)+1) \equiv 0 \pmod{2},$$

for a prime $p > 3$ such that $3^{-1}(j+3^{-1})$ is not a quadratic residue mod p,

$$c_5(32n+31) \equiv 0 \pmod{4},$$

$$c_9(8n+4) \equiv 0 \pmod{2}, \text{ if } n \text{ cannot be}$$

represented in the form $n = 10k^2 - 4k$ for integer k.

Keith further posed two conjectures on congruences modulo 2, 4, and 8 for some coefficients.

Conjecture 1.2

For $n \geq 0$,

$$
\begin{aligned}
c_5(32n+31) &\equiv 0 \pmod{8},\\
c_5(128n+123) &\equiv 0 \pmod{8},\\
c_5(512n+491) &\equiv 0 \pmod{8},\\
c_5(64n+19) &\equiv 0 \pmod{4},\\
c_5(256n+75) &\equiv 0 \pmod{4},\\
c_5(196n+7j+5) &\equiv 0 \pmod{4},
\end{aligned}
$$

where $j \in \{2, 6, 10, 14, 15, 19, 22, 26, 27\}$.

This conjecture was proved by Jin, Wang, and Yao[3]

[3] Jin, J., Wang, S., Yao, O.X.M.: *Proof of a conjecture of Keith on congruences of the reciprocal of a false theta function*. https://arxiv.org/abs/2508.01532.

Conjecture 1.3

For $n \geq 0$,

$$\begin{aligned} c_9(36n+14) &\equiv 0 \pmod{2}, \\ c_9(196n+j) &\equiv 0 \pmod{2}, \ \textit{where } j \in \{54, 166, 194\}, \\ c_{13}(32n+23) &\equiv 0 \pmod{2}, \\ c_{13}(64n+63) &\equiv 0 \pmod{2}, \\ c_{13}(72n+j) &\equiv 0 \pmod{2}, \ \textit{where } j \in \{15, 21, 39, 69\}, \\ c_{17}(128n+80) &\equiv 0 \pmod{2}. \end{aligned} \tag{2}$$

Congruence (17) is equivalent to

$$c_9(392n+j) \equiv 0 \pmod{2}, \ \textit{where } j \in \{54, 166, 194, 250, 362, 390\}.$$

Main Results

Theorem 2.1

For $n \geq 0$, we have

$$c_9(36n+14) \equiv 0 \pmod 2, \tag{3}$$
$$c_9(392n+j) \equiv 0 \pmod 2, \text{ where } j \in \{54, 166, 390\}, \tag{4}$$
$$c_{13}(32n+23) \equiv 0 \pmod 2, \tag{5}$$
$$c_{13}(64n+63) \equiv 0 \pmod 2, \tag{6}$$
$$c_{13}(72n+j) \equiv 0 \pmod 2, \text{ where } j \in \{15, 21, 39, 69\}, \tag{7}$$
$$c_{17}(128n+80) \equiv 0 \pmod 2. \tag{8}$$

Theorem 2.2

For $n \geq 0$, we have

$$\begin{aligned} c_9(392n+j) &\equiv 0 \pmod{2}, \ \textit{where } j \in \{110, 222, 278\}, \\ c_{13}(128n+15) &\equiv 0 \pmod{2}, \\ c_{13}(256n+175) &\equiv 0 \pmod{2}. \end{aligned}$$

Theorem 2.3

For $n \geq 0$, $k \geq 1$ the following holds:

$$c_{13}\left(2^{3k}n + \frac{5 \cdot 2^{3k} + 9}{7}\right) \equiv c_{13}(8n+7) \pmod 2,$$
$$c_{13}\left(9 \cdot 2^{3k}n + \frac{27 \cdot 2^{3k+1} + 9}{7}\right) \equiv c_{13}(16n+15) \pmod 2.$$

From Theorems 2.1–2.3, we deduce the following infinite families of congruences.

Corollary 2.4

For $n \geq 0$ and $k \geq 1$, the following holds:

$$c_{13}\left(2^{3k+2}n + \frac{19 \cdot 2^{3k} + 9}{7}\right) \equiv 0 \pmod{2}, \tag{9}$$

$$c_{13}\left(2^{3k+3}n + \frac{27 \cdot 2^{3k+1} + 9}{7}\right) \equiv 0 \pmod{2}, \tag{10}$$

$$c_{13}\left(2^{3k+4}n + \frac{3 \cdot 2^{3k+2} + 9}{7}\right) \equiv 0 \pmod{2}, \tag{11}$$

$$c_{13}\left(9 \cdot 2^{3k+3}n + \frac{27 \cdot 2^{3k+1} + 9}{7}\right) \equiv 0 \pmod{2}, \tag{12}$$

$$c_{13}\left(9 \cdot 2^{3k+4}n + \frac{171 \cdot 2^{3k+2} + 9}{7}\right) \equiv 0 \pmod{2}, \tag{13}$$

$$c_{13}\left(9 \cdot 2^{3k+2}n + \frac{243 \cdot 2^{3k} + 9}{7}\right) \equiv 0 \pmod{2}. \tag{14}$$

Proof of $c_9(36n+14) \equiv 0 \pmod{2}$

Note that

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}}$$

and

$$\Psi(a,b)=\sum_{n=0}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}} - \sum_{n=-\infty}^{-1} a^{\binom{n+1}{2}} b^{\binom{n}{2}}.$$

We have

$$\sum_{n=0}^{\infty} c_9(n) q^n = \frac{1}{\Psi(-q^9,q)} \equiv \frac{1}{f(-q^9,q)} \equiv \frac{f(-q,q^9)}{f(q,-q^9) f(-q,q^9)} \pmod 2. \tag{15}$$

Recall from Berndt[1] that

$$f(a,b) = f(ab^3, a^3b) + af(b/a, a^5b^3),$$
$$f(a,b)f(-a,-b) = f(-a^2,-b^2)\varphi(-ab),$$

where

$$\varphi(q) := f(q,q) = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}.$$

1 Berndt, B.C.: *Ramanujan's Notebooks Part III.* Springer (1991), [p. 46, Entry 30]

Therefore,

$$\sum_{n=0}^{\infty} c_9(n)q^n \equiv \frac{f(-q,q^9)}{f(q,-q^9)f(-q,q^9)}$$
$$\equiv \frac{f(-q^{12},-q^{28}) - qf(-q^8,-q^{32})}{f(-q^2,-q^{18})\varphi(q^{10})}$$
$$\equiv \frac{f(-q^{12},-q^{28}) - qf(-q^8,-q^{32})}{f(-q^2,-q^{18})} \pmod 2.$$

We have used the fact that

$$\varphi(q) := \frac{f_2^5}{f_1^2 f_4^2} \equiv 1 \pmod 2.$$

Extracting the even terms in

$$\sum_{n=0}^{\infty} c_9(n)q^n \equiv \frac{f(-q^{12},-q^{28}) - qf(-q^8,-q^{32})}{f(-q^2,-q^{18})} \pmod 2,$$

we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_9(2n)q^n \\
&\equiv \frac{f(-q^6,-q^{14})}{f(-q,-q^9)} \\
&\equiv \frac{f(-q^6,-q^{14})f(q,q^9)}{f(-q,-q^9)f(q,q^9)} \\
&\equiv \frac{f(-q^6,-q^{14})\Big(f(q^{12},q^{28}) + qf(q^8,q^{32})\Big)}{f(-q^2,-q^{18})\varphi(-q^{10})} \pmod 2.
\end{aligned}$$

Extracting the odd terms in

$$\sum_{n=0}^{\infty} c_9(2n)q^n \equiv \frac{f(-q^6,-q^{14})\Big(f(q^{12},q^{28})+qf(q^8,q^{32})\Big)}{f(-q^2,-q^{18})\varphi(-q^{10})} \pmod 2,$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty} c_9(4n+2)q^n &\equiv \frac{f(-q^3,-q^7)f(-q^4,-q^{16})}{f(-q,-q^9)}\\ &\equiv \frac{f^2(-q^2,-q^8)f(-q^3,-q^7)}{f(-q,-q^9)} \pmod 2.\end{aligned}$$

We need to find a 3-dissection of the right side.

We have

$$\sum_{n=0}^{\infty} c_9(4n+2)q^n \equiv \frac{f^2(-q^2,-q^8)f(-q^3,-q^7)}{f(-q,-q^9)}$$
$$\equiv \frac{f^2(-q^2,-q^8)f^2(-q^3,-q^7)}{f(-q,-q^9)f(-q^3,-q^7)} \pmod 2.$$

From

$$f(a,ab^2)f(b,a^2b) = f(a,b)\psi(ab),$$

we have

$$f(-q^2,-q^8)f(-q^3,-q^7) = f(-q^2,-q^3)\psi(q^5),$$

where

$$\psi(q) = f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}.$$

Jacobi's triple product identity is given by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Therefore,

$$\begin{aligned} f(-q,-q^9)(f(-q^3,-q^7) &= (q,q^3,q^7,q^9;q^{10})_\infty f_{10}^2 \\ &= \frac{(q;q^2)_\infty f_{10}^2}{(q^5;q^{10})_\infty}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{n=0}^{\infty} c_9(4n+2)q^n &\equiv \frac{f^2(-q^2,-q^8)f^2(-q^3,-q^7)}{f(-q,-q^9)f(-q^3,-q^7)} \\ &\equiv \frac{(q^5;q^{10})_\infty f^2(-q^2,-q^3)\psi^2(q^5)}{(q;q^2)_\infty f_{10}^2} \\ &\equiv f(-q^4,-q^6)f_1 f_5 \pmod 2. \end{aligned}$$

We seek a 3-dissection of $f(-q^4, -q^6)f_1 f_5$.
To that end, first we have

$$f_1 f_5 \equiv f_2^3 + q f_{10}^3 \pmod{2}.$$

This easily follows from Ramanujan's identity From [1, p. 262, Entry 10(v)], we have

$$\psi^2(q) - q\psi^2(q^5) = f(q, q^4)f(q^2, q^3).$$

Next, from Berndt[1], we have

$$f_1^3 = f_3 a(q^3) - 3q f_9^3,$$

which implies that

$$f_1^3 \equiv f_3 a(q^3) + q f_9^3 \pmod{2}.$$

Thus,

$$f_1 f_5 \equiv \left(f_6 a(q^6) + q^2 f_{18}^3\right)\left(f_{30} a(q^{30}) + q^{10} f_{90}^3\right) \pmod{2}.$$

1 Berndt, B.C.: *Ramanujan's Notebooks Part III.* Springer (1991), [p. 346, Entry 1(v)]

We were seeking a 3-dissection in

$$\sum_{n=0}^{\infty} c_9(4n+2)q^n \equiv f(-q^4,-q^6)f_1f_5 \pmod 2.$$

We recall another result from Berndt[1].

Lemma 3.1

Let $U_n = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$ *and* $V_n = a^{\frac{n(n-1)}{2}} b^{\frac{n(n+1)}{2}}$ *for each integer* n*. Then*

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

Setting $n = 3$, $a = -q^4$, and $b = -q^6$, we find that

$$f(-q^4, -q^6) = f(-q^{42}, -q^{48}) - q^4 f(-q^{18}, -q^{72}) - q^6 f(-q^{12}, -q^{78}),$$

which is a 3-dissection of $f(-q^4, -q^6)$.

1

Berndt, B.C.: *Ramanujan's Notebooks Part III.* Springer (1991), p. 48, Entry 31]

Employing the 3 dissections of $f_1 f_5$ and $f(-q^4, -q^6)$ in

$$\sum_{n=0}^{\infty} c_9(4n+2)q^n \equiv f(-q^4,-q^6) f_1 f_5 \pmod 2,$$

and then extracting the terms involving q^{3n}, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} & c_9(12n+2)q^n \\ &\equiv a(q^6)\Big(f(-q^{84},-q^{96}) - q^8 f(-q^{36},-q^{144}) - q^{12} f(-q^{24},-q^{156})\Big) \\ &\quad - q^2 f(-q^6,-q^{24})\Big(f_6^3 + q^3 f_{30}^3\Big) \pmod 2.\end{aligned}$$

Extracting the terms involving q^{3n+1}, we deduce that

$$c_9(36n+14) \equiv 0 \pmod 2.$$

Proof of $c_{17}(128n+80) \equiv 0 \pmod{2}$

We have

$$\begin{aligned}\sum_{n=0}^{\infty} c_{17}(n)q^n &= \frac{1}{\Psi(-q^{17},q)} \equiv \frac{1}{f(-q^{17},q)} \\ &\equiv \frac{f(-q,q^{17})}{f(q,-q^{17})f(-q,q^{17})} \\ &\equiv \frac{f(-q^{20},-q^{52}) - qf(-q^{16},-q^{56})}{f(-q^{2},-q^{34})\varphi(q^{18})} \pmod 2.\end{aligned}$$

Then

$$\sum_{n=0}^{\infty} c_{17}(2n)q^n \equiv \frac{f(-q^{10},-q^{26})}{f(-q,-q^{17})}$$
$$\equiv \frac{f(-q^{10},-q^{26})f(q,q^{17})}{f(-q,-q^{17})f(q,q^{17})}$$
$$\equiv \frac{f(-q^{10},-q^{26})\left(f(q^{20},q^{52})+qf(q^{16},q^{56})\right)}{f(-q^{2},-q^{34})\varphi(-q^{18})} \pmod 2,$$

$$\sum_{n=0}^{\infty} c_{17}(4n)q^n \equiv \frac{f(-q^{5},-q^{13})f(q^{10},q^{26})}{f(-q,-q^{17})}$$
$$\equiv \frac{f(q^{10},q^{26})f(-q^{5},-q^{13})f(q,q^{17})}{f(-q,-q^{17})f(q,q^{17})} \pmod 2,$$

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_{17}(8n)q^n \\
&\equiv \frac{f(q^5,q^{13})f(-q^3,-q^{15})f(-q^7,-q^{11})}{f(-q,-q^{17})} \\
&\equiv \frac{f(q^5,q^{13})f(-q^3,-q^{15})f(-q^7,-q^{11})f(-q,-q^{17})}{f(q,q^{17})f(-q,-q^{17})} \\
&\equiv \frac{f(-q,-q^{17})f(-q^3,-q^{15})f(-q^5,-q^{13})f(-q^7,-q^{11})}{f(-q^2,-q^{34})} \pmod{2}.
\end{aligned}$$

By Jacobi Triple Product Identity, we have

$$\begin{aligned}
&f(-q,-q^{17})f(-q^3,-q^{15})f(-q^5,-q^{13})f(-q^7,-q^{11})\\
&=(q,q^3,q^5,q^7,q^{11},q^{13},q^{15},q^{17};q^{18})_\infty f_{18}^4\\
&=\frac{(q;q^2)_\infty f_{18}^4}{(q^9;q^{18})_\infty}\\
&=\frac{f_1 f_{18}^5}{f_2 f_9}\\
&\equiv\frac{f_{72} f_9}{f_1} \pmod 2.
\end{aligned}$$

We have a 2-dissection of f_9/f_1 given by Xia and Yao[2], namely,

$$\frac{f_9}{f_1} \equiv \frac{f_{12}^3}{f_4 f_6 f_{18}} + q\frac{f_4 f_{36}}{f_2 f_6} \pmod 2.$$

Therefore,

$$\begin{aligned} &f(-q,-q^{17})f(-q^3,-q^{15})f(-q^5,-q^{13})f(-q^7,-q^{11}) \\ &\equiv \frac{f_{72} f_9}{f_1} \\ &\equiv \frac{f_{12}^3 f_{72}}{f_4 f_6 f_{18}} + q\frac{f_4 f_{36} f_{72}}{f_2 f_6} \pmod 2. \end{aligned}$$

2

Xia, E.X.W., Yao, O.X.M.: *Parity results for 9-regular partitions.* Ramanujan J. **34**, 109–117 (2014)

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_{17}(8n)q^n \\
&\equiv \frac{f(-q,-q^{17})f(-q^3,-q^{15})f(-q^5,-q^{13})f(-q^7,-q^{11})}{f(-q^2,-q^{34})} \\
&\equiv \frac{f_{72}}{f(-q^2,-q^{34})} \cdot \frac{f_9}{f_1} \\
&\equiv \frac{f_{72}}{f(-q^2,-q^{34})} \left(\frac{f_{12}^3 f_{72}}{f_4 f_6 f_{18}} + q\frac{f_4 f_{36} f_{72}}{f_2 f_6} \right) \pmod{2}.
\end{aligned}$$

Extracting the even terms, we have

$$\begin{aligned}\sum_{n=0}^{\infty} c_{17}(16n)q^n &\equiv \frac{f_{36}f_6^3}{f_2f_3f_9f(-q,-q^{17})} \equiv \frac{f_9^3f_6^3}{f_2f_3f(-q,-q^{17})}\\ &\equiv \frac{f_6^3 f(q,q^{17})}{f_2f(-q^2,-q^{34})}\cdot\frac{f_9^3}{f_3}\\ &\equiv \frac{f_6^3\Big(f(q^{20},q^{52})+qf(q^{16},q^{56})\Big)}{f_2f(-q^2,-q^{34})}\cdot\frac{f_9^3}{f_3} \pmod 2.\end{aligned}$$

We seek a 2-dissection of f_9^3/f_3.

From Xia and Yao[3], we have

$$\frac{f_3^3}{f_1} \equiv f_8 + q\frac{f_{12}^3}{f_4} \pmod{2}.$$

Therefore,

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_{17}(16n)q^n \\
&\equiv \frac{f_6^3\left(f(q^{20},q^{52}) + qf(q^{16},q^{56})\right)}{f_2 f(-q^2,-q^{34})} \cdot \frac{f_9^3}{f_3} \\
&\equiv \frac{f_6^3\left(f(q^{20},q^{52}) + qf(q^{16},q^{56})\right)}{f_2 f(-q^2,-q^{34})} \left(f_{24} + q^3\frac{f_{36}^3}{f_{12}}\right) \pmod{2}.
\end{aligned}$$

3

Xia, E.X.W., Yao, O.X.M.: *New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions.* J. Number Theory **133**, 1932–1949 (2013)

Extracting the odd terms, we find that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} c_{17}(32n+16)q^n \\
&\equiv \frac{f_3^3}{f_1 f(-q,-q^{17})}\left(f_{12} f(q^8,q^{28}) + q\frac{f_{18}^3}{f_6} f(q^{10},q^{26})\right) \\
&\equiv \frac{f_{12} f(q^8,q^{28}) + q\frac{f_{18}^3}{f_6} f(q^{10},q^{26})}{f(-q^2,-q^{34})} \cdot \frac{f_3^3}{f_1} \cdot f(q,q^{17}) \\
&\equiv \frac{f_{12} f(q^8,q^{28}) + q\frac{f_{18}^3}{f_6} f(q^{10},q^{26})}{f(-q^2,-q^{34})}\left(f_8 + q\frac{f_{12}^3}{f_4}\right) \\
&\quad \times \left(f(q^{20},q^{52}) + qf(q^{16},q^{56})\right) \pmod{2}.
\end{aligned}
$$

Extracting the even terms, we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} c_{17}(64n+16)q^n \\
&\equiv \frac{1}{f(-q,-q^{17})}\Big(f_4f_6f(q^4,q^{14})f(q^8,q^{28}) + q\frac{f_6^4}{f_2}f(q^4,q^{14})f(q^8,q^{28}) \\
&\quad + q\frac{f_4f_9^3}{f_3}f(q^5,q^{13})f(q^8,q^{28}) + q\frac{f_6^3f_9^3}{f_2f_3}f(q^5,q^{13})f(q^{10},q^{26})\Big) \\
&\equiv \frac{f(q^{20},q^{52}) + qf(q^{16},q^{56})}{f(-q^2,-q^{34})}\Big(f_4f_6f(q^4,q^{14})f(q^8,q^{28}) \\
&\quad + q\frac{f_{24}}{f_2}f(q^4,q^{14})f(q^8,q^{28})\Big) \\
&\quad + \frac{qf_4f(q^8,q^{28}) + q\frac{f_6^3}{f_2}f(q^{10},q^{26})}{f(-q^2,-q^{34})}f(q,q^{17})f(q^5,q^{13})\frac{f_9^3}{f_3} \pmod 2.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} c_{17}(64n+16)q^n$$
$$\equiv \frac{f(q^{20},q^{52})+qf(q^{16},q^{56})}{f(-q^2,-q^{34})}\Big(f_4 f_6 f(q^4,q^{14})f(q^8,q^{28})$$
$$+q\frac{f_{24}}{f_2}f(q^4,q^{14})f(q^8,q^{28})\Big)$$
$$+\frac{1}{f(-q^2,-q^{34})}\Big(qf_4 f(q^8,q^{28})+q\frac{f_6^3}{f_2}f(q^{10},q^{26})\Big)\Big(f_{24}+q^3\frac{f_{36}^3}{f_{12}}\Big)$$
$$\times\Big(f(q^6,q^{30})f(q^{14},q^{22})+qf(q^{12},q^{24})f(q^4,q^{32})\Big) \pmod 2.$$

Extracting the odd terms, we find that

$$\sum_{n=0}^{\infty} c_{17}(128n+80)q^n$$
$$\equiv \frac{1}{f(-q,-q^{17})}\left(\frac{f_{12}}{f_1}f(q^2,q^7)f(q^4,q^{14})f(q^{10},q^{26})\right.$$
$$+ f_2f_3f(q^2,q^7)f(q^5,q^{13})f(q^8,q^{28}) + \left(f_2f(q^4,q^{14}) + \frac{f_3^3}{f_1}f(q^5,q^{13})\right)$$
$$\left.\times\left(f_{12}f(q^3,q^{15})f(q^7,q^{11}) + q^2\frac{f_{18}^3}{f_6}f(q^6,q^{12})f(q^2,q^{16})\right)\right)$$
$$\equiv \frac{1}{f(-q,-q^{17})}(A+B) \pmod 2,$$

where

$$\begin{aligned}A &= \frac{f_{12}}{f_1} f(q^2,q^7) f(q^4,q^{14}) f(q^{10},q^{26})\\ &+ \frac{f_3^3}{f_1} f(q^5,q^{13}) \Big(f_{12} f(q^3,q^{15}) f(q^7,q^{11})\\ &+ q^2 \frac{f_{18}^3}{f_6} f(q^6,q^{12}) f(q^2,q^{16}) \Big)\end{aligned}$$

and

$$\begin{aligned}B &= f_2 f_3 f(q^2,q^7) f(q^5,q^{13}) f(q^8,q^{28})\\ &+ f_2 f(q^4,q^{14}) \Big(f_{12} f(q^3,q^{15}) f(q^7,q^{11})\\ &+ q^2 \frac{f_{18}^3}{f_6} f(q^6,q^{12}) f(q^2,q^{16}) \Big).\end{aligned}$$

To arrive at

$$c_{17}(128n+80) \equiv 0 \pmod 2$$

from

$$\sum_{n=0}^{\infty} c_{17}(128n+80)q^n \equiv \frac{1}{f(-q,-q^{17})}(A+B),$$

it suffices to show that $A+B \equiv 0 \pmod 2$.

After some simplifications, we find that

$$A+B \equiv \left(\frac{f_{12} f(q^5,q^{13})}{f_1} + f_2 f_3 f(q^4,q^{14})\right) C \pmod 2, \qquad (16)$$

where

$$C = f^3(q^2,q^7) f(q^5,q^{13}) + f_3^3 f(q^3,q^{15}) f(q^7,q^{11}) + q^2 \frac{f_{18}^3}{f_3} f(q^2,q^{16}).$$

By Jacobi triple product identity

$$\begin{aligned} f(q^3, q^{15}) &= (-q^3, -q^{15}; q^{18})_\infty f_{18} \\ &= \frac{(-q^3; q^6)_\infty f_{18}}{(-q^9; q^{18})_\infty} \\ &\equiv \frac{(q^3; q^6)_\infty f_{18}}{(q^9; q^{18})_\infty} \\ &\equiv \frac{f_9 f_{18}}{f_3} \pmod 2. \end{aligned}$$

Therefore,

$$\begin{aligned} C &= f^3(q^2,q^7)f(q^5,q^{13}) + f_3^3 f(q^3,q^{15})f(q^7,q^{11}) + q^2\frac{f_{18}^3}{f_3}f(q^2,q^{16}) \\ &\equiv f(q^2,q^7)f(q^4,q^{14})f(q^5,q^{13}) + f_6f_9f_{18}f(q^7,q^{11}) \\ &\quad + q^2\frac{f_{18}^3}{f_3}f(q^2,q^{16}) \pmod 2. \end{aligned}$$

However, from $f(a,ab^2)f(b,a^2b) = f(a,b)\psi(ab)$, we have

$$f(q^4,q^{14})f(q^5,q^{13}) = f(q^4,q^5)\psi(q^9) \equiv f(q^4,q^5)f_9f_{18} \pmod 2.$$

Thus,

$$\begin{aligned} C \equiv f_9f_{18}\Big(& f(q^2,q^7)f(q^4,q^5) + f_6 f(q^7,q^{11}) \\ & + q^2\frac{f_9f_{18}}{f_3}f(q^2,q^{16})\Big) \pmod 2. \end{aligned}$$

Now,

$$f(q^2,q^7)f(q^4,q^5)=f(q^6,q^{12})f(q^7,q^{11})+q^2f(q^3,q^{15})f(q^2,q^{16})$$

Therefore,

$$\begin{aligned}
C &\equiv f_9f_{18}\Big(f(q^2,q^7)f(q^4,q^5)+f_6f(q^7,q^{11})+q^2\frac{f_9f_{18}}{f_3}f(q^2,q^{16})\Big)\\
&\equiv f_9f_{18}\Big(f(q^6,q^{12})f(q^7,q^{11})+q^2f(q^3,q^{15})f(q^2,q^{16})+f_6f(q^7,q^{11})\\
&\quad+q^2\frac{f_9f_{18}}{f_3}f(q^2,q^{16})\Big)\\
&\equiv f_9f_{18}\Big(f_6f(q^7,q^{11})+q^2\frac{f_9f_{18}}{f_3}f(q^2,q^{16})+f_6f(q^7,q^{11})\\
&\quad+q^2\frac{f_9f_{18}}{f_3}f(q^2,q^{16})\Big)\\
&\equiv 0 \pmod 2.
\end{aligned}$$

We proved

$$\sum_{n=0}^{\infty} c_{17}(128n+80)q^n \equiv \frac{1}{f(-q,-q^{17})}(A+B) \equiv 0 \pmod{2},$$

where

$$A+B \equiv \left(\frac{f_{12}f(q^5,q^{13})}{f_1} + f_2f_3f(q^4,q^{14})\right)C \pmod{2}$$

with

$$C = f^3(q^2,q^7)f(q^5,q^{13}) + f_3^3 f(q^3,q^{15})f(q^7,q^{11}) + q^2\frac{f_{18}^3}{f_3}f(q^2,q^{16})$$
$$\equiv 0 \pmod{2}.$$

Concluding Remark

Keith's conjecture:
For $n \geq 0$,

$$\begin{aligned} c_9(36n+14) &\equiv 0 \pmod 2, \\ c_9(196n+j) &\equiv 0 \pmod 2, \ \textit{where } j \in \{54, 166, 194\}, \qquad (17) \\ c_{13}(32n+23) &\equiv 0 \pmod 2, \\ c_{13}(64n+63) &\equiv 0 \pmod 2, \\ c_{13}(72n+j) &\equiv 0 \pmod 2, \ \textit{where } j \in \{15, 21, 39, 69\}, \\ c_{17}(128n+80) &\equiv 0 \pmod 2. \end{aligned}$$

Congruence (17) is equivalent to

$$c_9(392n+j) \equiv 0 \pmod 2, \ \textit{where } j \in \{54, 166, 194, 250, 362, 390\}.$$

We proved:

For $n \geq 0$, we have

$$c_9(36n+14) \equiv 0 \pmod 2, \tag{18}$$

$$c_9(392n+j) \equiv 0 \pmod 2, \text{ where } j \in \{54, 166, 390\}, \tag{19}$$

$$c_{13}(32n+23) \equiv 0 \pmod 2, \tag{20}$$

$$c_{13}(64n+63) \equiv 0 \pmod 2, \tag{21}$$

$$c_{13}(72n+j) \equiv 0 \pmod 2, \text{ where } j \in \{15, 21, 39, 69\}, \tag{22}$$

$$c_{17}(128n+80) \equiv 0 \pmod 2. \tag{23}$$

To complete the proof of Keith's conjecture, one has to show that

$$c_9(392n+j) \equiv 0 \pmod{2}, \text{ where } j \in \{194, 250, 362\}.$$

We find that

$$\sum_{n=0}^{\infty} c_9(8n+2)q^n \equiv f(-q^2,-q^3)f_1^3 \equiv f(-q^2,-q^3)\psi(q) \pmod{2}.$$

Employing the 7-dissections of $\psi(q)$ and $f(q^2, q^3)$ in

$$\sum_{n=0}^{\infty} c_9(8n+2)q^n \equiv f(-q^2, -q^3)\psi(q) \pmod 2$$

and then extracting the terms involving q^{7n+3} from the resulting identity, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_9(56n+26)q^n \equiv\ & f(q^3, q^4)\Big(f(q^{12}, q^{23}) + q^3 f(q^2, q^{33})\Big) \\ & + f(q^2, q^5)\Big(f(q^{13}, q^{22}) + qf(q^8, q^{27})\Big) \\ & + f(q, q^6)\Big(f(q^{17}, q^{18}) + q^3 f(q^3, q^{32})\Big) \\ & + q^2 f(q^7, q^{28})\psi(q^7) \pmod 2.\end{aligned}$$

Therefore, to prove the conjecture it suffices to show that there is no term involving q^{7n+r}, where $r \in \{3, 4, 6\}$, on the right side of the above.

However, we could not find an effective 7-dissection to that end.

References I

[1] Berndt, B.C.: Ramanujan's Notebooks Part III. Springer, New York (1991)

[2] Berndt, B.C.: Ramanujan's Notebooks Part V. Springer, New York (1998)

[3] Berndt, B.C.: Number Theory in the Spirit of Ramanujan. American Mathematical Society, Providence, RI (2006)

[4] Jin, J., Wang, S., Yao, O.X.M.: Proof of a conjecture of Keith on congruences of the reciprocal of a false theta function. https://arxiv.org/abs/2508.01532

[5] Keith, W.J.: Reciprocals of false theta functions. Ramanujan J. **68**, 27 (2025)

References II

[6] Xia, E.X.W., Yao, O.X.M.: New Ramanujan-like congruences modulo powers of 2 and 3 for overpartitions. J. Number Theory **133**, 1932–1949 (2013)

[7] Xia, E.X.W., Yao, O.X.M.: Parity results for 9-regular partitions. Ramanujan J. **34**, 109–117 (2014)

Thank You so much!