A Genus 2 Congruence Family

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(with Koustav Banerjee)

Michigan Tech Seminar in Partition Theory, q-Series, and Related Topics

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FWF Der Wissenschaftsfonds.

Congruences and Partitions

$p(n)$ counts the integer partitions of n. For example, $p(4) = 5$:

- 4
- $3 + 1$
- $2 + 2$
- $2 + 1 + 1$
- $1 + 1 + 1 + 1$

n	$p(n)$	n	$p(n)$	n	$p(n)$	n	$p(n)$
1	1	16	231	31	6842	46	105558
2	2	17	297	32	8349	47	124754
3	3	18	385	33	10143	48	147273
4	5	19	490	34	12310	49	173525
5	7	20	627	35	14883	50	204226
6	11	21	792	36	17977	51	239943
7	15	22	1002	37	21637	52	281589
8	22	23	1255	38	26015	53	329931
9	30	24	1575	39	31185	54	386155
10	42	25	1958	40	37338	55	451276
11	56	26	2436	41	44583	56	526823
12	77	27	3010	42	53174	57	614154
13	101	28	3718	43	63261	58	715220
14	135	29	4565	44	75175	59	831820
15	176	30	5604	45	89134	60	966467

Congruences and Partitions

For $24n \equiv 1 \bmod 5^{\alpha}, \quad p(n) \equiv 0 \bmod 5^{\alpha}.$

For $24n \equiv 1 \bmod 7^{\alpha}, \quad p(n) \equiv 0 \bmod 7^{\lfloor \alpha/2 \rfloor + 1}.$

For $24n \equiv 1 \bmod 11^{\alpha}, \quad p(n) \equiv 0 \bmod 11^{\alpha}.$

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For $24n \equiv 1 \bmod 11^{\alpha}, \quad p(n) \equiv 0 \bmod 11^{\alpha}.$

$$\prod_{m=1}^{\infty} \frac{1}{1-q^m} = \sum_{n=0}^{\infty} p(n) q^n$$

Congruence Families for Modular Forms

Given a modular form f with a Fourer expansion $f = \sum_{n \geq n_0} a(n)q^n$, we find a common pattern:

For $n \equiv \lambda_\alpha \bmod \ell^\alpha$, we have $a(n) \equiv 0 \bmod \ell^\alpha$.

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Modular forms very often count interesting arithmetical objects.

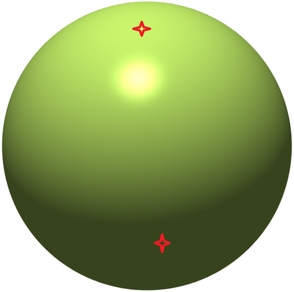
- Integer partitions, e.g., $p(n)$
- Representations of integers in terms of quadratic forms
- Important numbers related to elliptic curves on finite fields (Fermat's Last Theorem)

Arithmetic and Topology

Each congruence family for a modular form has an associated geometrical object.

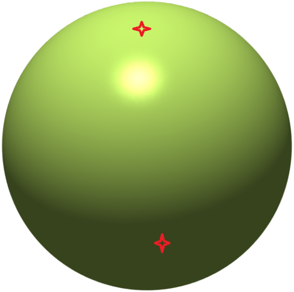
(Ramanujan)
If $24n-1$ is divisible by 5^{α}, then $p(n)$ is divisible by 5^{α}.

$\mathrm{X}_0(5)$ (genus 0, cusp count 2)



(Watson)
If $24n - 1$ is divisible by 7^{α}, then $p(n)$ is divisible by $7^{\lfloor \alpha/2 \rfloor + 1}$.

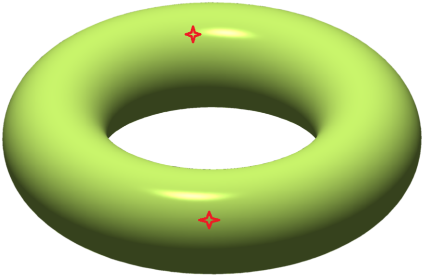
$\mathrm{X}_0(7)$ (genus 0, cusp count 2)



(Atkin)

If $24n - 1$ is divisible by 11^α, then $p(n)$ is divisible by 11^α.

$\mathrm{X}_0(11)$ (genus 1, cusp count 2)

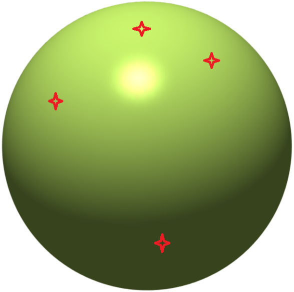


Arithmetic and Topology

$$\prod_{m=1}^{\infty} \frac{(1-q^{2m})^5}{(1-q^m)^{16}} = \sum_{n=0}^{\infty} d_5(n) q^n$$

(Banerjee, Me!)
If $4n-1$ is divisible by 5^{α}, then $d_5(n)$ is divisible by $5^{\lfloor \alpha/2 \rfloor + 1}$.

$\mathrm{X}_0(10)$ (genus 0, cusp count 4)

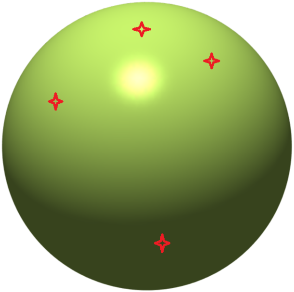


Arithmetic and Topology

$$\prod_{m=1}^{\infty}(1+x^m) = 1 + q(1)x^1 + q(2)x^2 + q(3)x^3 + \dots$$

(Gordon, Hughes)
If $24n+1$ is divisible by 5^{α}, then $q(n)$ is divisible by $5^{\lfloor(\alpha-1)/2\rfloor}$.

$\mathrm{X}_0(10)$ (genus 0, cusp count 4)

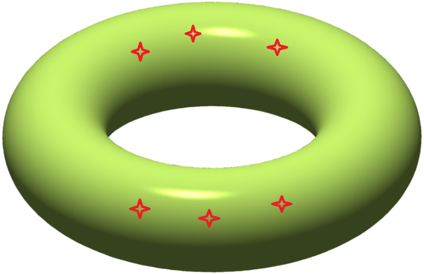


Arithmetic and Topology

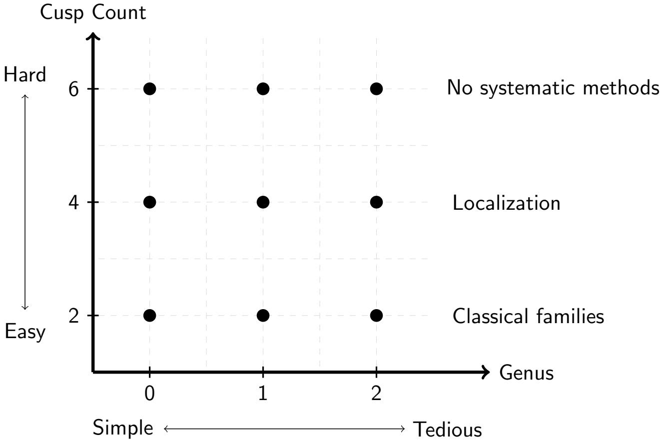
$$\prod_{m=1}^{\infty} \frac{(1-q^{2m})^5}{(1-q^m)^4(1-q^{4m})^2} = 1 + c\phi_2(1)q^1 + c\phi_2(2)q^2 + c\phi_2(3)q^3 + \dots$$

(Paule–Radu)
If $12n - 1$ is divisible by 5^{α}, then $c\phi_2(n)$ is divisible by 5^{α}.

$\mathrm{X}_0(20)$ (genus 1, cusp count 6)

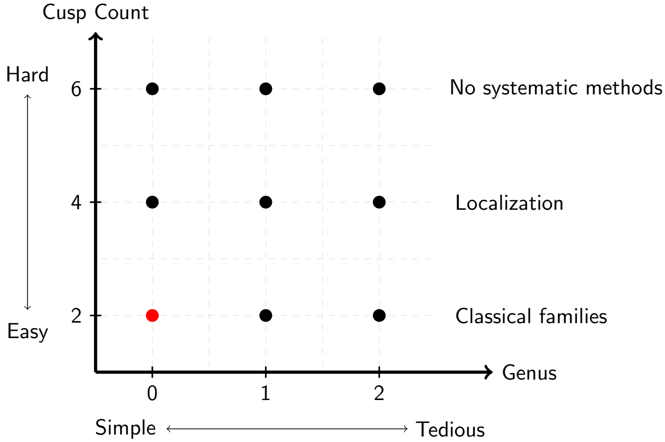


Classification



Classifying congruence families by the topology of the associated modular curve $\mathrm{X}_0(N)$

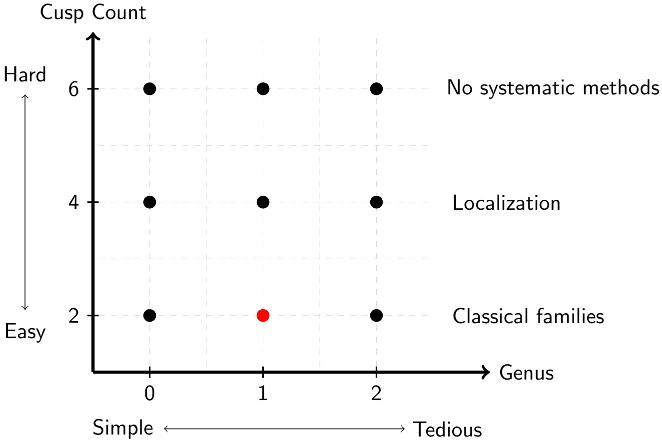
Classification



Classical families

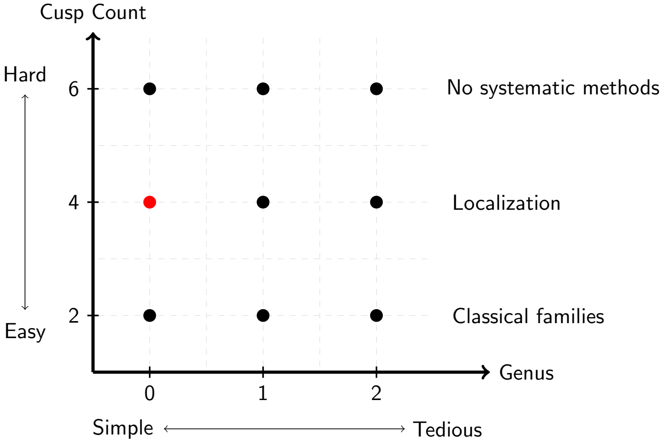
- Ramanujan's congruences for $p(n)$ by powers of 5, 7

Classification



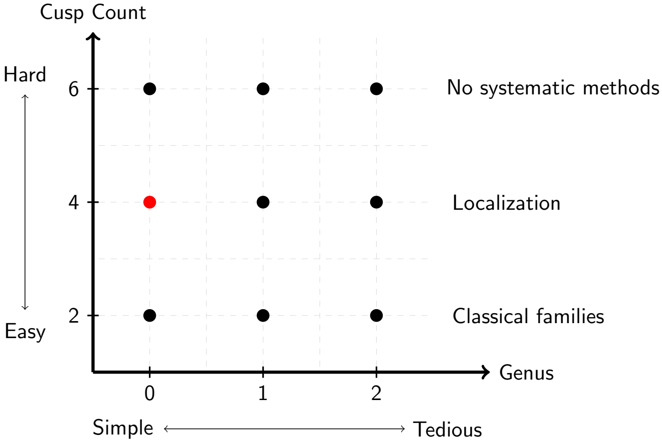
- Ramanujan's congruences for $p(n)$ by powers of 11

Classification



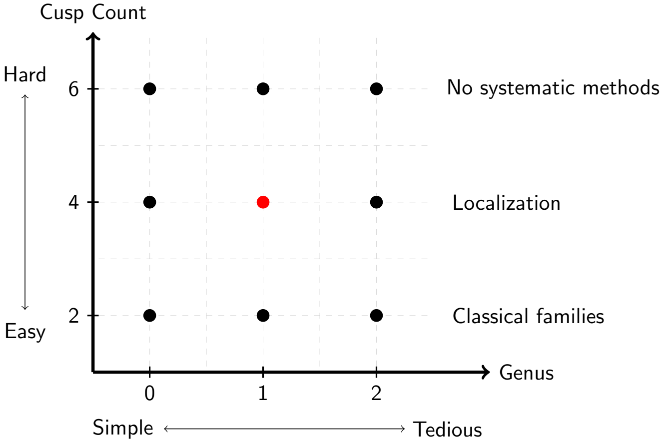
- $d_5(n)$ by powers of 5 (with Koustav Banerjee, "The Localization Method Applied to k-Elongated Plane Partitions and Divisibility by 5," *Mathematische Zeitschrift* 309 (46), 2025).

Classification



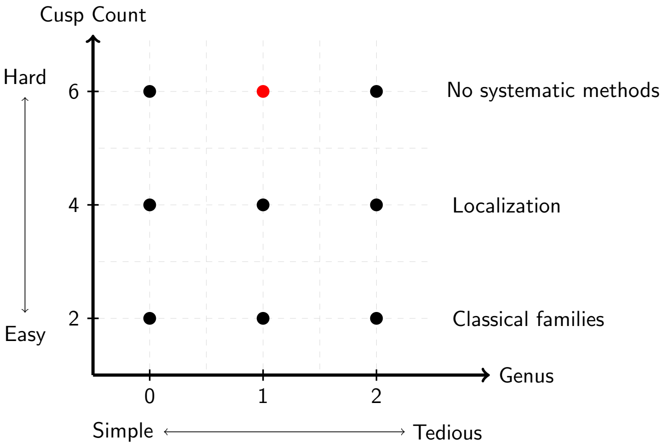
- $q(n)$ by powers of 5 (B. Gordon and K. Hughes, "Ramanujan Congruences," In: Knopp, M.I. (eds) Analytic Number Theory. Lecture Notes in Mathematics, 1981).

Classification



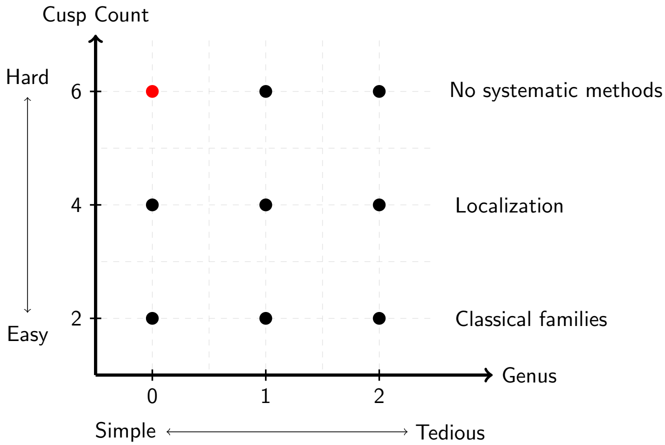
- $d_2(n)$ by powers of 7 (with Koustav Banerjee: "2-Elongated Plane Partitions and Powers of 7: The Localization Method Applied to a Genus 1 Congruence Family," *Canadian Journal of Mathematics*, 2025).

Classification



- $c\phi_2(n)$ by powers of 5 (Paule, Radu, "The Andrews–Sellers Family of Partition Congruences," *Advances in Mathematics* 230, pp. 819-838 (2012).)

Classification



Standing conjectures

- Hecke trace of singular moduli by powers of 3 (Beazer, *3-adic Properties of Hecke Traces of Singular Moduli*, Master Thesis, Brigham Young University (2021).)

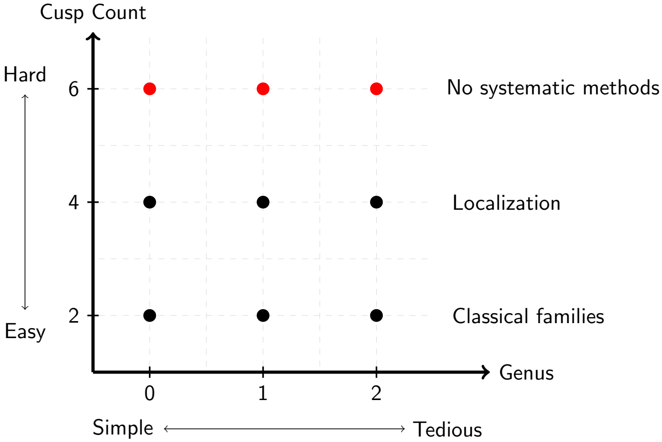
Beazer's Conjecture

Conjecture (Beazer)

Let $t_m(d)$ be the m-th Hecke trace function. If $m, \alpha \in \mathbb{Z}$ is coprime with 6 and $d \equiv 8, 11 \pmod{12}$, then

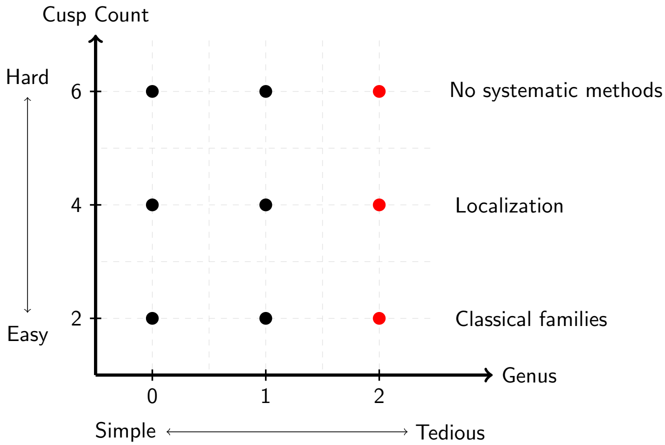
$$t_m(9^n d) \equiv 0 \pmod{3^{3n+2}}.$$

Classification



- Right now we just have “tricks.”
- Some open and extremely difficult conjectures.
- We want more systematic methods.

Classification



- What about here?

"As far as congruences for $q(n) \pmod{11^\alpha}$ are concerned, the problem is greatly complicated by the fact that the group $\Gamma_0(22)$ has genus 2. The computational difficulties seem too formidable to justify pursuing this problem here."

Basil Gordon, K. Hughes, "Ramanujan Congruences," In: Knopp, M.I. (eds) Analytic Number Theory. Lecture Notes in Mathematics, vol 899. Springer, Berlin, Heidelberg (1981). https://doi.org/10.1007/BFb0096473

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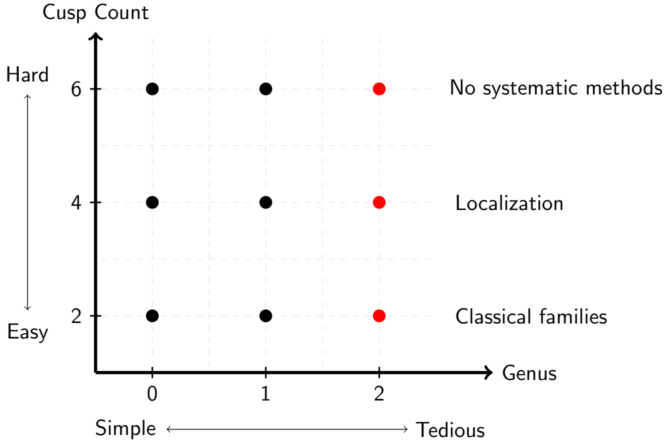
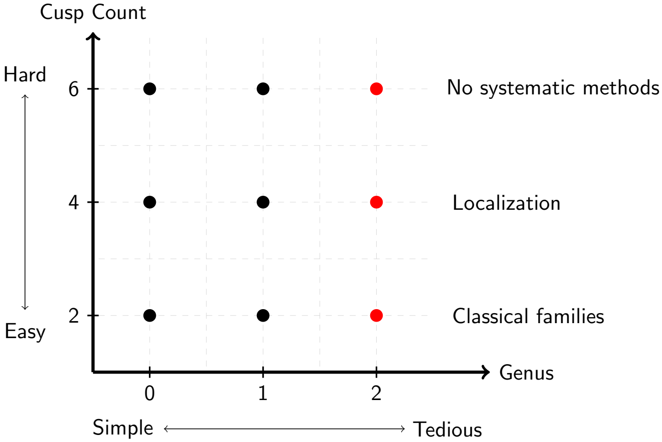


- No congruence families have been found in this domain

Classification



- No congruence families have been found in this domain
- ...until now.

A Genus 2 Congruence Family

$d_2(n)$ counts the 2-elongated plane partition diamonds of n.

$$\sum_{n=0}^{\infty} d_2(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2}{(1-q^m)^7}$$

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$$\sum_{n=0}^{\infty} d_2(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2}{(1-q^m)^7}$$

Theorem (Banerjee, Me!)

If $n, \alpha \in \mathbb{Z}$ with $8n \equiv 1 \pmod{11^\alpha}$, then

$$d_2(n) \equiv 0 \pmod{11^{\lfloor (\alpha+1)/2 \rfloor}}.$$

The associated modular curve is $\mathrm{X}_0(22)$ (genus 2, cusp count 4).

We build a sequence of functions

$$L_\alpha := \phi_\alpha \cdot \sum_{8n \equiv 1 \bmod 11^\alpha} d_2(n) q^{\lfloor n/11^\alpha \rfloor + 1},$$

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and a pair of Atkin U-operators such that

$$U^{(1)}\left(L_{2\alpha-1}\right) = L_{2\alpha},$$
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The first example:

$$L_1 = \prod_{m=1}^{\infty} \frac{(1-q^{11m})^7}{(1-q^{22m})^2} \cdot \sum_{n=0}^{\infty} d_2(11n+7) q^{n+2}.$$

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- These functions have $\mathrm{X}_0(22)$ as a domain.
- We want to represent L_α in terms of some useful basis functions.

Some Properties of $X_0(22)$

- Compact Riemann surface
- Genus 2
- Cusp count 4
- We can use the Weierstrass gap theorem

There exist functions x, y, z such that

$$\mathcal{M}^{(0)}\left(X_0(22)\right) = \mathbb{C}[x] \oplus y\mathbb{C}[x] \oplus z\mathbb{C}[x].$$

Some Properties of $\mathrm{X}_0(22)$

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$$x = \frac{1}{44}\Bigg(-7 + 11\frac{(q^2;q^2)_\infty^4 (q^{11};q^{11})_\infty^8}{(q;q)_\infty^8 (q^{22};q^{22})_\infty^4} - 4\frac{(q^2;q^2)_\infty^{11}(q^{11};q^{11})_\infty}{(q;q)_\infty^{11}(q^{22};q^{22})_\infty} + 132q^5\frac{(q^2;q^2)_\infty^3 (q^{22};q^{22})_\infty^7}{(q;q)_\infty^7 (q^{11};q^{11})_\infty^3}\Bigg),$$

$$y = \frac{1}{11}\left(-1 + \frac{(q^2;q^2)_\infty^{11}(q^{11};q^{11})_\infty}{(q;q)_\infty^{11}(q^{22};q^{22})_\infty} - 121q^5\frac{(q^2;q^2)_\infty^3 (q^{22};q^{22})_\infty^7}{(q;q)_\infty^7 (q^{11};q^{11})_\infty^3}\right),$$

$$z = q^5\frac{(q^2;q^2)_\infty^3 (q^{22};q^{22})_\infty^7}{(q;q)_\infty^7 (q^{11};q^{11})_\infty^3}.$$

$$\mathcal{M}^{(0)}\left(\mathrm{X}_0(22)\right) = \mathbb{C}[x] \oplus y\mathbb{C}[x] \oplus z\mathbb{C}[x].$$

$$L_1 \notin \mathbb{C}[x] \oplus y\mathbb{C}[x] \oplus z\mathbb{C}[x].$$

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$$x\left(\frac{\tau}{2\tau+1}\right) = -\frac{2}{11} + O\left(\tilde{q}\right).$$

Thus, $2+11x$ has positive order at $[1/2]$.

$$L_1 = \prod_{m=1}^{\infty} \frac{(1-q^{11m})^7}{(1-q^{22m})^2} \cdot \sum_{n=0}^{\infty} d_2(11n+7)q^{n+2}.$$

$$L_1 = \frac{1}{2(2+11x)^{15}}\left(11\left(xM_0 + yM_1 + zM_2\right) + (5x-1)r_L\right),$$
$$r_L = 5x + 4x^2 + x^3 + 6y + 6xy + 10x^2y + 2z + 7xz,$$
$$M_v \in \mathbb{Z}[x],\ \ v \in \{0,1,2\}.$$

We can show (e.g., with Sturm's theorem) that

$$r_L \equiv 0 \pmod{11}.$$

Nicolas, don't forget to put a Mathematica file here so you don't look silly.

Main Theorem

$$\text{Denote } \mathbb{Z}[x]_{(2)} := \left\{ \frac{1}{2^n} f(x) : f \in \mathbb{Z}[x] \right\}.$$

$$\text{For } \alpha \geq 1, \text{ let } \psi := \psi(\alpha) = \left\lfloor \frac{11^{\alpha+1}}{8} \right\rfloor + 1 - \gcd(\alpha, 2),$$

$$\text{and } \beta := \beta(\alpha) = \left\lfloor \frac{\alpha+1}{2} \right\rfloor.$$

There exist two integer sequences $(a(\alpha))_{\alpha\geq 1}$, $(b(\alpha))_{\alpha\geq 1}$ such that, for all $\alpha \geq 1$,

$$\frac{(2+11x)^{\psi}}{11^{\beta}} L_{\alpha} - \frac{(a+bx)}{11} r_L \in \bigoplus_{v\in\{x,y,z\}} v\mathbb{Z}[x]_{(2)}.$$

Sketch

$$\frac{1}{11^{\alpha}} L_{2\alpha} = \frac{1}{(2+11x)^n} \left(\sum_{0 \leq j+k \leq 1} \sum_{m \geq 1-(j+k)} s_{jk}(m) 11^{\theta^0_{jk}(m)} y^j z^k x^m \right).$$

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$$\begin{aligned} \frac{1}{11^\alpha} L_{2\alpha+1} &= \frac{1}{11^\alpha} U^{(0)}\left(L_{2\alpha} \right) \\ &= \sum_{0 \le j+k \le 1} \sum_{m \ge 1-(j+k)} s_{jk}(m) 11^{\theta^0_{jk}(m)} U^{(0)} \left(\frac{y^j z^k x^m}{(2+11x)^n} \right) \end{aligned}$$

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$$= \sum_{\substack{0\leq j+k\leq 1,\\ m\geq 1-(j+k)}} s_{jk}(m) 11^{\theta^0_{jk}(m)} \frac{1}{(2+11x)^{11n+15}c(x)^n}$$

$$\times \sum_{\substack{0\leq \gamma+\delta\leq 1,\\ r\geq 1-(\gamma+\delta)}} h^{0jk}_{\gamma\delta}(m,n,r) 11^{\pi^{0jk}_{\gamma\delta}(m,r)} y^{\gamma} z^{\delta} x^{r}$$

Sketch

$$\frac{1}{11^{\alpha}} L_{2\alpha} = \frac{1}{(2+11x)^n} \left(\sum_{0 \leq j+k \leq 1} \sum_{m \geq 1-(j+k)} s_{jk}(m) 11^{\theta^0_{jk}(m)} y^j z^k x^m \right)$$

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We want to show that

$$\pi^{0jk}_{\gamma\delta}(m,r)+\theta^0_{jk}(m)\geq\theta^1_{\gamma\delta}(r)+1.$$

Sketch

$$\frac{1}{11^{\alpha}} L_{2\alpha+1} = \frac{1}{(2+11x)^{11n+15} c(x)^n} \times \sum_{\substack{0\leq j+k\leq 1,\\ m\geq 1-(j+k),\\ 0\leq \gamma+\delta\leq 1,\\ r\geq 1-(\gamma+\delta)}} s_{jk}(m) h_{\gamma\delta}^{0jk}(m,n,r) 11^{\pi_{\gamma\delta}^{0jk}(m,r)+\theta_{jk}^{0}(m)} y^{\gamma} z^{\delta} x^{r}.$$

We want to show that $\pi_{\gamma\delta}^{0jk}(m,r) + \theta_{jk}^{0}(m) \geq \theta_{\gamma\delta}^{1}(r) + 1$.

Sketch

$$\frac{1}{11^\alpha}L_{2\alpha+1} = \frac{1}{(2+11x)^{11n+15}c(x)^n}$$
$$\times \sum_{\substack{0\le j+k\le 1,\\ m\ge 1-(j+k),\\ 0\le\gamma+\delta\le 1,\\ r\ge 1-(\gamma+\delta)}} s_{jk}(m)h_{\gamma\delta}^{0jk}(m,n,r)11^{\pi_{\gamma\delta}^{0jk}(m,r)+\theta_{jk}^{0}(m)}y^\gamma z^\delta x^r.$$

We want to show that $\pi_{\gamma\delta}^{0jk}(m,r)+\theta_{jk}^{0}(m)\ge\theta_{\gamma\delta}^{1}(r)+1$.

This is true *except* for the following:

- $\pi_{00}^{000}(1,r)$, $1\le r\le 4$,
- $\pi_{10}^{000}(1,r)$, $0\le r\le 3$,
- $\pi_{01}^{000}(1,r)$, $0\le r\le 2$,
- $\pi_{00}^{010}(0,r)$, $1\le r\le 4$,
- $\pi_{10}^{010}(0,r)$, $0\le r\le 3$,
- $\pi_{01}^{010}(0,r)$, $0\le r\le 2$.

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Sketch

$$\frac{1}{11^{\alpha}}L_{2\alpha+1} = \frac{1}{(2+11x)^{11n+15}c(x)^n}\sum_{\substack{0\leq\gamma+\delta\leq 1,\\ r\geq 1-(\gamma+\delta)}} t_{\gamma\delta}(r)11^{\theta^1_{\gamma\delta}(r)+1}y^{\gamma}z^{\delta}x^{r}$$
$$+\frac{(a+bx)r_L}{(2+11x)^{11n+15}c(x)^n}$$

Sketch

$$\frac{1}{11^{\alpha}}L_{2\alpha+1} = \frac{1}{(2+11x)^{11n+15}c(x)^n}\sum_{\substack{0\leq\gamma+\delta\leq 1,\\ r\geq 1-(\gamma+\delta)}} t_{\gamma\delta}(r)11^{\theta^1_{\gamma\delta}(r)+1}y^{\gamma}z^{\delta}x^{r} + \frac{(a+bx)r_L}{(2+11x)^{11n+15}c(x)^n}$$

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$$\frac{(2+11x)^{11n+15}c(x)^n}{11^{\alpha+1}}L_{2\alpha+1}-\frac{(a+bx)r_L}{11}$$
$$=\sum_{\substack{0\leq\gamma+\delta\leq 1,\\ r\geq 1-(\gamma+\delta)}} t_{\gamma\delta}(r)11^{\theta^1_{\gamma\delta}(r)}y^{\gamma}z^{\delta}x^r.$$

Main Theorem

$$\text{Denote } \mathbb{Z}[x]_{(2)} := \left\{ \frac{1}{2^n} f(x) : f \in \mathbb{Z}[x] \right\}.$$

$$\text{For } \alpha \geq 1, \text{ let } \psi := \psi(\alpha) = \left\lfloor \frac{11^{\alpha+1}}{8} \right\rfloor + 1 - \mathrm{gcd}(\alpha, 2),$$
$$\text{and } \beta := \beta(\alpha) = \left\lfloor \frac{\alpha+1}{2} \right\rfloor.$$

There exist two integer sequences $(a(\alpha))_{\alpha\geq 1}$, $(b(\alpha))_{\alpha\geq 1}$ such that, for all $\alpha \geq 1$,

$$\frac{(2+11x)^{\psi}}{11^{\beta}} L_{\alpha} - \frac{(a+bx)}{11} r_L \in \bigoplus_{v \in \{x,y,z\}} v\mathbb{Z}[x]_{(2)}.$$

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- Can we prove these congruence families without modular equation recurrences?
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- Are there any other methods?