5-adic Convergence Over Modular Curves of Genus 1: The Andrews–Sellers Congruences and Beyond

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Specialty Seminar in Partition Theory, q-Series, and Related Topics

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Definition

Definition

For any $n \in \mathbb{Z}_{\geq 0}$, a partition of n is a representation of n as a sum of positive integers, called parts. The number of partitions of a given n is denoted $p(n)$.

Partitions

For example, $p(6) = 11$:

- 6
- $5 + 1$
- $4 + 2$
- $4 + 1 + 1$
- $3 + 3$
- $3 + 2 + 1$
- $3 + 1 + 1 + 1$
- $2 + 2 + 2$
- $2 + 2 + 1 + 1$
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- $1 + 1 + 1 + 1 + 1 + 1$

Ramanujan's Conjectures

Let $n \geq 0$, $\alpha \geq 1$, and $24\lambda_{\ell,\alpha} \equiv 1 \pmod{\ell^\alpha}$.

Theorem (Ramanujan, Watson, Atkin)

$$p\left(5^\alpha n + \lambda_{5,\alpha}\right) \equiv 0 \pmod{5^\alpha},$$
$$p\left(7^\alpha n + \lambda_{7,\alpha}\right) \equiv 0 \pmod{7^{\left\lfloor \frac{\alpha}{2} \right\rfloor + 1}},$$
$$p\left(11^\alpha n + \lambda_{11,\alpha}\right) \equiv 0 \pmod{11^\alpha}.$$

Ramanujan's Conjectures

For $\alpha \geq 1$ and $\ell \in \{5, 7, 11\}$, let

$$L_{\ell,2\alpha-1} := q(q^{\ell}; q^{\ell})_{\infty} \sum_{n=0}^{\infty} p\left(\ell^{2\alpha-1} n + \lambda_{\ell,2\alpha-1}\right) q^n$$

$$L_{\ell,2\alpha} := q(q; q)_{\infty} \sum_{n=0}^{\infty} p\left(\ell^{2\alpha} n + \lambda_{\ell,2\alpha}\right) q^n.$$

With the right change of variables, these are modular functions.

Modular Group Action

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : N|c \right\}.$$

Let $\hat{\mathbb{H}} := \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. We define a group action

$$\Gamma_0(N) \times \hat{\mathbb{H}} \longrightarrow \hat{\mathbb{H}},$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau \right) \longmapsto \frac{a\tau + b}{c\tau + d}.$$

Define the orbits $[\tau]_N := \{\gamma\tau : \gamma \in \Gamma_0(N)\}$.

Modular Functions

Let $q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$.

Definition

A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is modular over $\Gamma_0(N)$ if

- For any $\tau, z \in \mathbb{H}$ such that $z \in [\tau]_N$, $f(z) = f(\tau)$,
- For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{n \geq n_\gamma} \alpha_\gamma(n) q^{n \cdot \gcd(c^2, N)/N},$$

with $n_\gamma \in \mathbb{Z}$, $\alpha_\gamma(n_\gamma) \neq 0$.

The set of modular functions over $\Gamma_0(N)$ is denoted $\mathcal{M}\left(\Gamma_0(N)\right)$.

Ramanujan's Conjectures

Let $q = e^{2\pi i\tau}$ with $\tau \in \mathbb{H}$, and take $L_{\ell,\alpha}$ as functions of τ. Then

$$L_{\ell,\alpha} \in \mathcal{M}\left(\Gamma_0(\ell)\right),$$

for all $\alpha \geq 1$. For example,

$$\frac{1}{5^\alpha} L_{5,\alpha} \in \mathbb{Z}[t_5], \text{ with } t_5 := q\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6}.$$

- This gives us $p\left(5^\alpha n + \lambda_{5,\alpha}\right) \equiv 0 \pmod{5^\alpha}$.
- We prove $p\left(7^\alpha n + \lambda_{7,\alpha}\right) \equiv 0 \pmod{7^{\left\lfloor \frac{\alpha}{2} \right\rfloor + 1}}$ in a similar manner.
- We *cannot* prove $p\left(11^\alpha n + \lambda_{11,\alpha}\right) \equiv 0 \pmod{11^\alpha}$ in this manner.

Modular Curves

Definition

For any $N \in \mathbb{Z}_{\geq 1}$, we define the classical modular curve of level N as the set of all orbits of $\Gamma_0(N)$ applied to $\hat{\mathbb{H}}$:

$$\mathrm{X}_0(N) := \left\{[\tau]_N : \tau \in \hat{\mathbb{H}}\right\}$$

Definition

For each $N \geq 1$ there exists some $d \geq 1$ and orbits $[r_k]_N$, $0 \leq k \leq d-1$, such that

$$\mathbb{Q} \cup \{\infty\} = \bigsqcup_{k=0}^{d-1} [r_k]_N.$$

The orbits $[r_k]_N$ are the cusps of $\mathrm{X}_0(N)$.

Modular Curves

Theorem

For every $N \in \mathbb{Z}_{\geq 1}$, $\mathrm{X}_0(N)$ is a compact Riemann surface.

For each partition congruence family, we can associate a compact Riemann surface.

$$p\left(5^{\alpha} n + \lambda_{5,\alpha}\right) \equiv 0 \pmod{5^{\alpha}} \longrightarrow \mathrm{X}_0(5),$$
$$p\left(7^{\alpha} n + \lambda_{7,\alpha}\right) \equiv 0 \pmod{7^{\left\lfloor \frac{\alpha}{2} \right\rfloor + 1}} \longrightarrow \mathrm{X}_0(7),$$
$$p\left(11^{\alpha} n + \lambda_{11,\alpha}\right) \equiv 0 \pmod{11^{\alpha}} \longrightarrow \mathrm{X}_0(11).$$

These are the classical modular curves of level 5, 7, 11 (resp.).

Genus

The genus of a compact Riemann surface X, denoted $g(\mathrm{X})$, is the number of handles in the surface.

- $g(\mathrm{X}_0(1)) = 0$,
- $g(\mathrm{X}_0(5)) = 0$,
- $g(\mathrm{X}_0(7)) = 0$,
- $g(\mathrm{X}_0(11)) = 1$,
- $g(\mathrm{X}_0(20)) = 1$.

Why is this important?

Weierstrass Gap Theorem

Theorem

Let X *be a compact Riemann surface, and let*

$$f : \mathrm{X} \to \mathbb{C}$$

be holomorphic over X*, except for a pole at a point* $p \in \mathrm{X}$*. Then the order of* f *at* p *can assume any negative integer, with exactly* $g(\mathrm{X})$ *exceptions.*

Weierstrass Gap Theorem

Because $g\left(\mathrm{X}_0(5)\right) = 0$, the set of all functions

$$\hat{f} : \mathrm{X}_0(5) \longrightarrow \mathbb{C}$$

which are holomorphic everywhere along $\mathrm{X}_0(5)$ except the cusp $[0]_5$ has the form $\mathbb{C}[\hat{t}]$. This suggests that there exists a function t such that

$$\mathcal{M}^0\left(\Gamma_0(5)\right) = \mathbb{C}[t].$$

On the other hand, $g\left(\mathrm{X}_0(20)\right) = 1$ suggests that there exist functions t, ρ such that

$$\mathcal{M}^0\left(\Gamma_0(20)\right) = \mathbb{C}[t] \oplus \rho\mathbb{C}[t].$$

Modular Curves

For each partition congruence family, we can associate a compact Riemann surface.

$$p\left(5^{\alpha}n+\lambda_{5,\alpha}\right)\equiv 0 \pmod{5^{\alpha}} \longrightarrow \mathrm{X}_0(5),$$
$$p\left(7^{\alpha}n+\lambda_{7,\alpha}\right)\equiv 0 \pmod{7^{\left\lfloor\frac{\alpha}{2}\right\rfloor+1}} \longrightarrow \mathrm{X}_0(7),$$
$$p\left(11^{\alpha}n+\lambda_{11,\alpha}\right)\equiv 0 \pmod{11^{\alpha}} \longrightarrow \mathrm{X}_0(11).$$

$c(n)$

Let $q = e^{2\pi i\tau}$, with $\tau \in \mathbb{H}$. Define $E_2(\tau)$ by

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}.$$

Definition

Define $c(n)$ with the generating function

$$\sum_{n=0}^{\infty} c(n)q^n := \frac{(2\cdot E_2(2\tau) - E_2(\tau))}{(q^2;q^2)_\infty}.$$

Congruences on $c(n)$

Theorem (Wang, Yang)

Let $n \geq 0$, $\alpha \geq 1$, *and* $12\delta_\alpha \equiv 1 \pmod{5^\alpha}$. *Then*

$$c\left(5^\alpha n + \delta_\alpha\right) \equiv 0 \pmod{5^\alpha},$$

Congruences on $\mathrm{spt}_\omega(n)$

Theorem (Wang, Yang)

Let $n \geq 0$, $\alpha \geq 1$, *and* $12\delta_\alpha \equiv 1 \pmod{5^\alpha}$. *Then*

$$\mathrm{spt}_\omega\left(2 \cdot 5^\alpha n + \delta_\alpha\right) \equiv 0 \pmod{5^\alpha},$$

with $\omega(q)$ *defined as Ramanujan's third order mock theta function.*

Proof in Terms of L_α

Let $\alpha \in \mathbb{Z}_{\geq 1}$, and define

$$L_\alpha := \Phi_\alpha(q) \sum_{n=0}^{\infty} c\left(5^\alpha n + \delta_\alpha\right) q^{n+1},$$

with $\Phi_\alpha \in \mathbb{Z}[[q]]$.

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with $\Phi_\alpha \in \mathbb{Z}[[q]]$. The game is to show that $L_\alpha \equiv 0 \pmod{5^\alpha}$.

Technique by Wang and Yang

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$$\frac{L_\alpha}{5^\alpha} \in \mathbb{Z}[t] \oplus \rho\mathbb{Z}[t].$$

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$$\frac{L_\alpha}{5^\alpha} \in \mathbb{Z}[t] \oplus \rho\mathbb{Z}[t].$$

This is characteristic when the associated modular curve has genus 1. However, the genus of $\mathrm{X}_0(10)$ is 0.

Comparison of Expressions for L_1

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$$L_1 = \frac{1}{(1+5y)^3} \cdot \left(120y + 1805y^2 + 12050y^3 + 39500y^4 + 50000y^5\right),$$

for a certain eta quotient y with an integer power series expansion over $\Gamma_0(10)$.

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$$(1+5y)^3 L_1 = 120y + 1805y^2 + 12050y^3 + 39500y^4 + 50000y^5.$$

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Similar identities hold for L_2, L_3, etc.

Weak Result

We let $\alpha \geq 1$,

$$\mathcal{S} := \{(1+5y)^n : n \in \mathbb{Z}_{\geq 0}\},$$
$$\mathbb{Z}[y]_{\mathcal{S}} := \text{ the localization of } \mathbb{Z}[y] \text{ at } \mathcal{S}.$$

Then we have the following:

Theorem (Me!)

$$\frac{1}{5^\alpha} \cdot L_\alpha \in \mathbb{Z}[y]_{\mathcal{S}}.$$

Strong Result

We let $\alpha \geq 1$,

$$\psi(\alpha) := \left\lfloor \frac{5^{\alpha+1}}{12} \right\rfloor + 1.$$

Then we have the following:

Theorem (Me!)

$$\frac{(1+5y)^{\psi(\alpha)}}{5^{\alpha}} \cdot L_{\alpha} \in \mathbb{Z}[y].$$

N. Smoot, "A Single-Variable Proof of the Omega SPT Congruence Family Over Powers of 5," (Submitted).

L_α

$$L_1 = \frac{1}{(1+5y)^3} \cdot \left(120y + 1805y^2 + 12050y^3 + 39500y^4 + 50000y^5\right).$$

We will prove that for all $\alpha \in \mathbb{Z}_{\geq 1}$ we have

$$\frac{1}{5^\alpha} \cdot L_\alpha = \sum_{m \geq 1} s(m) \cdot 5^{\theta(m)} \cdot \frac{y^m}{(1+5y)^n},$$

with $n \in \mathbb{Z}_{\geq 1}$ fixed, s, θ integer-valued functions, and s discrete.

U Operator

We can define linear operators $U^{(0)}$, $U^{(1)}$ such that

$$L_{\alpha+1} = U^{(i)}\left(L_{\alpha}\right),$$

for $i \equiv \alpha \pmod{2}$.

U Operator

$$\frac{1}{5^\alpha} \cdot L_\alpha = \sum_{m \geq 1} s(m) \cdot 5^{\theta(m)} \cdot \frac{y^m}{(1+5y)^n},$$

We study

$$U^{(i)} \left(\frac{y^m}{(1+5y)^n} \right).$$

General Relation

Theorem

There exist discrete arrays $h_1, h_0 : \mathbb{Z}^3 \to \mathbb{Z}$ *and functions* $\pi_i : \mathbb{Z}^2_{\geq 1} \to \mathbb{Z}_{\geq 0}$ *such that*

$$U^{(1)}\left(\frac{y^m}{(1+5y)^n}\right) = \frac{1}{(1+5y)^{5n-4}} \sum_{r \geq \lceil m/5 \rceil} h_1(m,n,r) \cdot 5^{\pi_1(m,r)} \cdot y^r,$$

$$U^{(0)}\left(\frac{y^m}{(1+5y)^n}\right) = \frac{1}{(1+5y)^{5n-2}} \sum_{r \geq \lceil (m+2)/5 \rceil} h_0(m,n,r) \cdot 5^{\pi_0(m,r)} \cdot y^r.$$

General Relation

$$\pi_1(m,r) := \begin{cases} 0, & 1 \leq m \leq 2 \text{ and } r = 1 \\ 3, & 1 \leq m \leq 2 \text{ and } r = 3 \\ \left\lfloor \frac{5r+1}{6} \right\rfloor, & 1 \leq m \leq 2 \text{ and } r \neq 1,3 \\ 2, & m = 3 \text{ and } r = 2 \\ \left\lfloor \frac{5r-2}{6} \right\rfloor, & m = 3 \text{ and } r \neq 2 \\ \left\lfloor \frac{5r-m+1}{6} \right\rfloor, & m \geq 4, \end{cases}$$

$$\pi_0(m,r) := \begin{cases} \left\lfloor \frac{5r+1}{6} \right\rfloor, & m = 1 \\ \left\lfloor \frac{5r+1}{6} \right\rfloor, & m = 2 \text{ and } r \neq 3,4,5 \\ \left\lfloor \frac{5r-5}{6} \right\rfloor, & m = 2 \text{ and } 3 \leq r \leq 5 \\ \left\lfloor \frac{5r-m-2}{6} \right\rfloor, & m \geq 3. \end{cases}$$

Proof Strategy

$$\mathcal{V}_n^{(1)} := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot y^m : s \text{ is discreet} \right\},$$

$$\mathcal{V}_n^{(0)} := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot y^m : s \text{ is discreet} \right\}.$$

$$\theta_1(m) := \begin{cases} \left\lfloor \frac{5m-5}{6} \right\rfloor, & 1\leq m\leq 2 \\ \left\lfloor \frac{5m-5}{6} \right\rfloor - 1, & m\geq 3, \end{cases}$$

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Show that $\frac{1}{5}L_1 \in \mathcal{V}_3^{(1)}$,

Show that for any $f \in \mathcal{V}_n^{(1)}$, $\frac{1}{5}U^{(1)}(f) \in \mathcal{V}_{5n-4}^{(0)}$,

Show that for any $f \in \mathcal{V}_n^{(0)}$, $\frac{1}{5}U^{(0)}(f) \in \mathcal{V}_{5n-2}^{(1)}$.

Even-to-Odd Index

Let $f \in \mathcal{V}_n^{(0)}$. Then

$$\begin{aligned}
U^{(0)}(f) =& U^{(0)}\left(\frac{1}{(1+5y)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot y^m\right)\\
=& \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot U^{(0)}\left(\frac{y^m}{(1+5y)^n}\right)\\
=& \frac{1}{(1+5y)^{5n-2}}\sum_{m\geq 1}\sum_{r\geq\lceil (m+2)/5\rceil} s(m)\cdot h_0(m,n,r)\cdot 5^{\theta_0(m)+\pi_0(m,r)}\cdot y^r\\
=& \frac{1}{(1+5y)^{5n-2}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_0(m,n,r)\cdot 5^{\theta_0(m)+\pi_0(m,r)}\cdot y^r
\end{aligned}$$

We want to show that

$$\theta_0(m)+\pi_0(m,r) \geq \theta_1(r)+1 \text{ for all } r\geq 1,$$
$$\text{so that } \frac{1}{5}U^{(0)}(f) \in \mathcal{V}_{5n-2}^{(1)}.$$

Odd-to-Even Index

Let $f \in \mathcal{V}_n^{(1)}$. Then

$$\begin{aligned} U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+5y)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot y^m\right)\\ =& \sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot U^{(1)}\left(\frac{y^m}{(1+5y)^n}\right)\\ =& \frac{1}{(1+5y)^{5n-4}}\sum_{m\geq 1}\sum_{r\geq\lceil m/5\rceil} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot y^r\\ =& \frac{1}{(1+5y)^{5n-4}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot y^r \end{aligned}$$

We want to show that

$$\theta_1(m)+\pi_1(m,r)\geq\theta_0(r)+1 \text{ for all } r\geq 1,$$
$$\text{so that } \frac{1}{5}U^{(1)}(f)\in\mathcal{V}_{5n-4}^{(0)}.$$

5-adic Irregularity

We are going to prove that

$$\theta_0(m) + \pi_0(m, r) \geq \theta_1(r) + 1 \text{ for all } r \geq 1,$$
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No we aren't.

$$\theta_0(m) + \pi_0(m, r) \geq \theta_1(r) + 1 \text{ for all } r \geq 1 \text{ is true.}$$
$$\theta_1(m) + \pi_1(m, r) \geq \theta_0(r) + 1, \text{ on the other hand...}$$

5-adic Irregularity

Let $f \in \mathcal{V}_n^{(1)}$. Then

$$\begin{aligned} U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+5y)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot y^m\right) \\ =& \sum_{m\geq 1} s(m)\cdot 5^{\theta_1(m)}\cdot U^{(1)}\left(\frac{y^m}{(1+5y)^n}\right) \\ =& \frac{1}{(1+5y)^{5n-4}}\sum_{m\geq 1}\sum_{r\geq\lceil m/5\rceil} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot y^r \\ =& \frac{1}{(1+5y)^{5n-4}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot y^r \end{aligned}$$

The coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ is

$$\sum_{m=1}^{5} s(m)\cdot h_1(m,n,1)\cdot 5^{\theta_1(m)+\pi_1(m,1)}.$$

5-adic Irregularity

The coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ has the form

$$= \sum_{m=1}^{5} s(m) \cdot h_1(m,n,1) \cdot 5^{\theta_1(m)+\pi_1(m,1)}$$

$$= \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) + s(4) \cdot h_1(4,n,1) \cdot 5 + s(5) \cdot h_1(5,n,1) \cdot 5^2$$

$$\equiv \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod{5}.$$

5-adic Irregularity

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$$\equiv \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod{5}.$$

$\theta_1(m) + \pi_1(m,r) \geq \theta_0(r) + 1$ cannot be true for $r = 1$.

5-adic Irregularity

Lemma

For all m, n such that $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq m \leq 3$ we have:

$$\begin{aligned}
h_0(1, n, 1) &\equiv 1 \pmod 5,\\
h_0(2, 5n-4, 1) &\equiv 0 \pmod 5,\\
h_0(3, n, 1) &\equiv 1 \pmod 5,\\
h_0(1, n, 2) &\equiv 4 \pmod 5,\\
h_0(2, 5n-4, 2) &\equiv 4 \pmod 5,\\
h_0(3, n, 2) &\equiv 4 \pmod 5,\\
h_0(2, 5n-4, 3) &\equiv 1 \pmod 5,\\
h_1(m, n, 1) &\equiv 1 \pmod 5.
\end{aligned}$$

5-adic Irregularity

Our coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ for $U^{(1)}(f)$ is

$$\equiv \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod{5}$$

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5-adic Irregularity

Our coefficient of $\frac{y^1}{(1+5y)^{5n-4}}$ for $U^{(1)}(f)$ is

$$\equiv \sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod 5$$
$$\equiv \sum_{m=1}^{3} s(m) \pmod 5.$$

Examine L_1:

$$L_1 = \frac{5}{(1+5y)^3} \cdot \left(24y + 361y^2 + 2410y^3 + 7900y^4 + 10000y^5\right).$$

5-adic Irregularity

Definition

$$\hat{\mathcal{V}}_n^{(1)} := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_1(m)} \cdot y^m : \sum_{m=1}^{3} s(m) \equiv 0 \bmod 5 \right\},$$

$$\mathcal{V}_n^{(0)} := \left\{ \frac{1}{(1+5y)^n} \sum_{m\geq 1} s(m) \cdot 5^{\theta_0(m)} \cdot y^m \right\}.$$

Here s again represents a discrete integer-valued function.

Resolving 5-adic Irregularity

Theorem

Suppose $f \in \hat{\mathcal{V}}_n^{(1)}$. *Then*

$$\frac{1}{5} \cdot U^{(1)}(f) \in \mathcal{V}_{5n-4}^{(0)},$$
$$\frac{1}{5^2} \cdot U^{(0)} \circ U^{(1)}(f) \in \hat{\mathcal{V}}_{25n-22}^{(1)}.$$

Sketch

Let $f \in \hat{\mathcal{V}}_n^{(1)}$. Then

$$\frac{1}{5^2} \cdot \left(U^{(0)} \circ U^{(1)} (f) \right) = \frac{1}{(1+5y)^{25n-22}} \sum_{w \geq 1} t(w) \cdot 5^{\theta_1(w)} y^w,$$

$$t(w) = \sum_{r=1}^{5w-2} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, w) \times 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_1(w)-2}.$$

Sketch

$$t(1) = \sum_{r=1}^{3} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, 1) \cdot 5^{\lambda(m,r,1)},$$

$$t(2) = \sum_{r=1}^{8} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, 2) \cdot 5^{\lambda(m,r,2)},$$

$$t(3) = \sum_{r=1}^{13} \sum_{m=1}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r, 5n-4, 3) \cdot 5^{\lambda(m,r,3)},$$

$$\lambda(m,r,w) := \theta_1(m) + \pi_1(m,r) + \pi_0(r,w) - 2.$$

We want to show that $t(1), t(2), t(3) \in \mathbb{Z}$, and that $t(1) + t(2) + t(3) \equiv 0 \pmod{5}$.

Sketch

$$t(1)+t(2)+t(3) \equiv \frac{1}{5}\cdot\left(\sum_{j=1}^{2} h_0(1,5n-4,j)\right)\cdot\left(\sum_{m=1}^{3} s(m)\cdot h_1(m,n,1)\right)$$
$$+h_0(1,5n-4,3)\cdot\left(\sum_{m=1}^{3} s(m)\cdot h_1(m,n,1)\right)$$
$$+\left(\sum_{j=1}^{2} h_0(1,5n-4,j)\right)\cdot s(4)\cdot h_1(4,n,1)$$
$$+\left(\sum_{j=1}^{3} h_0(2,5n-4,j)\right)\cdot\sum_{m=1}^{2} s(m)\cdot h_1(m,n,2)$$
$$+\left(\sum_{j=1}^{2} h_0(3,5n-4,j)\right)\cdot s(3)\cdot h_1(3,n,3) \pmod 5.$$

Sketch

It's That Lemma Again

For all m, n such that $n \in \mathbb{Z}_{\geq 1}$ and $1 \leq m \leq 3$ we have:

$$\begin{aligned} h_0(1, n, 1) &\equiv 1 \pmod 5, \\ h_0(2, 5n-4, 1) &\equiv 0 \pmod 5, \\ h_0(3, n, 1) &\equiv 1 \pmod 5, \\ h_0(1, n, 2) &\equiv 4 \pmod 5, \\ h_0(2, 5n-4, 2) &\equiv 4 \pmod 5, \\ h_0(3, n, 2) &\equiv 4 \pmod 5, \\ h_0(2, 5n-4, 3) &\equiv 1 \pmod 5, \\ h_1(m, n, 1) &\equiv 1 \pmod 5. \end{aligned}$$

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Therefore, $t(1) + t(2) + t(3) \equiv 0 \pmod 5$. $\square$

Proof of our Strong Result

Corollary [Me!]

For all $\alpha \in \mathbb{Z}_{\geq 1}$,

$$\frac{(1+5y)^{\psi(\alpha)}}{5^{\alpha}} \cdot L_{\alpha} \in \mathbb{Z}[y].$$

The Andrews–Sellers Congruences

Let

$$\mathrm{C}\Phi_2(q) := \sum_{n=0}^{\infty} c\phi_2(n)q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^4 (q^4;q^4)_\infty^2}.$$

Theorem

For all $\alpha \geq 1$ and $n \geq 0$,

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha},$$

with λ_α the minimum positive solution to $12x \equiv 1 \pmod{5^\alpha}$.

This was conjectured by James Sellers in 1994, and proved by Peter Paule and Cristian-Silviu Radu in 2012.

The Andrews–Sellers Congruences

$$L_{2\alpha-1} := \frac{q}{\mathrm{C}\Phi_2(q^5)} \sum_{n=0}^{\infty} c\phi_2\left(5^{2\alpha-1}n + \lambda_{2\alpha-1}\right) q^n$$

$$L_{2\alpha} := \frac{q}{\mathrm{C}\Phi_2(q)} \sum_{n=0}^{\infty} c\phi_2\left(5^{2\alpha}n + \lambda_{2\alpha}\right) q^n,$$

Theorem (Paule, Radu)

There exist modular functions $t, p_1, p_0 \in \mathcal{M}\left(\Gamma_0(20)\right)$ *with integer coefficients such that*

$$\frac{1}{5^{2\alpha-1}} L_{2\alpha-1} \in \mathbb{Z}[t] \oplus p_1\mathbb{Z}[t],$$

$$\frac{1}{5^{\alpha}} L_{2\alpha} \in \mathbb{Z}[t] \oplus p_0\mathbb{Z}[t].$$

...Why?

The Andrews–Sellers Congruences

There exist functions $f \in \mathcal{M}^0\left(\Gamma_0(20)\right)$ and linear operators $U^{(0)}, U^{(1)}$ with the following property:

$$\left(U^{(0)} \circ U^{(1)}\right)^n (f) \equiv f \pmod{5}.$$

for some $n \in \mathbb{Z}_{\geq 1}$.

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P. Paule, S. Radu, "The Andrews–Sellers Family of Partition Congruences," *Advances in Mathematics* 230, pp. 819-838 (2012).

- This is a problem in proving that $L_\alpha \equiv 0 \pmod{5^\alpha}$ for $(L_\alpha)_{\geq 0} \subseteq \mathcal{M}\left(\Gamma_0(20)\right)$.

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- This is a problem in proving that $L_\alpha \equiv 0 \pmod{5^\alpha}$ for $(L_\alpha)_{\geq 0} \subseteq \mathcal{M}\left(\Gamma_0(20)\right)$.
- Our problem is in $\mathcal{M}\left(\Gamma_0(10)\right)$.

An Analogy

Suppose you want to prove that

$$\lim_{\alpha \to \infty} L_\alpha = 0$$

for some sequnce of functions $(L_\alpha)_{\alpha \geq 0}$.

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for some sequnce of functions $(L_\alpha)_{\alpha \geq 0}$. If

$$L_\alpha = f_\alpha + g_\alpha,$$

then it is sufficient (but not necessary!) to prove that

$$\lim_{\alpha \to \infty} f_\alpha = \lim_{\alpha \to \infty} g_\alpha = 0.$$

Summary

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Summary

- Our proof reveals some interesting algebraic structure in the form of the localized polynomial ring.
- There are some extremely difficult steps in showing that going from L_α to $L_{\alpha+1}$ *always* picks up an extra power of 5.
- The steps needed to overcome these difficulties may be useful for other problems, especially for congruences associated with genus 1 Riemann surfaces.

To be continued...

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