Localization Applied to a Genus 1 Congruence Family

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Specialty Seminar in Partition Theory, q-Series and Related Topics

16 March, 2023

RISC RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION

JOHANNES KEPLER UNIVERSITY LINZ

FWF Der Wissenschaftsfonds.

Partitions

Definition

For any $n \in \mathbb{Z}_{\geq 0}$, a partition of n is a representation of n as a sum of other positive intergers, called parts. The number of partitions of a given n is denoted $p(n)$.

For example, $p(4) = 5$:

- 4
- $3 + 1$
- $2 + 2$
- $2 + 1 + 1$
- $1 + 1 + 1 + 1$

Partitions

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

The sequence for $p(n)$ begins

$$(p(n))_{n\geq 0} = (1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, \ldots)$$

Ramanujan's Congruences

$$(p(n))_{n\geq 0} = \big(1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135,\\ 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, \ldots\big)$$

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- $p(25n+24) \equiv 0 \pmod{25}$.

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- $p(5n+4) \equiv 0 \pmod{5}$.
- $p(25n+24) \equiv 0 \pmod{25}$.
- $p(125n+99) \equiv 0 \pmod{125}$.

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&176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, \ldots)
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- $p(5n+4) \equiv 0 \pmod{5}$.
- $p(25n+24) \equiv 0 \pmod{25}$.
- $p(125n+99) \equiv 0 \pmod{125}$.

Theorem (Ramanujan, 1918)

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$ such that $24n \equiv 1 \pmod{5^\alpha}$. Then

$$p(n) \equiv 0 \pmod{5^\alpha}.$$

Ramanujan's Congruences

Theorem (Ramanujan, Watson, Atkin)

Let $\ell \in \{5, 7, 11\}$ *and* $n, \alpha \in \mathbb{Z}_{\geq 0}$ *such that* $24n \equiv 1 \pmod{\ell^{\alpha}}$. *Then*

$$p(n) \equiv 0 \pmod{\ell^{\beta}},$$
$$\beta = \begin{cases} \alpha, & \ell \in \{5, 11\}, \\ \lfloor \alpha/2 \rfloor + 1, & \ell = 7. \end{cases}$$

Similar Congruences

$$\text{Suppose } \sum_{n=0}^{\infty} a(n)q^n = \mathcal{G},$$

with $\mathcal{G}$ usually a modular form or similar. A *congruence family* for $a(n)$ modulo powers of a prime ℓ is a set of divisibilities

$$a(n) \equiv 0 \pmod{\ell^\beta} \text{ when } \Lambda n \equiv 1 \pmod{\ell^\alpha},$$

with $\Lambda \in \mathbb{Z}$ fixed and

$$\beta \to \infty \text{ as } \alpha \to \infty.$$

j Invariant

$$G_k(\tau) := \sum_{\substack{(m,n)\in\mathbb{Z}^2,\\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k},$$

$$\Delta(\tau) := (2\pi)^{12} q \prod_{m=1}^{\infty} (1-q^m)^{24}.$$

With these functions, we may define the modular j invariant:

$$j := j(\tau) = 1728\frac{60^3 G_4(\tau)^3}{\Delta(\tau)} = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n.$$

j Invariant

Theorem (Lehner, Atkin)

For $\alpha \geq 1$ and $n \geq 0$, the following apply:

$$\begin{aligned}
\text{If } n &\equiv 0 \pmod{2^\alpha}, \text{ then } c(n) \equiv 0 \pmod{2^{3\alpha+8}};\\
\text{If } n &\equiv 0 \pmod{3^\alpha}, \text{ then } c(n) \equiv 0 \pmod{3^{2\alpha+3}};\\
\text{If } n &\equiv 0 \pmod{5^\alpha}, \text{ then } c(n) \equiv 0 \pmod{5^{\alpha+1}};\\
\text{If } n &\equiv 0 \pmod{7^\alpha}, \text{ then } c(n) \equiv 0 \pmod{7^{\alpha}};\\
\text{If } n &\equiv 0 \pmod{11^\alpha}, \text{ then } c(n) \equiv 0 \pmod{11^{\alpha}}.
\end{aligned}$$

k-Colored Partitions

$$P_k(\tau) := \sum_{n=0}^{\infty} p_k(n) q^n = \prod_{m=1}^{\infty} (1 - q^m)^{-k}.$$

Theorem (D. Tang)

For $\alpha \geq 1$ and $n \geq 0$, the following applies:

$$\textit{If } 12n \equiv 1 \pmod{5^{\alpha}}, \textit{ then } p_2(n) \equiv 0 \pmod{5^{\lfloor \alpha/2 \rfloor + 1}}.$$

A Smallest Parts Function Related to Mock Theta Functions

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)^2_{n+1}}, \qquad \sum_{n=1}^{\infty} p_\omega(n)q^n = q\omega(q),$$

where $p_\omega(n)$ as the counting function of partitions of n in which all odd parts are less than twice the smallest part.

Theorem (Wang, Yang)

Let $\lambda_\alpha \in \mathbb{Z}$ be the minimal positive solution to $12x \equiv 1 \pmod{5^\alpha}$. Then

$$\mathrm{spt}_\omega(2\cdot 5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}.$$

Generalized 2-Color Frobenius Partitions

$$\mathrm{C}\Phi_2(\tau) := \sum_{n=0}^{\infty} c\phi_2(n)q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^5}{(1-q^m)^4(1-q^{4m})^2}.$$

Theorem (Paule, Radu)

For all integers $n \geq 0$, $\alpha \geq 1$, such that $12n \equiv 1 \pmod{5^\alpha}$, we have

$$c\phi_2(n) \equiv 0 \pmod{5^\alpha}.$$

2-Elongated Plane Partition Diamonds

$$D_k(\tau) := \sum_{n=0}^{\infty} d_k(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^k}{(1-q^m)^{3k+1}}.$$

Theorem (Smoot)

For all integers $n \geq 0$, $\alpha \geq 1$, such that $8n \equiv 1 \pmod{3^\alpha}$, we have

$$d_2(n) \equiv 0 \pmod{3^{2\lfloor \alpha/2 \rfloor + 1}}.$$

Theorem (Sellers, Smoot)

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ such that $3n \equiv 1 \pmod{4^\alpha}$. Then

$$d_7(n) \equiv 0 \pmod{8^\alpha}.$$

Why?

Theory of Partition Congruences

If $\Lambda n \equiv 1 \pmod{\ell^{\alpha}}$, then $a(n) \equiv 0 \pmod{\ell^{\beta}}$.

$$L_{\alpha} := \phi_{\alpha} \cdot \sum_{\substack{n \geq 0 \\ \Lambda n \equiv 1 \bmod \ell^{\alpha}}} a(n)\, q^{\lfloor n/\ell^{\alpha} \rfloor + \delta} \in \mathcal{M}\left(\mathrm{X}_0(N)\right),$$

$$q := e^{2\pi i \tau}, \tau \in \mathbb{H}.$$

$(L_{\alpha})_{\alpha \geq 1}$ is a sequence of meromorphic functions on the classical modular curve $\mathrm{X}_0(N)$.

Theory of Partition Congruences

$\mathrm{X}_0(N)$ is a compact Riemann surface, diffeomorphic to a 2 dimensional $\mathcal{C}^{\infty}$ real manifold. The two key topological properties important to us are:

- The genus $\mathfrak{g}$;
- The cusp count ϵ_{∞}.

Theory of Partition Congruences

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- The genus $\mathfrak{g}$;
- The cusp count ϵ_∞.
- $\mathfrak{g} \geq 0$.
- $\epsilon_\infty = 1, 3$ for $N = 1, 4$ (respectively).
- $\epsilon_\infty \geq 2$ and even for all other N.

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e.g., Beazer's conjectures

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- For $\mathfrak{g} = 1$, $\epsilon_\infty = 4$...
- Localization currently fails for $\mathfrak{g} = 1$, $\epsilon_\infty = 6$, but a modification is likely possible

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What about $\mathfrak{g} = 1$, $\epsilon_\infty = 4$?

$\mathfrak{g} = 1$

- For $\mathfrak{g} = 1$, $\epsilon_\infty = 4$... localization succeeds!

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The critical number is ϵ_∞.

Classification

$\epsilon_\infty\left(\mathrm{X}_0(N)\right)$	$\mathfrak{g}\left(\mathrm{X}_0(N)\right)=0$	$\mathfrak{g}\left(\mathrm{X}_0(N)\right)=1$	$\mathfrak{g}\left(\mathrm{X}_0(N)\right)=2$
2	Classical ($p(n)$ mod 5^α)	$p(n)$ mod 11^α	?
4	Localization (spt_ω mod 5^α, d_2 mod 3^α)	Localization (d_2 mod 7^α)	Localization (conjectured)
6	Localization (conjectured) (Beazer)	Localization (conjectured) (Andrews–Sellers)	?
≥ 8	?	?	?

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Counterexamples:

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- Sellers–Smoot: $d_7(n)$ mod 8^α over $\mathrm{X}_0(8)$ ($\epsilon_\infty = 4$, $\mathfrak{g} = 0$), localization unnecessary
- $N = 1$ ($\epsilon_\infty = 1$, $\mathfrak{g} = 0$)
- $N = 4$ ($\epsilon_\infty = 3$, $\mathfrak{g} = 0$)

Theory of Partition Congruences

$$L_\alpha := \phi_\alpha \cdot \sum_{\substack{n\geq 0\\ \Lambda n\equiv 1 \bmod \ell^\alpha}} a(n)\, q^{\lfloor n/\ell^\alpha \rfloor + \delta} \in \mathcal{M}\left(\mathrm{X}_0(N)\right).$$

We build a sequence of modified U_ℓ operators, $\left(U^{(\alpha)}\right)_{\alpha\geq 1}$ such that

$$U^{(\alpha)}\left(L_\alpha\right) = L_{\alpha+1}.$$

Theory of Partition Congruences

$$\frac{1}{\ell^\beta} L_\alpha := \sum_{m \geq 1} s_\alpha(m) \cdot \ell^{\theta_\alpha(m)} \cdot \varphi_\alpha(m).$$

- $s_\alpha(m) \in \mathbb{Z}$.
- $\varphi_\alpha(m)$ are some convenient reference functions.

Theory of Partition Congruences

$$\begin{aligned}\frac{1}{\ell^{\beta}} L_{\alpha+1} &= U^{(\alpha)}\left(\frac{1}{\ell^{\beta}} L_{\alpha}\right)\\ &= \sum_{m\geq 1} s_{\alpha}(m)\cdot \ell^{\theta_{\alpha}(m)}\cdot U^{(\alpha)}\left(\varphi_{\alpha}(m)\right).\end{aligned}$$

- Most $U^{(\alpha)}\left(\varphi_{\alpha}(m)\right)$ will gain a power of ℓ.
- For some exceptional m, this will *not* happen.
- The $\varphi_{\alpha}(m)$ must interact so that exceptional terms cancel out.

Stability Under the Congruence Kernel

$$\frac{1}{\ell^{\beta}} L_{\alpha} := \sum_{m\geq 1} s_{\alpha}(m)\cdot \ell^{\theta_{\alpha}(m)}\cdot \varphi_{\alpha}(m).$$

We construct a linear transformation

$$\Omega : \bigoplus_{m=1}^{\infty} \mathbb{Z} \longrightarrow \bigoplus_{k=1}^{R} \mathbb{Z}/\ell^{M}\mathbb{Z}.$$

We need to prove: $\mathbf{s}_1 := \left(s_1(m)\right)_{m\geq 1} \in \ker\left(\Omega\right),$

and $\mathbf{s}_{\alpha} \in \ker\left(\Omega\right) \implies \mathbf{s}_{\alpha+1} \in \ker\left(\Omega\right)$

- For $\epsilon_\infty = 2$ (classical case):

$$\Omega : \bigoplus_{m=1}^{\infty} \mathbb{Z} \longrightarrow \{0\}.$$

  Interaction of basis functions is important.
- For $\epsilon_\infty = 4$, Ω is nontrivial; interaction of basis functions must be accounted for.
- For $\epsilon_\infty = 6$, Ω is difficult to construct; complex interaction between basis functions.
- $\mathfrak{g}$ appears to have no affect on Ω.

$d_k(n)$: k-Elongated Plane Partitions of n

Define $D_k(q)$ by

$$D_k(q) := \sum_{n=0}^{\infty} d_k(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^k}{(1-q^m)^{3k+1}},$$

in which $d_k(n)$ counts the number of k-elongated plane partitions of n.

$d_k(n)$: k-Elongated Plane Partitions of n

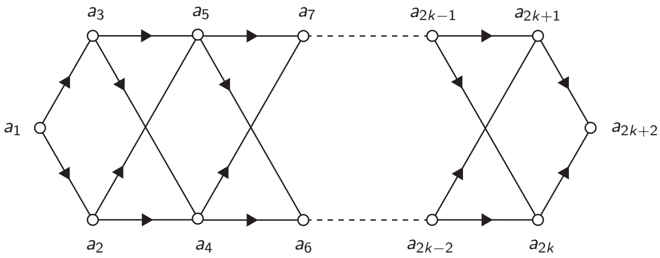

Figure: A length 1 k-elongated partition diamond.

- $a_j \in \mathbb{Z}_{\geq 0}$
- $a_b \to a_c$ indicates that $a_b \geq a_c$
- $a_1 + a_{2k+2} = n$

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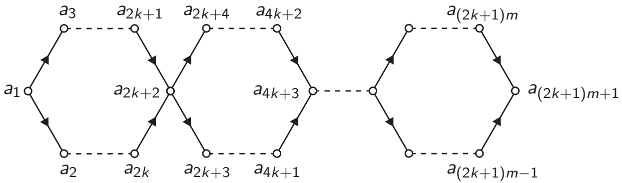

Figure: A length m k-elongated partition diamond.

- $a_j \in \mathbb{Z}_{\geq 0}$
- $a_b \to a_c$ indicates that $a_b \geq a_c$
- $a_1 + a_{2k+2} + ... + a_{(2k+1)m+1} = n$

Congruences on $d_2(n)$

This was conjectured by Koustav Banerjee:

Theorem (Banerjee, Smoot)

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ *such that* $8n \equiv 1 \pmod{7^{\alpha}}$. *Then*

$$d_2(n) \equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor}}.$$

Theorem (Banerjee, Smoot)

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ such that $8n \equiv 1 \pmod{7^{\alpha}}$. Then

$$d_2(n) \equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor}}.$$

- Associated Riemann survace is the modular curve $\mathrm{X}_0(14)$
- $\mathfrak{g}\left(\mathrm{X}_0(14)\right) = 1 > 0$, indicating really tedious work ahead.
- $\epsilon_\infty\left(\mathrm{X}_0(14)\right) = 4$.

$$L_{2\alpha-1}(\tau) = \frac{(q^7;q^7)_\infty^7}{(q^{14};q^{14})_\infty^2} \cdot \sum_{n=0}^{\infty} d_2(7^{2\alpha-1}n + \lambda_{2\alpha-1})q^{n+1},$$

$$L_{2\alpha}(\tau) = \frac{(q;q)_\infty^7}{(q^2;q^2)_\infty^2} \cdot \sum_{n=0}^{\infty} d_2(7^{2\alpha}n + \lambda_{2\alpha})q^{n+1},$$

$$
\begin{aligned}
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L_{2\alpha}(\tau) &= \frac{(q;q)_\infty^7}{(q^2;q^2)_\infty^2} \cdot \sum_{n=0}^{\infty} d_2(7^{2\alpha}n + \lambda_{2\alpha})q^{n+1},
\end{aligned}
$$

$$
\begin{aligned}
U^{(1)}\left(L_{2\alpha-1}\right) &= L_{2\alpha}, \\
U^{(0)}\left(L_{2\alpha}\right) &= L_{2\alpha+1}.
\end{aligned}
$$

First Example

$$L_1 = \frac{(q^7;q^7)^7}{(q^{14};q^{14})^2}\sum_{n=0}^{\infty} d_2(7n+1)q^{n+1}.$$

There should exist modular functions x, y with integer coefficients, such that

$$L_1 \in \mathbb{Z}[x] \oplus y\mathbb{Z}[x].$$

First Problem

$$L_1 = \frac{(q^7;q^7)^7_\infty}{(q^{14};q^{14})^2_\infty}\sum_{n=0}^{\infty} d_2(7n+1)q^{n+1}.$$

$$\begin{aligned} L_1 = & \frac{1}{(1+7x)^3}\Bigg(\frac{1}{7}\cdot 320013737x + \frac{1}{7}\cdot 29164229489x^2 + \frac{1}{7}\cdot 1226655768017x^3 + 4505536916704x^4 + 79044206825472x^5 \\ & + 999877459130368x^6 + 9391378522824704x^7 + 66411983644131328x^8 + 354409645379944448x^9 \\ & + 1415208166316048384x^{10} + 4140177110624894976x^{11} + 8532124891883765760x^{12} + 11539756946659737600x^{13} \\ & + 8913467434661314560x^{14} + 2773078757450186752x^{15} - \frac{1}{7}\cdot 320013688y - \frac{1}{7}\cdot 28844055074xy - 171156188528x^2y \\ & - 4337927987008x^3y - 74846829673728x^4y - 928384597776384x^5y - 8516830910414848x^6y - 58508210959679488x^7y \\ & - 300982634640572416x^8y - 1145123381897592832x^9y - 3131903931035156480x^{10}y - 5830893280174276608x^{11}y \\ & - 6623201496588615680x^{12}y - 3466348446812733440x^{13}y\Bigg) \end{aligned}$$

First Problem

$$L_1 = \frac{(q^7;q^7)^7}{(q^{14};q^{14})^2}\sum_{n=0}^{\infty} d_2(7n+1)q^{n+1}.$$

$$\begin{aligned}
L_1 =& \frac{1}{(1+7x)^3}\bigg(\frac{1}{7}\cdot 320013737x + \frac{1}{7}\cdot 29164229489x^2 + \frac{1}{7}\cdot 1226655768017x^3 + 4505536916704x^4 + 79044206825472x^5 \\
&+ 999877459130368x^6 + 9391378522824704x^7 + 66411983644131328x^8 + 354409645379944448x^9 \\
&+ 1415208166316048384x^{10} + 4140177110624894976x^{11} + 8532124891883765760x^{12} + 11539756946659737600x^{13} \\
&+ 8913467434661314560x^{14} + 2773078757450186752x^{15} - \frac{1}{7}\cdot 320013688y - \frac{1}{7}\cdot 28844055074xy - 171156188528x^2y \\
&- 4337927987008x^3y - 74846829673728x^4y - 928384597776384x^5y - 8516830910414848x^6y - 58508210959679488x^7y \\
&- 300982634640572416x^8y - 1145123381897592832x^9y - 3131903931035156480x^{10}y - 5830893280174276608x^{11}y \\
&- 6623201496588615680x^{12}y - 3466348446812733440x^{13}y\bigg)
\end{aligned}$$

$$x + 3x^2 + x^3 + 6y + 5xy \equiv 0 \pmod 7.$$

Second Problem

$$L_\alpha \in \mathbb{Q}[x] \oplus y\mathbb{Q}[x].$$

$$1+7x = \frac{(q^2;q^2)_\infty^7 (q^7;q^7)_\infty}{(q;q)_\infty^7 (q^{14};q^{14})_\infty}, \quad 1+8y = \frac{(q^2;q^2)_\infty^4 (q^7;q^7)_\infty^8}{(q;q)_\infty^8 (q^{14};q^{14})_\infty^4}$$

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$$x = ? \qquad\qquad y = ?$$

Setup

$$V_n^{(\alpha)} := \left\{ \frac{1}{(1+7x)^n} \sum_{\substack{0\le\beta\le1,\\ m\ge1-\beta}} s_\beta(m) 7^{\theta_\beta^{(\alpha)}(m)} y^\beta x^m \right\}$$

$$\theta_0^{(1)}(m) = \begin{cases} -1, & 1 \le m \le 3, \\ \left\lfloor \frac{7m-1}{9} \right\rfloor - 3 & m \ge 4 \end{cases}, \quad \theta_1^{(1)}(m) = \begin{cases} -1, & 0 \le m \le 1, \\ \left\lfloor \frac{7m-4}{9} \right\rfloor - 1 & m \ge 2 \end{cases}$$

$$\theta_0^{(0)}(m) = \begin{cases} 0, & 1 \le m \le 2, \\ \left\lfloor \frac{7m-7}{9} \right\rfloor - 1 & m \ge 3 \end{cases}, \quad \theta_1^{(0)}(m) = \begin{cases} 0 & 0 \le m \le 1, \\ \left\lfloor \frac{7m-5}{9} \right\rfloor & m \ge 2 \end{cases}$$

$$x + 3x^2 + x^3 + 6y + 5xy \equiv 0 \pmod{7}.$$

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$$\begin{aligned} s_0(1)\big(x &+ s_0(1)^{-1}s_0(2)x^2 + s_0(1)^{-1}s_0(3)x^3 \\ &+ s_0(1)^{-1}s_1(0)y + s_0(1)^{-1}s_1(1)xy\big) \equiv 0 \pmod 7. \end{aligned}$$

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$$\begin{aligned} s_0(2) &\equiv 3s_0(1) \pmod 7, & s_0(3) &\equiv s_0(1) \pmod 7, \\ s_1(0) &\equiv 6s_0(1) \pmod 7, & s_1(1) &\equiv 5s_0(1) \pmod 7. \end{aligned}$$

Congruence Kernel

$$V_n^{(0)} := \left\{ \frac{1}{(1+7x)^n} \sum_{\substack{0\leq\beta\leq 1,\\ m\geq 1-\beta}} s_\beta(m) 7^{\theta_\beta^{(0)}(m)} y^\beta x^m \right\}$$

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$$\Omega.\mathbf{v} := \begin{pmatrix} 43 & 48 & 15 & 42 & 7 & 8 & 27 & 7 & 42 \\ 0 & 7 & 0 & 0 & 0 & 0 & 35 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 28 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 21 & 0 & 0 \end{pmatrix}.\mathbf{v} \in \bigoplus_{k=1}^{4} \mathbb{Z}/49\mathbb{Z}$$

General Relation

Theorem

For $m, n \in \mathbb{Z}_{\geq 1}$ and $\alpha, \beta, \gamma \in \{0, 1\}$, there exists discrete arrays $h_{\beta\gamma}^{(\alpha)} : \mathbb{Z}^3 \to \mathbb{Z}$ and functions $\pi_{\beta\gamma}^{(\alpha)} : \mathbb{Z}^2 \to \mathbb{Z}_{\geq -1}$ such that

$$U^{(\alpha)}\left(\frac{y^\beta x^m}{(1+7x)^n}\right) = \frac{1}{(1+7x)^{7n+3(1-\alpha)}} \sum_{\substack{0\leq\gamma\leq 1,\\ r\geq 1-\gamma}} h_{\beta\gamma}^{(\alpha)}(m,n,r) 7^{\pi_{\beta\gamma}^{(\alpha)}(m,r)} y^\gamma x^r.$$

Moreover, we have

$$h_{\beta\gamma}^{(\alpha)}(m,n,r) \equiv h_{\beta\gamma}^{(\alpha)}(m,n-7,r) \pmod{49}.$$

Outline

$$L_1 \in V_3^{(1)}.$$

We have to prove the following:

- $$\text{If } f \in V_n^{(1)} \text{ with } n \equiv 3 \pmod 7, \text{then } \frac{1}{7} U^{(1)}(f) \in V_{7n}^{(0)}.$$
- $$\text{If } f \in V_n^{(1)} \text{ with } n \equiv 3 \pmod 7, \text{then } \frac{1}{7} U^{(0)} \circ U^{(1)}(f) \in V_{49n+3}^{(1)}.$$

Outline

We let $f \in V_n^{(1)}$.

$$f = \frac{1}{(1+7x)^n} \sum_{\substack{0\leq\beta\leq 1,\\ m\geq 1-\beta}} s_\beta(m) 7^{\theta_\beta^{(1)}(m)} y^\beta x^m.$$

$$U^{(1)}(f) = \frac{1}{(1+7x)^{7n}} \sum_{\substack{0\leq\beta\leq 1,\\ m\geq 1-\beta,\\ 0\leq\gamma\leq 1,\\ r\geq 1-\gamma}} s_\beta(m) h_{\beta\gamma}^{(1)}(m,n,r) 7^{\theta_\beta^{(1)}(m)+\pi_{\beta\gamma}^{(\alpha)}(m,r)} y^\gamma x^r.$$

We need to determine when

$$\theta_\beta^{(1)}(m) + \pi_{\beta\gamma}^{(1)}(m,r) \geq \theta_\beta^{(0)}(r) + 1$$

Outline

$$U^{(1)}(f) = \frac{1}{(1+7x)^{7n}} \sum_{\substack{0\leq\beta\leq 1,\\ m\geq 1-\beta,\\ 0\leq\gamma\leq 1,\\ r\geq 1-\gamma}} s_\beta(m) h_{\beta\gamma}^{(1)}(m,n,r) 7^{\theta_\beta^{(1)}(m)+\pi_{\beta\gamma}^{(\alpha)}(m,r)} y^\gamma x^r.$$

Recall:

$$s_\beta(m) h_{\beta\gamma}^{(1)}(m,n,r) \equiv s_\beta(m) h_{\beta\gamma}^{(1)}(m,3,r) \pmod{49}.$$

Mathematica!

$$\frac{1}{7}U^{(0)} \circ U^{(1)}(f) = \frac{1}{(1+7x)^{49n+3}} \sum_{\substack{0 \le \delta \le 1, \\ w \ge 1-\delta,}} q_\delta(w) 7^{\theta_\delta^{(1)}(w)} y^\delta x^w.$$

We need to check that

$$\mathbf{q} := \left(q_0(1), q_0(2), q_0(3), q_0(4), q_0(5), q_1(0), q_1(1), q_1(2), q_1(3)\right),$$
$$\mathbf{q} \in \ker\left(\Omega\right).$$

Main Theorem

Theorem

$$\text{Let } \psi := \psi(\alpha) = \left\lfloor \frac{7^{\alpha+1}}{16} \right\rfloor, \beta := \beta(\alpha) = \left\lfloor \frac{\alpha}{2} \right\rfloor,$$
$$\text{and } r_L := \frac{1}{7}\left(x + 3x^2 + x^3 + 6y + 5xy\right).$$

There exists an integer sequence $(k_\alpha)_{\alpha\geq 1}$ *such that, for all* $\alpha \geq 1$,

$$\frac{(1+7x)^{\psi}}{7^{\beta}} \cdot L_\alpha + k_\alpha \cdot r_L \in \mathbb{Z}[x] \oplus y\mathbb{Z}[x].$$

Conjecture

We just proved

Theorem (Banerjee, Smoot)

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ *such that* $8n \equiv 1 \pmod{7^{\alpha}}$. *Then*

$$d_2(n) \equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor}}.$$

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Conjecture (Banerjee)

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ such that $6n \equiv 1 \pmod{7^{\alpha}}$. Then

$$d_3(n) \equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor}}.$$

Conjecture

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Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ *such that* $8n \equiv 1 \pmod{7^\alpha}$. *Then*

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We leave this as an exercise for the interested reader.

Future Work

- What about when $\epsilon_\infty = 6$?

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- What about when $\epsilon_\infty = 6$?
- What about when $\mathfrak{g} \geq 2$?

Classification

$\epsilon_\infty(\mathrm{X}_0(N))$	$\mathfrak{g}(\mathrm{X}_0(N)) = 0$	$\mathfrak{g}(\mathrm{X}_0(N)) = 1$	$\mathfrak{g}(\mathrm{X}_0(N)) = 2$
2	Classical ($p(n)$ mod 5^α)	$p(n)$ mod 11^α	?
4	Localization (spt_ω mod 5^α, d_2 mod 3^α)	Localization (d_2 mod 7^α)	Localization (conjectured)
6	Localization (conjectured) (Beazer)	Localization (conjectured) (Andrews–Sellers)	?
≥ 8	?	?	?

Counterexamples:

- Sellers–Smoot: $d_7(n)$ mod 8^α over $\mathrm{X}_0(8)$ ($\epsilon_\infty = 4$, $\mathfrak{g} = 0$), localization unnecessary
- $N = 1$ ($\epsilon_\infty = 1$, $\mathfrak{g} = 0$)
- $N = 4$ ($\epsilon_\infty = 3$, $\mathfrak{g} = 0$)

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