

Frobenius partitions and automorphisms

Nicolas Allen Smoot
(Joint work with James A. Sellers)

University of Vienna

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FWF

Der Wissenschaftsfonds.

Congruence Families for Modular Forms

Given a modular form f with a Fourier expansion $f = \sum_{n \geq n_0} a(n)q^n$, we find a common pattern:

For $\Lambda n \equiv 1 \pmod{\ell^\alpha}$, we have $a(n) \equiv 0 \pmod{\ell^\beta}$.

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- For $24n \equiv 1 \pmod{5^\alpha}$, $p(n) \equiv 0 \pmod{5^\alpha}$
- For $24n \equiv 1 \pmod{7^\alpha}$, $p(n) \equiv 0 \pmod{7^{\lfloor \alpha/2 \rfloor + 1}}$
- For $24n \equiv 1 \pmod{11^\alpha}$, $p(n) \equiv 0 \pmod{11^\alpha}$

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For $\Lambda n \equiv 1 \pmod{\ell^\alpha}$, we have $a(n) \equiv 0 \pmod{\ell^\beta}$.

- This is a commonplace phenomenon.
- Sometimes these families are easy and routine to prove.
- In some cases these families are standing conjectures.

Congruence Families for Modular Forms

$$a(n) \equiv 0 \pmod{\ell^\beta} \text{ when } \Lambda n \equiv 1 \pmod{\ell^\alpha}.$$

Construct a sequence of functions

$$L_\alpha = \phi_\alpha \cdot \sum_{\substack{n \geq 0, \\ \Lambda n \equiv 1 \pmod{\ell^\alpha}}} a(n) q^{\lfloor n/\ell^\alpha \rfloor},$$

meromorphic on $X_0(N)$ with possible poles only at the cusps.

Construct an operator sequence $U^{(\alpha)}$ such that

$$U^{(\alpha)}(L_\alpha) = L_{\alpha+1}.$$

Example: Ramanujan's Congruences for $p(n)$

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Theorem (Ramanujan, 1918)

If $24n \equiv 1 \pmod{5^{\alpha}}$, then $p(n) \equiv 0 \pmod{5^{\alpha}}$.

The associated modular curve is $X_0(5)$.

Example: Andrews–Sellers Congruences

$$C\Phi_2(q) := \sum_{n=0}^{\infty} c\phi_2(n)q^n = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^4 (q^4; q^4)_{\infty}^2}.$$

Example: Andrews–Sellers Congruences

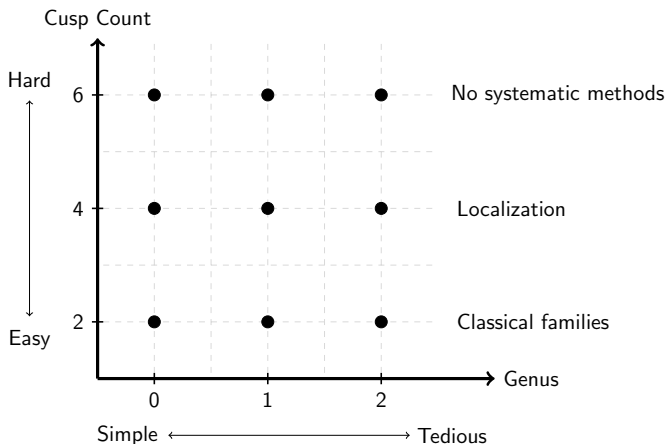
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Theorem (Paule and Radu, 2012)

If $12n \equiv 1 \pmod{5^\alpha}$, then $c\phi_2(n) \equiv 0 \pmod{5^\alpha}$.

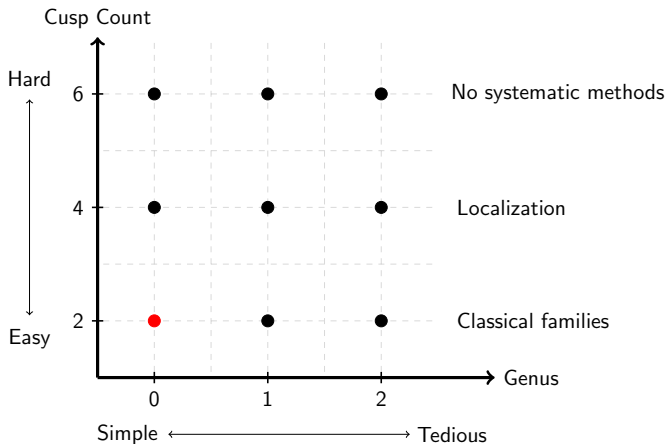
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Classification



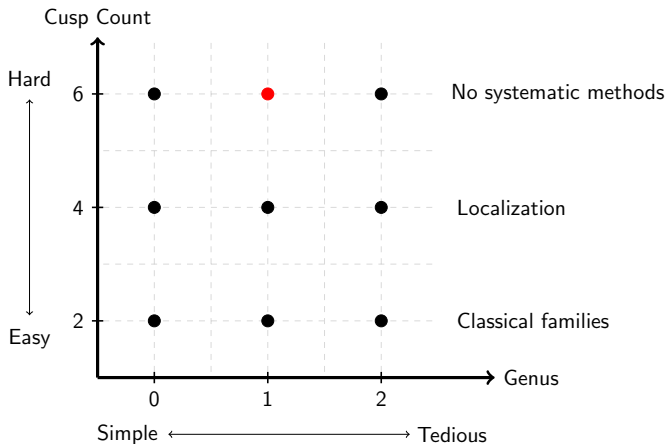
Classifying congruence families by the topology of the associated modular curve $X_0(N)$

Classification



Ramanujan's congruences for $p(n)$ by powers of 5 lie here.

Classification



The Andrews–Sellers congruences lie here.

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If $12n \equiv 1 \pmod{5^\alpha}$, then $c\phi_2(n) \equiv 0 \pmod{5^\alpha}$.

The associated modular curve is $X_0(20)$.

$$L_{\alpha}^{(1)} := \frac{1}{C\Phi_2(q^{1+4a})} \cdot \sum_{12n \equiv 1 \pmod{5^{\alpha}}} c\phi_2(n) q^{\lfloor n/5^{\alpha} \rfloor + 1},$$

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- $a \in \{0, 1\}$, $\alpha \equiv a \pmod{2}$.
- We want to show that $L_{\alpha}^{(1)} \equiv 0 \pmod{5^{\alpha}}$.

$$L_1^{(1)} := \frac{(q^5; q^5)_{\infty}^4 (q^{20}; q^{20})_{\infty}^2}{(q^{10}; q^{10})_{\infty}^5} \sum_{n=0}^{\infty} c\phi_2(5n+3) q^{n+1},$$

$$\mathcal{M}^0(X_0(20)) = \mathbb{C}[x] + y\mathbb{C}[x].$$

$$x = q \frac{(q^2; q^2)_\infty (q^{10}; q^{10})_\infty^3}{(q; q)_\infty^3 (q^5; q^5)_\infty},$$

$$y = q^2 \frac{(q^2; q^2)_\infty^2 (q^4; q^4)_\infty (q^5; q^5)_\infty (q^{20}; q^{20})_\infty^3}{(q; q)_\infty^5 (q^{10}; q^{10})_\infty^2}.$$

$$L_1^{(1)} = \frac{5}{(1+5x)^2} \left(4x + 137x^2 + 1704x^3 + 10080x^4 + 28800x^5 \right. \\ \left. + 32000x^6 - y(20 + 400x + 3040x^2 + 10240x^3 + 12800x^4) \right).$$

$$\mathcal{M}^0(X_0(5)) = \mathbb{C}[t].$$

$$\frac{1}{5^{2\alpha-1}} L_{2\alpha-1}^{(1)} \in \mathbb{Z}[t] + p_1^{(1)} \mathbb{Z}[t]$$

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$$\begin{aligned} L_1^{(1)} &:= \frac{(q^5; q^5)_\infty^4 (q^{20}; q^{20})_\infty^2}{(q^{10}; q^{10})_\infty^5} \sum_{n=0}^{\infty} c\phi_2(5n+3) q^{n+1} \\ &= -5t + 25p_1^{(1)}. \end{aligned}$$

This function was studied by Drake.

$$\sum_{n=0}^{\infty} c\psi_2(n)q^n = \frac{(q^4; q^4)_{\infty}^2}{(q; q)_{\infty}^2 (q^2; q^2)_{\infty}}.$$

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$$\sum_{n=0}^{\infty} c\psi_2(n)q^n = \frac{(q^4; q^4)_{\infty}^2}{(q; q)_{\infty}^2 (q^2; q^2)_{\infty}}.$$

Theorem (Us!)

If $6n \equiv -1 \pmod{5^\alpha}$, then $c\psi_2(n) \equiv 0 \pmod{5^\alpha}$.

This was proposed by James Sellers in 2023 and proved by Sellers and Smoot in the same year.

$$\sum_{n=0}^{\infty} c\phi_2(n)q^n = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^4 (q^4; q^4)_{\infty}^2}.$$

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Generalized Frobenius Partitions

Definition (Jiang–Rolen–Woodbury)

Let $n, k \in \mathbb{Z}_{\geq 1}$ and $\beta \in \mathbb{Z} + \frac{k}{2}$ nonnegative. A (k, β) -colored generalized Frobenius partition of n is an array of the form

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{pmatrix}$$

with the following:

- Each $a_i, b_j \in \mathbb{Z}_{\geq 0}$, and any integer occurs at most k times,
- Each row is decreasing with respect to lexicographic ordering,
- $r + s \neq 0$
- $r - s = \beta - \frac{k}{2}$
- $n = r + \sum_{0 \leq i \leq r} a_i + \sum_{0 \leq j \leq s} b_j$.

Denote the number of such partitions of n as $c\psi_{k,\beta}(n)$.

Generalized Frobenius Partitions

A $(2, 1)$ -colored generalized Frobenius partition of n is an array

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with the following:

- Each $a_i, b_j \in \mathbb{Z}_{\geq 0}$, and any integer occurs at most twice,
- Each row is decreasing with respect to lexicographic ordering,
- $r \geq 1$
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$$c\phi_2(n) = c\psi_{2,1}(n)$$

Generalized Frobenius Partitions

A $(2, 0)$ -colored generalized Frobenius partition of n is an array

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$$c\psi_2(n) = c\psi_{2,0}(n)$$

$$C\Psi_{2,1}(q) := \sum_{n=0}^{\infty} c\psi_{2,1}(n)q^n = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^4 (q^4; q^4)_{\infty}^2}.$$

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Theorem

If $3 \cdot 2^{\beta+1}n \equiv (-1)^{\beta+1} \pmod{5^{\alpha}}$, then $c\psi_{2,\beta}(n) \equiv 0 \pmod{5^{\alpha}}$.

$$L_{\alpha}^{(\beta)} := \frac{1}{C\Psi_{2,\beta}(q^{1+4a})} \cdot \sum_{3 \cdot 2^{\beta+1} n \equiv (-1)^{\beta+1} \pmod{5^{\alpha}}} c\psi_{2,\beta}(n) q^{\lfloor n/5^{\alpha} \rfloor + \beta},$$

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Proof Idea for $c\psi_{2,1}(n)$ (Andrews–Sellers Congruences)

One proves that

$$\frac{1}{5^\alpha} L_\alpha^{(1)} \in \mathbb{Z}[t] + p_a^{(1)} \mathbb{Z}[t],$$

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Proof Sketch for $c\psi_{2,0}(n)$ (Sellers–Smoot Congruences)

One proves that

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The First Coincidence

$$\frac{1}{5^\alpha} L_\alpha^{(1)} \in \mathbb{Z}[t] + p_a^{(1)} \mathbb{Z}[t],$$
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The proofs are (formally) *exactly identical*.

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Main Theorem

Theorem

Define

$$\mathcal{R} := \mathbb{C}[t] + p_0^{(0)}\mathbb{C}[t] + p_1^{(0)}\mathbb{C}[t] + p_0^{(1)}\mathbb{C}[t] + p_1^{(1)}\mathbb{C}[t],$$

and the automorphisms

$$\sigma^{(\beta)} : \mathcal{R} \longrightarrow \mathcal{R}$$
$$: \begin{cases} z \longmapsto z, \quad \forall z \in \mathbb{C}, \\ t \longmapsto t, \\ p_a^{(\beta)} \longmapsto p_a^{(1-\beta)}. \end{cases}$$

Then

$$\sigma^{(\beta)} \left(L_{\alpha}^{(\beta)} \right) = L_{\alpha}^{(1-\beta)}.$$

The Second Coincidence

$$\frac{1}{5^\alpha} L_\alpha^{(\beta)} \in \mathbb{Z}[t] + p_a^{(\beta)} \mathbb{Z}[t],$$

with

- $a \in \{0, 1\}$, $\alpha \equiv a \pmod{2}$
- $t = \eta(5\tau)^6 / \eta(\tau)^6$ (Hauptmodul for $\Gamma_0(5)$)
- $p_0^{(\beta)}, p_1^{(\beta)}$ are modular functions with \mathbb{Z} coefficients over $\Gamma_0(20)$.

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$$\mathcal{M}^0(X_0(20)) = \mathbb{C}[x] + y\mathbb{C}[x].$$

$$x = q \frac{(q^2; q^2)_\infty (q^{10}; q^{10})_\infty^3}{(q; q)_\infty^3 (q^5; q^5)_\infty},$$

$$y = q^2 \frac{(q^2; q^2)_\infty^2 (q^4; q^4)_\infty (q^5; q^5)_\infty (q^{20}; q^{20})_\infty^3}{(q; q)_\infty^5 (q^{10}; q^{10})_\infty^2}.$$

The Second Coincidence

$$\frac{1}{5^\alpha} L_\alpha^{(1)} \in \mathbb{Z}[t] + p_a^{(1)} \mathbb{Z}[t],$$

$$p_1^{(1)} = \frac{1}{(1+5x)^2} \left(x + 30x^2 + 352x^3 + 2032x^4 + 5760x^5 + 6400x^6 \right. \\ \left. - 4y - 80xy - 608x^2y - 2048x^3y - 2560x^4y \right),$$

$$p_0^{(1)} = \frac{1}{(1+5x)^2} \left(61x + 1426x^2 + 13520x^3 + 65200x^4 + 160000x^5 \right. \\ \left. + 160000x^6 - 4y - 48xy - 160x^2y \right).$$

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$$\frac{1}{5^\alpha} L_\alpha^{(0)} \in \mathbb{Z}[t] + p_a^{(0)} \mathbb{Z}[t],$$

$$p_1^{(0)} = \frac{1}{(1+5x)^2} \left(1 + 27x + 302x^2 + 1776x^3 + 5744x^4 + 9600x^5 \right. \\ \left. + 6400x^6 + 4y + 80xy + 608x^2y + 2048x^3y + 2560x^4y \right),$$

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The Second Coincidence

$$p_a^{(1)} + p_a^{(0)} \in \mathbb{Z}[x]_S.$$

$$x = q \frac{(q^2; q^2)_\infty (q^{10}; q^{10})_\infty^3}{(q; q)_\infty^3 (q^5; q^5)_\infty}.$$

$$L_\alpha^{(1)} + L_\alpha^{(0)} \in \mathbb{Z}[x]_S \subseteq \mathcal{M}(X_0(10)).$$

Conjecture

A proof of

$$L_\alpha^{(1)} + L_\alpha^{(0)} \equiv 0 \pmod{5^\alpha}$$

is accessible by the localization method.

The Third Coincidence

To prove the Andrews–Sellers congruences, Paule and Radu show that

$$\frac{1}{5^\alpha} L_\alpha^{(1)} \in \mathbb{Z}[t] + p_a^{(1)} \mathbb{Z}[t],$$

with

- $a \in \{0, 1\}$, $\alpha \equiv a \pmod{2}$
- $t = \eta(5\tau)^6 / \eta(\tau)^6$ (Hauptmodul for $\Gamma_0(5)$)
- $p_0^{(1)}, p_1^{(1)}$ are modular functions with \mathbb{Z} coefficients over $\Gamma_0(20)$.

The Third Coincidence

To prove the Andrews–Sellers congruences, Paule and Radu show that

$$\frac{1}{5^\alpha} L_\alpha^{(1)} \in \mathbb{Z}[t] + p_a^{(1)} \mathbb{Z}[t],$$

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How are $p_0^{(1)}, p_1^{(1)}$ found?

The Third Coincidence

$$W := \begin{pmatrix} 4 & -1 \\ 100 & -24 \end{pmatrix}.$$

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Paule and Radu discovered that

$$\frac{(q^{10}; q^{10})_{\infty}^5}{(q^5; q^5)_{\infty}^2} L_{2\alpha-1}^{(1)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].$$

$$\frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2} L_{2\alpha}^{(1)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].$$

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They then construct $p_0^{(1)}, p_1^{(1)}$ with a similar behavior with respect to W .

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This is one way to find $p_0^{(0)}, p_1^{(0)}$.

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$$\begin{aligned}\frac{(q^{10}; q^{10})_{\infty}^5}{(q^5; q^5)_{\infty}^2} \cdot L_{2\alpha-1}^{(1)}(W(\tau)) &\in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]], \\ \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2} \cdot L_{2\alpha}^{(1)}(W(\tau)) &\in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]], \\ \frac{(q^5; q^5)_{\infty}^2}{(q^{10}; q^{10})_{\infty}} \cdot L_{2\alpha-1}^{(0)}(W(\tau)) &\in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]], \\ \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}} \cdot L_{2\alpha}^{(0)}(W(\tau)) &\in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].\end{aligned}$$

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$$\begin{aligned}\varphi(q) \cdot L_{2\alpha}^{(1)}(W(\tau)) &\in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]], \\ \varphi(-q) \cdot L_{2\alpha}^{(0)}(W(\tau)) &\in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].\end{aligned}$$

$$\varphi(q) = \frac{(q^2; q^2)_\infty^5}{(q^4; q^4)_\infty^2 (q; q)_\infty^2}, \quad \varphi(-q) = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty}.$$

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Theorem

For $\alpha \in \mathbb{Z}_{\geq 1}$, $a, \beta \in \{0, 1\}$, $a \equiv \alpha \pmod{2}$, we have

$$\varphi\left((-1)^{\beta+1} q^{1+4a}\right) \cdot L_{\alpha}^{(\beta)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]],$$

in which $\varphi(q)$ is Ramanujan's phi function.

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Thank you!