Plane Partitions and the Localization Method

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RISC RESEARCH INSTITUTE FOR SYMBOLIC COMPUTATION

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FWF Der Wissenschaftsfonds.

Partitions

Definition

For any $n \in \mathbb{Z}_{\geq 0}$, a partition of n is a representation of n as a sum of other natural numbers, called parts. The number of partitions of a given n is denoted $p(n)$.

For example, $p(4) = 5$:

- 4
- $3 + 1$
- $2 + 2$
- $2 + 1 + 1$
- $1 + 1 + 1 + 1$

Partitions

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

Ramanujan's Congruences

Theorem

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \cdot \prod_{m=1}^{\infty} \frac{(1-q^{5m})^5}{(1-q^m)^6}.$$

Notice that

$$p(5n+4) \equiv 0 \pmod{5}.$$

Ramanujan's Congruences

Theorem

$$\sum_{n=0}^{\infty} p(25n+24)q^n$$
$$=5^{12} \cdot q^4 \prod_{m=1}^{\infty} \frac{(1-q^{5m})^{30}}{(1-q^m)^{31}} + 5^{10} \cdot 6 \cdot q^3 \prod_{m=1}^{\infty} \frac{(1-q^{5m})^{24}}{(1-q^m)^{25}}$$
$$+ 5^7 \cdot 63 \cdot q^2 \prod_{m=1}^{\infty} \frac{(1-q^{5m})^{18}}{(1-q^m)^{19}} + 5^5 \cdot 52 \cdot q \prod_{m=1}^{\infty} \frac{(1-q^{5m})^{12}}{(1-q^m)^{13}}$$
$$+ 5^2 \cdot 63 \cdot \prod_{m=1}^{\infty} \frac{(1-q^{5m})^{6}}{(1-q^m)^{7}}.$$

Notice that $p(25n+24) \equiv 0 \pmod{25}$.

Ramanujan's Congruences

Theorem

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$, *and* $\lambda_\alpha \in \mathbb{Z}$ *such that* $24\lambda_\alpha \equiv 1 \pmod{5^\alpha}$. *Then*

$$p\left(5^\alpha n + \lambda_\alpha\right) \equiv 0 \pmod{5^\alpha}.$$

- $p(5n+4) \equiv 0 \pmod{5}$.
- $p(25n+24) \equiv 0 \pmod{25}$.
- $p(125n+99) \equiv 0 \pmod{125}$.

$d_2(n)$

Let $q = e^{2\pi i\tau}$, with $\tau \in \mathbb{H}$. Define $D_2(q)$ by

$$D_2(q) := \sum_{n=0}^{\infty} d_2(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^2}{(1-q^m)^7}.$$

From G.E. Andrews, P. Paule, "MacMahon's Partition Analysis XIII."

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Conjecture (G.E. Andrews, P. Paule)

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$, and $\lambda_\alpha \in \mathbb{Z}$ such that $8\lambda_\alpha \equiv 1 \pmod{3^\alpha}$. Then

$$d_2\left(3^\alpha n + \lambda_\alpha\right) \equiv 0 \pmod{3^\alpha}.$$

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$$d_2\left(3^\alpha n + \lambda_\alpha\right) \equiv 0 \pmod{3^\alpha}.$$

"... the Conjectures... seem to be particulary challenging, especially the infinite family of Ramanujan type congruences."

Congruences on $d_2(n)$

Theorem (Me)

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$, and $\lambda_\alpha \in \mathbb{Z}$ such that $8\lambda_\alpha \equiv 1 \pmod{3^\alpha}$. Then

$$d_2\left(3^{2\alpha-1}n + \lambda_{2\alpha-1}\right) \equiv 0 \pmod{3^{2\alpha-1}},$$
$$d_2\left(3^{2\alpha}n + \lambda_{2\alpha}\right) \equiv 0 \pmod{3^{2\alpha+1}}.$$

Setup

Define

$$
\begin{aligned}
L_1 &:= 1,\\
L_{2\alpha-1}(\tau) &:= \frac{(q^3;q^3)_\infty^7}{(q^6;q^6)_\infty^2}\cdot\sum_{n=0}^{\infty} d_2(3^{2\alpha-1}n+\lambda_{2\alpha-1})q^{n+1},\\
L_{2\alpha}(\tau) &:= \frac{(q;q)_\infty^7}{(q^2;q^2)_\infty^2}\cdot\sum_{n=0}^{\infty} d_2(3^{2\alpha}n+\lambda_{2\alpha})q^{n+1}.
\end{aligned}
$$

Theorem

For all $\alpha \in \mathbb{Z}_{\geq 1}$, $L_\alpha \in \mathcal{M}\left(\Gamma_0(6)\right)$.

Setup

Define

$$\mathcal{A}(q) := q\frac{D_2(q)}{D_2(q^9)},$$

$$U_\ell\left(\sum_{n\geq N} a(n)q^n\right) := \sum_{\ell n\geq N} a(\ell n)q^n.$$

$$U^{(0)}(f) := U_3(\mathcal{A}\cdot f),$$
$$U^{(1)}(f) := U_3(f).$$

Setup

$$
\begin{aligned}
L_1 &= 1, \\
L_{2\alpha-1}(\tau) &= \frac{(q^3;q^3)_\infty^7}{(q^6;q^6)_\infty^2} \cdot \sum_{n=0}^{\infty} d_2(3^{2\alpha-1}n + \lambda_{2\alpha-1}) q^{n+1}, \\
L_{2\alpha}(\tau) &= \frac{(q;q)_\infty^7}{(q^2;q^2)_\infty^2} \cdot \sum_{n=0}^{\infty} d_2(3^{2\alpha}n + \lambda_{2\alpha}) q^{n+1}.
\end{aligned}
$$

Lemma

$$
\begin{aligned}
L_{2\alpha} &= U^{(1)}\left(L_{2\alpha-1}\right), \\
L_{2\alpha+1} &= U^{(0)}\left(L_{2\alpha}\right).
\end{aligned}
$$

Genus

- The congruence family is associated with the congruence subgroup $\Gamma_0(6)$, and the compact Riemann surface $X_0(6)$.
- Each L_α is a modular function for $\Gamma_0(6)$.

$$\mathfrak{g}\left(X_0(6)\right) = 0.$$

Lemma

The space of modular functions for $\Gamma_0(6)$ with a pole only at the cusp $[0]_6$ has the form $\mathbb{C}[x]$ for a function x.

What We Want

For some modular function $z \in \mathcal{M}^0\left(\Gamma_0(6)\right)$,

$$z^{n(\alpha)} \cdot L_\alpha \in \mathcal{M}^0\left(\Gamma_0(6)\right) = \mathbb{C}[x],$$

with $n(\alpha)$ some integer-valued function of α.

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with $n(\alpha)$ some integer-valued function of α.

$$z = \frac{(q^2;q^2)_\infty^9 (q^3;q^3)_\infty^3}{(q;q)_\infty^9 (q^6;q^6)_\infty^3}, \qquad x = q\frac{(q^2;q^2)_\infty (q^6;q^6)_\infty^5}{(q;q)_\infty^5 (q^3;q^3)_\infty}.$$

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$$z = 1 + 9x.$$

This suggests that

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$$z \in \mathcal{M}^0\left(\Gamma_0(6)\right) = \mathbb{C}[x].$$

$$z = 1 + 9x.$$

This suggests that

$$L_\alpha \in \mathbb{Z}[x]_{\mathcal{S}},$$

$$\mathcal{S} := \left\{(1+9x)^n : n \geq 0\right\}.$$

Example

To begin with, examine L_1.

$$L_1 = \frac{(q^3;q^3)_\infty^7}{(q^6;q^6)_\infty^2} \cdot \sum_{n=0}^{\infty} d_2(3n+2)q^{n+1}.$$

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$$L_1 = \frac{1}{1+9x} \cdot \left(33x + 1392x^2 + 21120x^3 + 138240x^4 + 331776x^5\right).$$

Example

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$$L_1 = \frac{(q^3;q^3)_\infty^7}{(q^6;q^6)_\infty^2} \cdot \sum_{n=0}^{\infty} d_2(3n+2)q^{n+1}.$$

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Similar identities hold for L_2, L_3.

Main Theorem

Theorem (Me)

Let $\alpha \geq 1$ *and*

$$\psi := \left\lfloor \frac{3^{\alpha+1}}{8} \right\rfloor \text{ and } \beta := 2\lfloor \alpha/2 \rfloor + 1.$$

Then

$$\frac{(1+9x)^{\psi}}{3^{\beta}} L_{\alpha} \in \mathbb{Z}[x], \text{ for all } \alpha \geq 1.$$

L_α

$$L_1 = \frac{1}{1+9x} \cdot \left(33x + 1392x^2 + 21120x^3 + 138240x^4 + 331776x^5\right).$$

We will prove that

$$\frac{1}{3^\alpha} \cdot L_\alpha = \sum_{m \geq 1} s(m) \cdot 3^{\theta(m)} \cdot \frac{x^m}{(1+9x)^n},$$

with $n \in \mathbb{Z}_{\geq 1}$ fixed, s, θ integer-valued functions, and s discrete.

U Operator

$$\frac{1}{3^{\alpha}} \cdot L_{\alpha} = \sum_{m \geq 1} s(m) \cdot 3^{\theta(m)} \cdot \frac{x^m}{(1+9x)^n},$$

We study

$$U^{(i)} \left(\frac{x^m}{(1+9x)^n} \right).$$

General Relation

Theorem

There exist discrete arrays $h_1, h_0 : \mathbb{Z}^3 \to \mathbb{Z}$ *and functions* $\pi_i : \mathbb{Z}^2_{\geq 1} \to \mathbb{Z}_{\geq 0}$ *such that*

$$U^{(1)}\left(\frac{x^m}{(1+9x)^n}\right) = \frac{1}{(1+9x)^{3n}} \sum_{r \geq \lceil m/3 \rceil} h_1(m,n,r) \cdot 3^{\pi_1(m,r)} \cdot x^r,$$

$$U^{(0)}\left(\frac{y^m}{(1+9x)^n}\right) = \frac{1}{(1+9x)^{3n+1}} \sum_{r \geq \lceil (m+1)/3 \rceil} h_0(m,n,r) \cdot 3^{\pi_0(m,r)} \cdot x^r.$$

General Relation

$$\pi_0(m,r) := \max\left(0, \left\lfloor \frac{3r-m}{4} \right\rfloor - 1\right),$$

$$\pi_1(m,r) := \begin{cases} 0, & 1 \leq m \leq 3 \text{ and } r = 1, \\ \left\lfloor \frac{3r+1}{4} \right\rfloor, & 1 \leq m \leq 3 \text{ and } r \geq 2, \\ \max\left(0, \left\lfloor \frac{3r-m+1}{4} \right\rfloor\right), & m \geq 4 \end{cases}$$

Proof Strategy

$$\mathcal{V}_n := \left\{ \frac{1}{(1+9x)^n} \sum_{m \geq 1} s(m) \cdot 3^{\theta(m)} \cdot x^m : s \text{ is discreet} \right\}.$$

$$\theta(m) := \begin{cases} 0, & 1 \leq m \leq 3, \\ 2, & 4 \leq m \leq 6, \\ \left\lfloor \frac{3m-3}{4} \right\rfloor - 1, & m \geq 7, \end{cases}$$

Proof Strategy

$$\mathcal{V}_n := \left\{ \frac{1}{(1+9x)^n} \sum_{m\geq 1} s(m) \cdot 3^{\theta(m)} \cdot x^m : s \text{ is discreet} \right\}.$$

Show that $\frac{1}{3}L_1 \in \mathcal{V}_1$,

Show that for any $f \in \mathcal{V}_n$, $\frac{1}{9}U^{(1)}(f) \in \mathcal{V}_{3n}$,

Show that for any $f \in \mathcal{V}_n$, $U^{(0)}(f) \in \mathcal{V}_{3n+1}$.

Even-to-Odd Index

Let $f \in \mathcal{V}_n$. Then

$$\begin{aligned}
U^{(0)}(f) =& U^{(0)}\left(\frac{1}{(1+9x)^n}\sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot x^m\right)\\
=& \sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot U^{(0)}\left(\frac{x^m}{(1+9x)^n}\right)\\
=& \frac{1}{(1+9x)^{3n+1}}\sum_{m\geq 1}\sum_{r\geq\lceil (m+1)/3\rceil} s(m)\cdot h_0(m,n,r)\cdot 3^{\theta(m)+\pi_0(m,r)}\cdot x^r\\
=& \frac{1}{(1+9x)^{3n+1}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_0(m,n,r)\cdot 3^{\theta(m)+\pi_0(m,r)}\cdot x^r
\end{aligned}$$

We want to show that

$$\theta(m)+\pi_0(m,r)\geq\theta(r) \text{ for all } r\geq 1,$$

$$\text{so that } U^{(0)}(f)\in\mathcal{V}_{3n+1}.$$

Odd-to-Even Index

Let $f \in \mathcal{V}_n$. Then

$$\begin{aligned}
U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+9x)^n}\sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot x^m\right)\\
=& \sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot U^{(1)}\left(\frac{y^m}{(1+9x)^n}\right)\\
=& \frac{1}{(1+9x)^{3n}}\sum_{m\geq 1}\sum_{r\geq\lceil m/3\rceil} s(m)\cdot h_1(m,n,r)\cdot 3^{\theta(m)+\pi_1(m,r)}\cdot x^r\\
=& \frac{1}{(1+9x)^{3n}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_1(m,n,r)\cdot 3^{\theta(m)+\pi_1(m,r)}\cdot x^r
\end{aligned}$$

We want to show that

$$\theta(m)+\pi_1(m,r)\geq\theta(r)+2 \text{ for all } r\geq 1,$$

$$\text{so that } \frac{1}{9}U^{(1)}(f)\in\mathcal{V}_{3n}.$$

3-adic Irregularity

We are going to prove that

$$\theta(m) + \pi_0(m, r) \geq \theta(r) \text{ for all } r \geq 1,$$
$$\theta(m) + \pi_1(m, r) \geq \theta(r) + 2 \text{ for all } r \geq 1.$$

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$$\theta(m) + \pi_1(m, r) \geq \theta(r) + 2 \text{ for all } r \geq 1.$$

No we aren't.

$$\theta(m) + \pi_0(m, r) \geq \theta(r) \text{ for all } r \geq 1 \text{ is true.}$$
$$\theta(m) + \pi_1(m, r) \geq \theta(r) + 2, \text{ on the other hand...}$$

3-adic Irregularity

Let $f \in \mathcal{V}_n$. Then

$$\begin{aligned}
U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+9y)^n}\sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot x^m\right)\\
=& \sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot U^{(1)}\left(\frac{x^m}{(1+9x)^n}\right)\\
=& \frac{1}{(1+9x)^{3n}}\sum_{m\geq 1}\sum_{r\geq \lceil m/3\rceil} s(m)\cdot h_1(m,n,r)\cdot 3^{\theta(m)+\pi_1(m,r)}\cdot x^r\\
=& \frac{1}{(1+9x)^{3n}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_1(m,n,r)\cdot 3^{\theta(m)+\pi_1(m,r)}\cdot x^r
\end{aligned}$$

The coefficient of $\frac{x^1}{(1+9x)^{3n}}$ is

$$\sum_{m=1}^{3} s(m)\cdot h_1(m,n,1)\cdot 3^{\theta(m)+\pi_1(m,1)}.$$

3-adic Irregularity

The coefficient of $\frac{x^1}{(1+9x)^{3n}}$ has the form

$$
\begin{aligned}
&= \sum_{m=1}^{3} s(m) \cdot h_1(m, n, 1) \cdot 3^{\theta(m)+\pi_1(m,1)} \\
&= s(1) \cdot h_1(1, n, 1) + s(2) \cdot h_1(2, n, 1) + s(3) \cdot h_1(3, n, 1).
\end{aligned}
$$

3-adic Irregularity

Theorem

For all $m, n, r \in \mathbb{Z}_{\geq 1}$ with $1 \leq m \leq 3r$ and $1 \leq r \leq 5$ we have:

$$h_0(m, 3n, r) \equiv h_0(m, 3, r) \pmod 9,$$
$$h_1(m, 3n+1, r) \equiv h_1(m, 1, r) \pmod 9.$$

In particular, for $m = 1, 2, 3$,

$$h_1(m, 3n+1, 1) \equiv 1 \pmod 9.$$

3-adic Irregularity

Our coefficient of $\frac{x^1}{(1+9x)^{3n}}$ for $U^{(1)}(f)$ is

$$\begin{aligned} &\sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod 9 \\ \equiv &\sum_{m=1}^{3} s(m) \pmod 9. \end{aligned}$$

3-adic Irregularity

Our coefficient of $\frac{x^1}{(1+9x)^{3n}}$ for $U^{(1)}(f)$ is

$$\sum_{m=1}^{3} s(m) \cdot h_1(m,n,1) \pmod 9$$
$$\equiv \sum_{m=1}^{3} s(m) \pmod 9.$$

Examine L_1:

$$\frac{1}{3}L_1 = \frac{1}{1+9x} \cdot \left(11x + 464x^2 + 7040x^3 + 46080x^4 + 110592x^5\right).$$

Notice that $11 + 464 + 7040 = 9 \cdot 835$.

3-adic Irregularity

Definition

$$\mathcal{V}_n^{(1)} := \left\{ \frac{1}{(1+9x)^n} \sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot x^m : \sum_{m=1}^{3} s(m) \equiv 0 \bmod 9 \right\},$$

$$\mathcal{V}_n^{(0)} := \left\{ \frac{1}{(1+9x)^n} \sum_{m\geq 1} s(m)\cdot 3^{\theta(m)}\cdot x^m \right\}.$$

Here s again represents a discrete integer-valued function.

Resolving 3-adic Irregularity

Theorem

Suppose $f \in \mathcal{V}_n^{(1)}$. *Then*

$$\frac{1}{9} \cdot U^{(1)}(f) \in \mathcal{V}_{3n},$$
$$\frac{1}{9} \cdot U^{(0)} \circ U^{(1)}(f) \in \mathcal{V}_{9n+1}.$$

Sketch

Let $f \in \mathcal{V}_n^{(1)}$. Then

$$\frac{1}{9} \cdot \left(U^{(0)} \circ U^{(1)} \left(f \right) \right) = \frac{1}{(1+9x)^{9n+1}} \sum_{w \geq 1} t(w) \cdot 3^{\theta(w)} x^w,$$

$$t(w) = \sum_{r=1}^{3w-1} \sum_{m=1}^{3r} s(m) \cdot h_1(m,n,r) \cdot h_0(r,3n,w) \\ \times 3^{\theta(m)+\pi_1(m,r)+\pi_0(r,w)-\theta(w)-2}.$$

Sketch

$$t(1) = \sum_{r=1}^{2} \sum_{m=1}^{3r} s(m) \cdot h_1(m,n,r) \cdot h_0(r,3n,1) \cdot 3^{\lambda(m,r,1)},$$

$$t(2) = \sum_{r=1}^{5} \sum_{m=1}^{3r} s(m) \cdot h_1(m,n,r) \cdot h_0(r,3n,2) \cdot 3^{\lambda(m,r,2)},$$

$$t(3) = \sum_{r=1}^{8} \sum_{m=1}^{3r} s(m) \cdot h_1(m,n,r) \cdot h_0(r,3n,3) \cdot 3^{\lambda(m,r,3)},$$

$$\lambda(m,r,w) := \theta(m) + \pi_1(m,r) + \pi_0(r,w) - 2.$$

We want to show that $t(1), t(2), t(3) \in \mathbb{Z}$, and that $t(1) + t(2) + t(3) \equiv 0 \pmod 9$.

Sketch

$$t(1) + t(2) + t(3) \equiv 6 \cdot s(1) + 6 \cdot s(2) + 6 \cdot s(3) \pmod{9}.$$

Sketch

$$t(1) + t(2) + t(3) \equiv 6 \cdot s(1) + 6 \cdot s(2) + 6 \cdot s(3) \pmod{9}.$$

$\square$

Proof of our Strong Result

Proof (I)

$$\frac{1}{3} \cdot L_1 \in \mathcal{V}_1^{(1)}.$$

Suppose that for some $\alpha \in \mathbb{Z}_{\geq 1}$, there exists some $n \in \mathbb{Z}_{\geq 1}$ such that

$\frac{1}{3^{2\alpha-1}} \cdot L_{2\alpha-1} \in \mathcal{V}_n^{(1)}$. Then

$L_{2\alpha-1} = 3^{2\alpha-1} \cdot f_{2\alpha-1}$, for $f_{2\alpha-1} \in \mathcal{V}_n^{(1)}$. Now,

$L_{2\alpha} = U_3\left(L_{2\alpha-1}\right) = U_3\left(3^{2\alpha-1} \cdot f_{2\alpha-1}\right) = 3^{2\alpha-1} \cdot U^{(1)}(f_{2\alpha-1})$.

There exists some $f_{2\alpha} \in \mathcal{V}_{3n}^{(0)}$ such that $U^{(1)}(f_{2\alpha-1}) = 9 \cdot f_{2\alpha}$. Therefore,

$$L_{2\alpha} = 3^{2\alpha+1} \cdot f_{2\alpha}, \text{ and } \frac{1}{3^{2\alpha+1}} \cdot L_{2\alpha} \in \mathcal{V}_{3n}^{(0)}.$$

Proof of our Strong Result

Proof (II)

$$L_{2\alpha+1} = U_3\left(\mathcal{A} \cdot L_{2\alpha}\right) = U_3\left(3^{2\alpha+1} \cdot \mathcal{A} \cdot f_{2\alpha}\right) = 3^{2\alpha+1} \cdot U^{(0)}(f_{2\alpha}).$$

There exists some $f_{2\alpha+1} \in \mathcal{V}^{(1)}_{9n+1}$ such that $U^{(0)}(f_{2\alpha}) = f_{2\alpha+1}$. Therefore,

$$L_{2\alpha+1} = 3^{2\alpha+1} \cdot f_{2\alpha+1}, \text{ and } \frac{1}{3^{2\alpha+1}} \cdot L_{2\alpha+1} \in \mathcal{V}^{(1)}_{9n+1}.$$

Proof of our Strong Result

Proof (III)

$$\psi(\alpha) = \left\lfloor \frac{3^{\alpha+1}}{8} \right\rfloor.$$

Establishing that $\psi(\alpha)$ give the appropriate indices for $\mathcal{V}_n^{(1)}, \mathcal{V}_n^{(0)}$ is an elementary exercise in number theory. Prove that

$$\psi(1) = 1,$$
$$3\psi(2\alpha - 1) = \psi(2\alpha),$$
$$3\psi(2\alpha) + 1 = \psi(2\alpha + 1).$$

□

Complications for Proving Congruence Families by ℓ^{α}

Topological Difficulties:

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Topological Difficulties:

- The genus

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- The genus (Number of necessary "basis" functions)

Complications for Proving Congruence Families by ℓ^α

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- The number of cusps

Complications for Proving Congruence Families by ℓ^α

Topological Difficulties:

- The genus (Number of necessary "basis" functions)
- The number of cusps (Resolved with Localization)

Complications for Proving Congruence Families by ℓ^α

Topological Difficulties:

- The genus (Number of necessary "basis" functions)
- The number of cusps (Resolved with Localization)

Other Difficulties:

Complications for Proving Congruence Families by ℓ^{α}

Topological Difficulties:

- The genus (Number of necessary “basis” functions)
- The number of cusps (Resolved with Localization)

Other Difficulties:

- Failure of eta quotient representation

Complications for Proving Congruence Families by ℓ^α

Topological Difficulties:

- The genus (Number of necessary "basis" functions)
- The number of cusps (Resolved with Localization)

Other Difficulties:

- Failure of eta quotient representation (Resolved by a number of different methods)

Complications for Proving Congruence Families by ℓ^α

Topological Difficulties:

- The genus (Number of necessary "basis" functions)
- The number of cusps (Resolved with Localization)

Other Difficulties:

- Failure of eta quotient representation (Resolved by a number of different methods)
- Existence of nontrivial eigenfunctions mod ℓ

Complications for Proving Congruence Families by ℓ^α

Topological Difficulties:

- The genus (Number of necessary "basis" functions)
- The number of cusps (Resolved with Localization)

Other Difficulties:

- Failure of eta quotient representation (Resolved by a number of different methods)
- Existence of nontrivial eigenfunctions mod ℓ (Hard)

Future Work

- Extending methods to arbitrary congruence problems on a genus 0 modular curve.
- (Long-term) Extending methods to congruence problems on a genus 1 modular curve.

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