Partitions, Kernels, and the Localization Method

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Partitions

Definition

For any $n \in \mathbb{Z}_{\geq 0}$, a partition of n is a representation of n as a sum of other positive intergers, called parts. The number of partitions of a given n is denoted $p(n)$.

For example, $p(4) = 5$:

- 4
- $3 + 1$
- $2 + 2$
- $2 + 1 + 1$
- $1 + 1 + 1 + 1$

Partitions

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

The sequence for $p(n)$ begins

$$(p(n))_{n\geq 0} = (1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135,\\ 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, ...)$$

Ramanujan's Congruences

$$(p(n))_{n\geq 0} = \big(1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, \ldots\big)$$

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- $p(5n+4) \equiv 0 \pmod 5$.

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- $p(5n+4) \equiv 0 \pmod 5$.
- $p(25n+24) \equiv 0 \pmod{25}$.

Ramanujan's Congruences

$$(p(n))_{n\geq 0} = (1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, ...)$$

- $p(5n+4) \equiv 0 \pmod 5$.
- $p(25n+24) \equiv 0 \pmod{25}$.
- $p(125n+99) \equiv 0 \pmod{125}$.

Ramanujan's Congruences

$$(p(n))_{n\geq 0} = (1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 57, 77, 101, 135,\\ 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, 1958, ...)$$

- $p(5n+4) \equiv 0 \pmod{5}$.
- $p(25n+24) \equiv 0 \pmod{25}$.
- $p(125n+99) \equiv 0 \pmod{125}$.

Theorem (Ramanujan, 1918)

Let $n, \alpha \in \mathbb{Z}_{\geq 0}$ such that $24n \equiv 1 \pmod{5^\alpha}$. Then

$$p(n) \equiv 0 \pmod{5^\alpha}.$$

Ramanujan's Congruences

Theorem (Ramanujan, Watson, Atkin)

Let $\ell \in \{5, 7, 11\}$ *and* $n, \alpha \in \mathbb{Z}_{\geq 0}$ *such that* $24n \equiv 1 \pmod{\ell^\alpha}$. *Then*

$$p(n) \equiv 0 \pmod{\ell^\beta},$$
$$\beta = \begin{cases} \alpha, & \ell \in \{5, 11\}, \\ \lfloor \alpha/2 \rfloor + 1, & \ell = 7. \end{cases}$$

Similar Congruences

Suppose

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{\delta|M} (q^{\delta};q^{\delta})_{\infty}^{r_{\delta}}.$$

A *congruence family* for $a(n)$ modulo powers of a prime ℓ is a set of divisibilities

$$a(n) \equiv 0 \pmod{\ell^{\beta}} \text{ when } \Lambda n \equiv 1 \pmod{\ell^{\alpha}},$$

with $\Lambda \in \mathbb{Z}$ fixed and

$$\beta \to \infty \text{ as } \alpha \to \infty.$$

$d_k(n)$: k-Elongated Plane Partitions of n

Define $D_k(q)$ by

$$D_k(q) := \sum_{n=0}^{\infty} d_k(n) q^n = \prod_{m=1}^{\infty} \frac{(1-q^{2m})^k}{(1-q^m)^{3k+1}},$$

in which $d_k(n)$ counts the number of k-elongated plane partitions of n.

$d_k(n)$: k-Elongated Plane Partitions of n

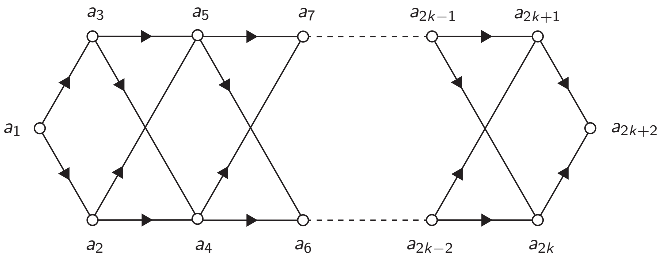

Figure: A length 1 k-elongated partition diamond.

- $a_j \in \mathbb{Z}_{\geq 0}$
- $a_b \to a_c$ indicates that $a_b \geq a_c$
- $a_1 + a_{2k+2} = n$

$d_k(n)$: k-Elongated Plane Partitions of n

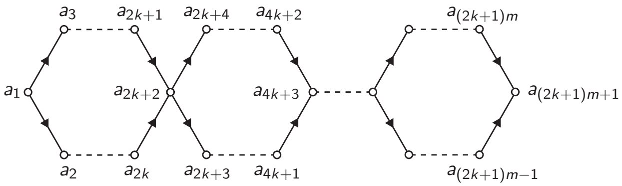

Figure: A length m k-elongated partition diamond.

- $a_j \in \mathbb{Z}_{\geq 0}$
- $a_b \to a_c$ indicates that $a_b \geq a_c$
- $a_1 + a_{2k+2} + \ldots + a_{(2k+1)m+1} = n$

Congruences on $d_5(n)$

Theorem (da Silva, Hirschhorn, Sellers)

For all $j, n \geq 0$,

$$d_{5j+5}(5n+4) \equiv 0 \pmod{5}.$$

Note that $d_5(5n+4) \equiv 0 \pmod{5}$.

Congruences on $d_5(n)$

This was conjectured by Koustav Banerjee:

Theorem (Banerjee, Me)

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ such that $4n \equiv 1 \pmod{5^\alpha}$. Then

$$d_5(n) \equiv 0 \pmod{5^{\lfloor \alpha/2 \rfloor + 1}}.$$

Theory of Partition Congruences

If $\Lambda n \equiv 1 \pmod{\ell^\alpha}$, then $a(n) \equiv 0 \pmod{\ell^\beta}$.

$$L_\alpha := \phi_\alpha \cdot \sum_{n=0}^{\infty} a\left(\ell^\alpha n + \lambda_\alpha\right) q^{n+c} \in \mathcal{M}\left(\Gamma_0(N)\right), \text{ for } q := e^{2\pi i \tau}, \tau \in \mathbb{H}.$$

L_α is equivalently a meromorphic function on the classical modular curve $\mathrm{X}_0(N)$.

Theory of Partition Congruences

$\mathrm{X}_0(N)$ is a compact Riemann surface, diffeomorphic to a 2 dimensional $\mathcal{C}^\infty$ real manifold. The two key topological properties important to us are:

- The genus $\mathfrak{g}$;
- The cusp count ϵ_∞.

Poles

Because $\mathrm{X}_0(N)$ is compact, any holomorphic function $f : \mathrm{X}_0(N) \to \mathbb{C}$ must be constant.

L_α has to have a pole somewhere.

The Genus

Let $\mathcal{M}^0\left(\mathrm{X}_0(N)\right)$ be the set of modular functions with a pole only at the cusp [0]. If $\mathfrak{g}\left(\mathrm{X}_0(N)\right) = 0$, then

$$\mathcal{M}^0\left(\mathrm{X}_0(N)\right) = \mathbb{C}[x],$$

for a function x. This is a consequence of the Weierstrass gap theorem.

The Cusp Count

Let $N = \ell$ be a prime. Then $\epsilon_\infty\left(\mathrm{X}_0(\ell)\right) = 2$. Denote the cusps as $[0]$ and $[\infty]$.

$$L_\alpha := \phi_\alpha \cdot \sum_{n=0}^{\infty} a\left(\ell^\alpha n + \lambda_\alpha\right) q^{n+c}.$$

Notice: because $c > 0$, L_α has positive order at $[\infty]$, and is holomorphic everywhere besides the cusps.

$$L_\alpha \in \mathcal{M}^0\left(\Gamma_0(\ell)\right) = \mathbb{C}[x].$$

The Cusp Count

Suppose N is not prime. Then $\epsilon_\infty\left(\mathrm{X}_0(N)\right) > 2$.

$$L_\alpha := \phi_\alpha \cdot \sum_{n=0}^{\infty} a\left(\ell^\alpha n + \lambda_\alpha\right) q^{n+c}.$$

Notice: L_α has positive order at $[\infty]$, but it may have poles at cusps besides $[0]$ and $[\infty]$.

The Cusp Count

Suppose N is not prime, and $\epsilon_\infty\left(\mathrm{X}_0(N)\right) > 2$.
There exists a function $z \in \mathcal{M}^0\left(\Gamma_0(N)\right)$ with positive order at every cusp except at [0].

$$z^m \cdot L_\alpha \in \mathcal{M}^0\left(\Gamma_0(N)\right) = \mathbb{C}[x],$$
$$L_\alpha \in \mathbb{C}[x]_{\mathcal{S}},$$

with

$$\mathcal{S} := \left\{z^n : n \geq 0\right\}.$$

Theory When $\mathfrak{g}\left(\mathrm{X}_0(N)\right) = 0$

- If $\epsilon_\infty\left(\mathrm{X}_0(N)\right) = 2$, then the classical techniques of Ramanujan and Watson will apply.
- If $\epsilon_\infty\left(\mathrm{X}_0(N)\right) > 2$, then we use localization.

Theory When $\mathfrak{g}\left(\mathrm{X}_0(N)\right) > 0$

- We're working on it.

Theorem

Let $n, \alpha \in \mathbb{Z}_{\geq 1}$ such that $4n \equiv 1 \pmod{5^{\alpha}}$. Then

$$d_5(n) \equiv 0 \pmod{5^{\lfloor \alpha/2 \rfloor + 1}}.$$

- Associated Riemann survace is the modular curve $\mathrm{X}_0(10)$
- $\mathfrak{g}\left(\mathrm{X}_0(10)\right) = 0$, indicating the existence of Hauptmoduln
- $\epsilon_{\infty}\left(\mathrm{X}_0(10)\right) = 4 > 2$, indicating complex behavior at the cusps for the function sequence associated with the congruence family.
- All of this necessitates the localization method.

First Example

$$L_1 = \frac{(q^5;q^5)_\infty^{16}}{(q^{10};q^{10})_\infty^5} \sum_{n=0}^{\infty} d_5(5n+4)q^{n+2}.$$

$$\begin{aligned}
L_1 = &\frac{1}{(1+5x)^6} \cdot \Big(5705x^2 + 6840120x^3 + 2034152125x^4 + 280484938650x^5 + 22921365211325x^6 + 1260917405154520x^7 \\
&+ 50400843190048480x^8 + 1539115922208139200x^9 + 37183654303328448000x^{10} + 728924483359472640000x^{11} \\
&+ 11816089262411136000000x^{12} + 160681440628058880000000x^{13} + 1853291134193264640000000x^{14} \\
&+ 18284160727362809856000000x^{15} + 155286793010086625280000000x^{16} + 1140657222505472000000000000x^{17} \\
&+ 7269894420215070720000000000x^{18} + 40277647277404979200000000000x^{19} \\
&+ 194099187864646451200000000000x^{20} + 813054581193729638400000000000x^{21} \\
&+ 2954545150241538048000000000000x^{22} + 9282005730758492160000000000000x^{23} \\
&+ 25080951875200614400000000000000x^{24} + 57872525958316032000000000000000x^{25} \\
&+ 112916020309524480000000000000000x^{26} + 183812885074411520000000000000000x^{27} \\
&+ 245082228994867200000000000000000x^{28} + 260725452832768000000000000000000x^{29} \\
&+ 212837104353280000000000000000000x^{30} + 125198296678400000000000000000000x^{31} \\
&+ 47244640256000000000000000000000x^{32} + 8589934592000000000000000000000x^{33} \Big).
\end{aligned}$$

$$L_0 := 1,$$

$$L_{2\alpha-1}(\tau) = \frac{(q^5;q^5)_\infty^{16}}{(q^{10};q^{10})_\infty^5} \cdot \sum_{n=0}^{\infty} d_5(5^{2\alpha-1}n + \lambda_{2\alpha-1})q^{n+2},$$

$$L_{2\alpha}(\tau) = \frac{(q;q)_\infty^{16}}{(q^2;q^2)_\infty^5} \cdot \sum_{n=0}^{\infty} d_5(5^{2\alpha}n + \lambda_{2\alpha})q^{n+1},$$

Main Theorem

Theorem

Let

$$\psi := \psi(\alpha) := \left\lfloor \frac{5^{\alpha+1}}{4} \right\rfloor + 1 - \gcd(\alpha, 2),$$
$$\beta := \beta(\alpha) = \lfloor \alpha/2 \rfloor + 1.$$

Then for all $\alpha \geq 1$, we have

$$\frac{(1+5x)^{\psi}}{5^{\beta}} \cdot L_\alpha \in \mathbb{Z}[x].$$

L_α

$$L_1 = \frac{1}{(1+5x)^6}\cdot\Big(5705x^2 + 6840120x^3 + \dots + 858993459200000000000000000000x^{33}\Big).$$

We will express

$$L_\alpha = \sum_{m\geq 1} s(m)\cdot 5^{\theta_i(m)}\cdot\frac{x^m}{(1+5x)^n},$$

with $n\in\mathbb{Z}_{\geq 1}$ fixed, s, θ_i integer-valued functions, s discrete, and $i=0,1$ depending on the parity of α.

U Operator

$$L_\alpha = \sum_{m\geq 1} s(m)\cdot 5^{\theta_i(m)}\cdot \frac{x^m}{(1+5x)^n},$$

We study

$$U^{(i)}\left(\frac{x^m}{(1+5x)^n}\right).$$

General Relation

Theorem

There exist discrete arrays $h_1, h_0 : \mathbb{Z}^3 \to \mathbb{Z}$ and functions $\pi_i : \mathbb{Z}^2_{\geq 1} \to \mathbb{Z}_{\geq 0}$ such that

$$U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n}} \sum_{r\geq\lceil m/5\rceil} h_1(m,n,r)\cdot 5^{\pi_1(m,r)}\cdot x^r,$$

$$U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right) = \frac{1}{(1+5x)^{5n+6}} \sum_{r\geq\lceil (m+1)/5\rceil} h_0(m,n,r)\cdot 5^{\pi_0(m,r)}\cdot x^r.$$

General Relation

$$\pi_0(m,r) := \max\left(0, \left\lfloor \frac{5r-m+2}{7} \right\rfloor - 5\right),$$

$$\pi_1(m,r) := \left\lfloor \frac{5r-m}{7} \right\rfloor.$$

Proof Strategy

$$\mathcal{V}_n^{(i)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_i(m)} \cdot x^m : s \text{ is discrete} \right\},$$

$i \in \{0,1\}$

$$\theta_1(m) := \begin{cases} 0, & 1 \leq m \leq 7, \\ \left\lfloor \frac{5m-2}{7} \right\rfloor - 5, & m \geq 8, \end{cases}$$

$$\theta_0(m) := \begin{cases} 0, & 1 \leq m \leq 4, \\ \left\lfloor \frac{5m-1}{7} \right\rfloor - 2, & m \geq 5, \end{cases}$$

Proof Strategy

$$\mathcal{V}_n^{(i)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_i(m)} \cdot x^m : s \text{ is discrete} \right\}.$$

$$\text{Show that } \frac{1}{5}L_1 \in \mathcal{V}_1,$$

$$\text{Show that for any } f \in \mathcal{V}_n,\ \frac{1}{5}U^{(1)}(f) \in \mathcal{V}_{5n},$$

$$\text{Show that for any } f \in \mathcal{V}_n,\ U^{(0)}(f) \in \mathcal{V}_{5n+6}.$$

Even-to-Odd Index

Let $f \in \mathcal{V}_n^{(0)}$. Then

$$\begin{aligned}
U^{(0)}(f) =& U^{(0)}\left(\frac{1}{(1+5x)^n}\sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot x^m\right)\\
=& \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot U^{(0)}\left(\frac{x^m}{(1+5x)^n}\right)\\
=& \frac{1}{(1+5x)^{5n+6}}\sum_{m\geq 1}\sum_{r\geq\lceil (m+1)/5\rceil} s(m)\cdot h_0(m,n,r)\cdot 5^{\theta_0(m)+\pi_0(m,r)}\cdot x^r\\
=& \frac{1}{(1+5x)^{5n+6}}\sum_{r\geq 1}\sum_{m\geq 1} s(m)\cdot h_0(m,n,r)\cdot 5^{\theta_0(m)+\pi_0(m,r)}\cdot x^r
\end{aligned}$$

We want to show that

$$\theta_0(m) + \pi_0(m,r) \geq \theta_1(r) \text{ for all } r\geq 1,$$

$$\text{so that } U^{(0)}(f) \in \mathcal{V}_{5n+6}^{(1)}.$$

Odd-to-Even Index

Let $f \in \mathcal{V}_n^{(1)}$. Then

$$\begin{aligned}
U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+5x)^n}\sum_{m\geq 2} s(m)\cdot 5^{\theta_1(m)}\cdot x^m\right)\\
=& \sum_{m\geq 2} s(m)\cdot 5^{\theta_1(m)}\cdot U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right)\\
=& \frac{1}{(1+5x)^{5n}}\sum_{m\geq 2}\sum_{r\geq\lceil 5/3\rceil} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot x^r\\
=& \frac{1}{(1+5x)^{5n}}\sum_{r\geq 1}\sum_{m\geq 2} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot x^r
\end{aligned}$$

We want to show that

$$\theta_1(m)+\pi_1(m,r)\geq\theta_0(r)+1 \text{ for all } r\geq 1,$$
$$\text{so that } \frac{1}{5}U^{(1)}(f)\in\mathcal{V}_{5n}^{(0)}.$$

5-adic Irregularity

We are going to prove that

$$\theta_0(m) + \pi_0(m,r) \geq \theta_1(r) \text{ for all } r \geq 1,$$
$$\theta_1(m) + \pi_1(m,r) \geq \theta_0(r) + 1 \text{ for all } r \geq 1.$$

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No we aren't.

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$$\theta_1(m) + \pi_1(m,r) \geq \theta_0(r) + 1 \text{ for all } r \geq 1.$$

No we aren't.

$$\theta_0(m) + \pi_0(m,r) \geq \theta_1(r) \text{ for all } r \geq 1 \text{ is true.}$$
$$\theta_1(m) + \pi_1(m,r) \geq \theta_0(r) + 1, \text{ on the other hand...}$$

5-adic Irregularity

Let $f \in \mathcal{V}_n$. Then

$$\begin{aligned}
U^{(1)}(f) =& U^{(1)}\left(\frac{1}{(1+5x)^n}\sum_{m\geq 2} s(m)\cdot 5^{\theta_1(m)}\cdot x^m\right)\\
=& \sum_{m\geq 2} s(m)\cdot 5^{\theta_1(m)}\cdot U^{(1)}\left(\frac{x^m}{(1+5x)^n}\right)\\
=& \frac{1}{(1+5x)^{5n}}\sum_{m\geq 2}\sum_{r\geq\lceil m/5\rceil} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot x^r\\
=& \frac{1}{(1+5x)^{5n}}\sum_{r\geq 1}\sum_{m\geq 2} s(m)\cdot h_1(m,n,r)\cdot 5^{\theta_1(m)+\pi_1(m,r)}\cdot x^r
\end{aligned}$$

The coefficient of $\frac{x^1}{(1+5x)^{5n}}$ is

$$\sum_{m=2}^{5} s(m)\cdot h_1(m,n,1)\cdot 5^{\theta_1(m)+\pi_1(m,1)}.$$

5-adic Irregularity

The coefficient of $\frac{x^1}{(1+5x)^{5n}}$ has the form

$$
\begin{aligned}
&\sum_{m=2}^{5} s(m)\cdot h_1(m,n,1)\cdot 5^{\theta_1(m)+\pi_1(m,1)}\\
=&s(2)\cdot h_1(2,n,1)+s(3)\cdot h_1(3,n,1)+s(4)\cdot h_1(4,n,1)\\
&+s(5)\cdot h_1(5,n,1).
\end{aligned}
$$

5-adic Irregularity

The coefficient of $\frac{x^2}{(1+5x)^{5n}}$ has the form

$$\begin{aligned}&\sum_{m=2}^{8} s(m)\cdot h_1(m,n,2)\cdot 5^{\theta_1(m)+\pi_1(m,2)}\\ =&5s(2)\cdot h_1(2,n,2)+5s(3)\cdot h_1(3,n,2)+s(4)\cdot h_1(4,n,2)\\ &+s(5)\cdot h_1(5,n,2)+s(6)\cdot h_1(6,n,2)+s(7)\cdot h_1(7,n,2)\\ &+s(8)\cdot h_1(8,n,2).\end{aligned}$$

5-adic Irregularity

Theorem

For all $m, n, r \in \mathbb{Z}_{\geq 1}$, and $i = 0, 1$, we have:

$$h_i(m, n, r) \equiv h_i(m, n-5, r) \pmod{5}.$$

5-adic Irregularity

In particular, for $m = 2, 3, 5,$

$$h_1(m, n, 1) \equiv 1 \pmod{5}.$$

In particular, for $m = 6, 7, 8,$

$$h_1(m, n, 2) \equiv 1 \pmod{5}.$$

Finally,

$$\begin{aligned} h_1(4, n, 1) &\equiv 2 \pmod{5}, \\ h_1(4, n, 2) &\equiv 4 \pmod{5}, \\ h_1(5, n, 2) &\equiv 0 \pmod{5}. \end{aligned}$$

5-adic Irregularity

Our coefficient of $\frac{x^1}{(1+5x)^{5n}}$ for $U^{(1)}(f)$ is

$$\sum_{m=2}^{5} s(m) \cdot h_1(m,n,1) \cdot 5^{\theta_1(m)+\pi_1(m,1)}$$
$$\equiv s(2) + s(3) + 2s(4) + s(5) \pmod{5}.$$

5-adic Irregularity

Our coefficient of $\frac{x^1}{(1+5x)^{5n}}$ for $U^{(1)}(f)$ is

$$\sum_{m=2}^{5} s(m)\cdot h_1(m,n,1)\cdot 5^{\theta_1(m)+\pi_1(m,1)}$$
$$\equiv s(2)+s(3)+2s(4)+s(5) \pmod 5.$$

Examine L_1:

$$L_1 = \frac{5}{(1+5x)^6}\Big(1141x^2 + 1368024x^3 + 406830425x^4 + 56096987730x^5 +$$

Notice that $1141 + 1368024 + 2\cdot 406830425 + 56096987730 \equiv 0 \pmod 5$.

5-adic Irregularity

Our coefficient of $\frac{x^2}{(1+5x)^{5n}}$ for $U^{(1)}(f)$ is

$$\sum_{m=2}^{8} s(m) \cdot h_1(m,n,2) \cdot 5^{\theta_1(m)+\pi_1(m,2)}$$
$$\equiv 4s(4) + s(6) + s(7) + s(8) \pmod{5}.$$

5-adic Irregularity

Definition

$$\mathcal{V}_n^{(1)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 2} s(m)\cdot 5^{\theta_1(m)}\cdot x^m : (s(m))_{2\leq m\leq 8} \in \ker(\Omega) \right\},$$

$$\mathcal{V}_n^{(0)} := \left\{ \frac{1}{(1+5x)^n} \sum_{m\geq 1} s(m)\cdot 5^{\theta_0(m)}\cdot x^m \right\}.$$

$\Omega : \mathbb{Z}^7 \to \mathbb{Z}/5\mathbb{Z}^2$

$$\Omega(\mathbf{s}) := \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 & 4 & 4 \end{pmatrix} \cdot \mathbf{s}.$$

Resolving 5-adic Irregularity

Theorem

Suppose $f \in \mathcal{V}_n^{(1)}$. *Then*

$$\frac{1}{5} \cdot U^{(1)}(f) \in \mathcal{V}_{5n}^{(1)},$$
$$\frac{1}{5} \cdot U^{(0)} \circ U^{(1)}(f) \in \mathcal{V}_{25n+6}^{(1)}.$$

Sketch

Let $f \in \mathcal{V}_n^{(1)}$. Then

$$\frac{1}{5} \cdot \left(U^{(0)} \circ U^{(1)}(f) \right) = \frac{1}{(1+5x)^{5n+6}} \sum_{w \geq 1} t(w) \cdot 5^{\theta_1(w)} x^w,$$

$$t(w) = \sum_{r=1}^{5w-6} \sum_{m=2}^{5r} s(m) \cdot h_1(m,n,r) \cdot h_0(r,5n,w) \\ \times 5^{\theta_1(m)+\pi_1(m,r)+\pi_0(r,w)-\theta_1(w)-1}.$$

It can be shown that $(t(w))_{2 \leq w \leq 8} \in \ker(\Omega)$.

Proof of our Strong Result

Proof (I)

$$\frac{1}{5} \cdot L_1 \in \mathcal{V}_1^{(1)}.$$

Suppose that for some $\alpha \in \mathbb{Z}_{\geq 1}$, there exists some $n \in \mathbb{Z}_{\geq 1}$ such that

$$\frac{1}{5^\alpha} \cdot L_{2\alpha-1} \in \mathcal{V}_n^{(1)}. \text{ Then}$$

$$L_{2\alpha-1} = 5^\alpha \cdot f_{2\alpha-1}, \text{ for } f_{2\alpha-1} \in \mathcal{V}_n^{(1)}. \text{ Now,}$$

$$L_{2\alpha} = U_5\left(L_{2\alpha-1}\right) = U_5\left(5^\alpha \cdot f_{2\alpha-1}\right) = 5^\alpha \cdot U^{(1)}(f_{2\alpha-1}).$$

There exists some $f_{2\alpha} \in \mathcal{V}_{5n}^{(0)}$ such that $U^{(1)}(f_{2\alpha-1}) = 5 \cdot f_{2\alpha}$. Therefore,

$$L_{2\alpha} = 5^{\alpha+1} \cdot f_{2\alpha}, \text{ and } \frac{1}{5^{\alpha+1}} \cdot L_{2\alpha} \in \mathcal{V}_{5n}^{(0)}.$$

Proof of our Strong Result

Proof (II)

$$L_{2\alpha+1} = U_5\left(\mathcal{A} \cdot L_{2\alpha}\right) = U_5\left(5^{\alpha+1} \cdot \mathcal{A} \cdot f_{2\alpha}\right) = 5^{\alpha+1} \cdot U^{(0)}(f_{2\alpha}).$$

There exists some $f_{2\alpha+1} \in \mathcal{V}^{(1)}_{5n+1}$ such that $U^{(0)}(f_{2\alpha}) = f_{2\alpha+1}$. Therefore,

$$L_{2\alpha+1} = 5^{\alpha+1} \cdot f_{2\alpha+1}, \text{ and } \frac{1}{5^{\alpha+1}} \cdot L_{2\alpha+1} \in \mathcal{V}^{(1)}_{5n+1}.$$

Therefore, etc. $\square$

Classification

$$\text{If } \Lambda n \equiv 1 \pmod{\ell^\alpha}, \text{ then } a(n) \equiv 0 \pmod{\ell^\beta}.$$

Suppose we are working over $\mathrm{X}_0(2\ell)$ with genus 0.

$$\begin{aligned} L_\alpha &= \phi_\alpha \cdot \sum_{n=0}^{\infty} a\left(\ell^\alpha n + \lambda_\alpha\right) q^{n+c} \\ &= \frac{1}{z^{n(\alpha)}} \sum_{m\geq 0} s(m)\ell^{\theta(m)} x^m. \end{aligned}$$

$$\left(s(m)\right)_{m\geq 0} \in \ker(\Omega)$$

for some linear operator Ω.

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