The Splitting of Ramanujan Congruences Over Modular Curves

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Congruence Families for Modular Forms

Given a modular form f with a Fourer expansion $f = \sum_{n \geq n_0} a(n)q^n$, we find a common pattern:

For $\Lambda n \equiv 1 \bmod \ell^{\alpha}$, we have $a(n) \equiv 0 \bmod \ell^{\beta}$.

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- For $24n \equiv 1 \bmod 5^{\alpha}$, $p(n) \equiv 0 \bmod 5^{\alpha}$
- For $24n \equiv 1 \bmod 7^{\alpha}$, $p(n) \equiv 0 \bmod 7^{\lfloor \alpha/2 \rfloor + 1}$
- For $24n \equiv 1 \bmod 11^{\alpha}$, $p(n) \equiv 0 \bmod 11^{\alpha}$

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For $\Lambda n \equiv 1 \bmod \ell^{\alpha}$, we have $a(n) \equiv 0 \bmod \ell^{\beta}$.

- This is a commonplace phenomenon.
- Sometimes these families are easy and routine to prove.
- In some cases these families are standing conjectures.

$$a(n) \equiv 0 \pmod{\ell^\beta} \text{ when } \Lambda n \equiv 1 \pmod{\ell^\alpha}.$$

Construct a sequence of functions

$$L_\alpha = \phi_\alpha \cdot \sum_{\substack{n \geq 0, \\ \Lambda n \equiv 1 \bmod \ell^\alpha}} a(n) q^{\lfloor n/\ell^\alpha \rfloor},$$

meromorphic on $\mathrm{X}_0(N)$ with possible poles only at the cusps. Construct an operator sequence $U^{(\alpha)}$ such that

$$U^{(\alpha)}(L_\alpha) = L_{\alpha+1}.$$

Example: Ramanujan's Congruences for $p(n)$

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Theorem (Ramanujan, 1918)

If $24n \equiv 1 \pmod{5^\alpha}$, then $p(n) \equiv 0 \pmod{5^\alpha}$.

The associated modular curve is $\mathrm{X}_0(5)$.

Example: Andrews–Sellers Congruences

$$\mathrm{C}\Phi_2\left(q\right) := \sum_{n=0}^{\infty} c\phi_2(n)q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^4(q^4;q^4)_\infty^2}.$$

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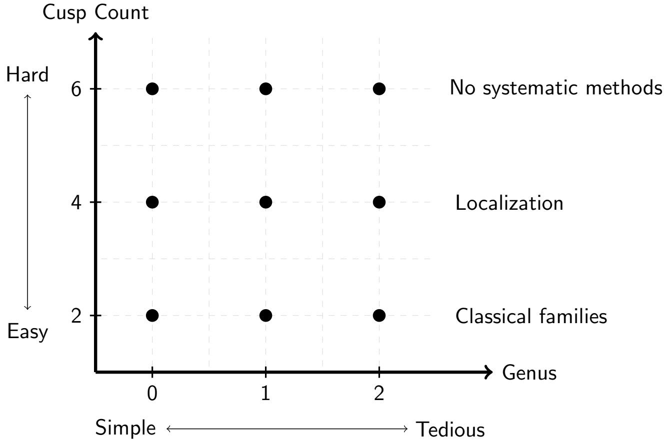
Theorem (Paule and Radu, 2012)

If $12n \equiv 1 \pmod{5^\alpha}$, then $c\phi_2(n) \equiv 0 \pmod{5^\alpha}$.

The associated modular curve is $\mathrm{X}_0(20)$.

Classification



Classifying congruence families by the topology of the associated modular curve $\mathrm{X}_0(N)$

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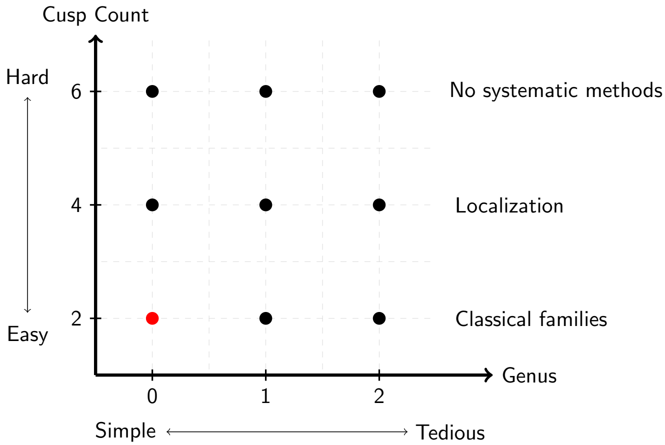
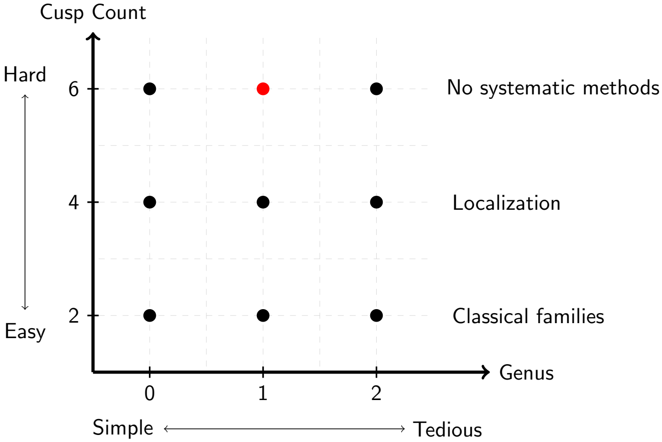


Ramanujan's congruences for $p(n)$ by powers of 5 lie here.

Classification



The Andrews–Sellers congruences lie here.

Andrews–Sellers Congruences

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If $12n \equiv 1 \pmod{5^\alpha}$, then $c\phi_2(n) \equiv 0 \pmod{5^\alpha}$.

The associated modular curve is $\mathrm{X}_0(20)$.

Proof Idea

$$L_{\alpha}^{(1)} := \frac{1}{\mathrm{C}\Phi_2\left(q^{1+4a}\right)} \cdot \sum_{12n\equiv 1 \bmod 5^{\alpha}} c\phi_2(n) q^{\lfloor n/5^{\alpha} \rfloor + 1},$$

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- $a \in \{0, 1\}$, $\alpha \equiv a \pmod 2$.

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- $a \in \{0, 1\}$, $\alpha \equiv a \pmod{2}$.
- We want to show that $L_{\alpha}^{(1)} \equiv 0 \pmod{5^{\alpha}}$.

$$L_{1}^{(1)} := \frac{(q^5; q^5)_{\infty}^4 (q^{20}; q^{20})_{\infty}^2}{(q^{10}; q^{10})_{\infty}^5} \sum_{n=0}^{\infty} c\phi_2(5n+3) q^{n+1},$$

Andrews–Sellers Congruences

$$\mathcal{M}^0\left(\mathrm{X}_0(20)\right) = \mathbb{C}[x] + y\mathbb{C}[x].$$

$$x = q\frac{(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3}{(q;q)_\infty^3 (q^5;q^5)_\infty},$$

$$y = q^2\frac{(q^2;q^2)_\infty^2 (q^4;q^4)_\infty (q^5;q^5)_\infty (q^{20};q^{20})_\infty^3}{(q;q)_\infty^5 (q^{10};q^{10})_\infty^2}.$$

$$L_1^{(1)} = \frac{5}{(1+5x)^2}\Big(4x + 137x^2 + 1704x^3 + 10080x^4 + 28800x^5$$
$$+ 32000x^6 - y\left(20 + 400x + 3040x^2 + 10240x^3 + 12800x^4\right)\Big).$$

Proof Idea

$$\mathcal{M}^0\left(\mathrm{X}_0(5)\right) = \mathbb{C}[t].$$

$$\frac{1}{5^{2\alpha-1}} L_{2\alpha-1}^{(1)} \in \mathbb{Z}[t] + p_1^{(1)}\mathbb{Z}[t]$$

$$\frac{1}{5^{2\alpha}} L_{2\alpha}^{(1)} \in \mathbb{Z}[t] + p_0^{(1)}\mathbb{Z}[t]$$

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- $a \in \{0,1\}$, $\alpha \equiv a \pmod 2$
- $t = \eta(5\tau)^6/\eta(\tau)^6$ is a modular function over $\mathrm{X}_0(5)$
- $p_a^{(1)}$ are modular functions over $\mathrm{X}_0(20)$
- $t, p_a^{(1)}$ have $\mathbb{Z}$ coefficients.

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$$L_1^{(1)} := \frac{(q^5;q^5)_\infty^4 (q^{20};q^{20})_\infty^2}{(q^{10};q^{10})_\infty^5} \sum_{n=0}^{\infty} c\phi_2(5n+3)q^{n+1}$$
$$= -5t + 25p_1^{(1)}.$$

This function was studied by Drake.

$$\sum_{n=0}^{\infty} c\psi_2(n)q^n = \frac{(q^4;q^4)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty}.$$

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Theorem (Us!)

If $6n \equiv -1 \pmod{5^\alpha}$, then $c\psi_2(n) \equiv 0 \pmod{5^\alpha}$.

This was proposed by James Sellers in 2023 and proved by Sellers and Smoot in the same year.

$$\sum_{n=0}^{\infty} c\phi_2(n) q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^4 (q^4;q^4)_\infty^2}.$$

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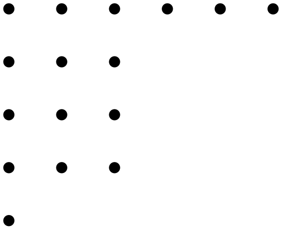
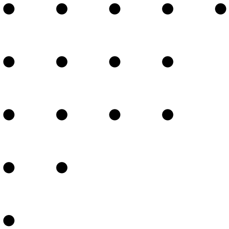
Frobenius Coordinates for Partitions



$$\begin{pmatrix} 5 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 4 & 2 & 1 \\ 4 & 2 & 0 \end{pmatrix}$$

Frobenius Coordinates for Partitions

Definition

A Frobenius array has the form

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

with the following:

- $r \geq 1$,
- Each a_i, b_j belongs to $\mathbb{Z}_{\geq 0}$,
- Each row is strictly decreasing,
- $n = r + \sum_{1 \leq i \leq r}(a_i + b_i)$.

The number of Frobenius arrays with fixed n is $p(n)$.

Generalized Frobenius Partitions

Definition

A $(2,1)$-*colored generalized Frobenius partition* of n is an array

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_r \end{pmatrix}$$

with the following:

- $r \geq 1$,
- Each a_i, b_j belongs to one of 2 copies of $\mathbb{Z}_{\geq 0}$,
- Each row is strictly decreasing (w.r.t. lexicographic ordering),
- $n = r + \sum_{1 \leq i \leq r}(a_i + b_i)$.

The number of such arrays with fixed n is $c\phi_2(n) = c\psi_{2,1}(n)$.

Generalized Frobenius Partitions

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A $(2,0)$-*colored generalized Frobenius partition* of n is an array

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with the following:

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The number of such arrays with fixed n is $c\psi_2(n) = c\psi_{2,0}(n)$.

Definition (Jiang–Rolen–Woodbury)

Let $n, k \in \mathbb{Z}_{\geq 1}$ and $\beta \in \mathbb{Z} + \frac{k}{2}$ nonnegative. A (k, β)-*colored generalized Frobenius partition* of n is an array of the form

$$\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{pmatrix}$$

with the following:

- $r + s \neq 0$ and $r - s = \beta - \frac{k}{2}$,
- Each a_i, b_j belongs to one of k copies of $\mathbb{Z}_{\geq 0}$,
- Each row is decreasing with respect to lexicographic ordering,
- $n = r + \sum_{0 \leq i \leq r} a_i + \sum_{0 \leq j \leq s} b_j$.

Denote the number of such partitions of n as $c\psi_{k,\beta}(n)$.

$$\mathrm{C}\Psi_{2,1}(q) := \sum_{n=0}^{\infty} c\psi_{2,1}(n)q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^4 (q^4;q^4)_\infty^2}.$$

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Theorem (Paule and Radu)

If $12n \equiv 1 \pmod{5^\alpha}$, then $c\psi_{2,1}(n) \equiv 0 \pmod{5^\alpha}$.

Theorem (Us!)

If $6n \equiv -1 \pmod{5^\alpha}$, then $c\psi_{2,0}(n) \equiv 0 \pmod{5^\alpha}$.

$$\mathrm{C}\Psi_{2,1}(q) = \sum_{n=0}^{\infty} c\psi_{2,1}(n) q^n = \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^4 (q^4;q^4)_\infty^2}.$$

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Theorem

If $3 \cdot 2^{\beta+1} n \equiv (-1)^{\beta+1} \pmod{5^\alpha}$, then $c\psi_{2,\beta}(n) \equiv 0 \pmod{5^\alpha}$.

Proof Idea

$$L_{\alpha}^{(\beta)} := \frac{1}{\mathrm{C}\Psi_{2,\beta}\left(q^{1+4a}\right)} \cdot \sum_{3\cdot 2^{\beta+1}n\equiv(-1)^{\beta+1} \bmod 5^{\alpha}} c\psi_{2,\beta}(n)q^{\lfloor n/5^{\alpha}\rfloor+\beta},$$

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Proof Idea for $c\psi_{2,1}(n)$ (Andrews–Sellers Congruences)

One proves that

$$\frac{1}{5^{\alpha}} L_{\alpha}^{(1)} \in \mathbb{Z}[t] + p_a^{(1)} \mathbb{Z}[t],$$

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Proof Sketch for $c\psi_{2,0}(n)$ (Sellers–Smoot Congruences)

One proves that

$$\frac{1}{5^\alpha} L_\alpha^{(0)} \in \mathbb{Z}[t] + p_a^{(0)} \mathbb{Z}[t],$$

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The First Coincidence

$$\frac{1}{5^\alpha} L_\alpha^{(1)} \in \mathbb{Z}[t] + p_a^{(1)}\mathbb{Z}[t],$$
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The proofs are (formally) *exactly identical*.

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Main Theorem

Theorem

For $\beta \in \{0,1\}$, define

$$\mathcal{R}^{(\beta)} := \mathbb{C}(t) + p_0^{(\beta)}\mathbb{C}(t) + p_1^{(\beta)}\mathbb{C}(t),$$

and the mappings

$$\sigma^{(\beta)} : \mathcal{R}^{(\beta)} \longrightarrow \mathcal{R}^{(1-\beta)}$$
$$: \begin{cases} t \longmapsto t, \\ p_a^{(\beta)} \longmapsto p_a^{(1-\beta)}. \end{cases}$$

Then

$$\sigma^{(\beta)}\left(L_\alpha^{(\beta)}\right) = L_\alpha^{(1-\beta)}.$$

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Then

$$\sigma^{(\beta)}\left(L_\alpha^{(\beta)}\right) = L_\alpha^{(1-\beta)}.$$

Corollary

For all $\alpha \geq 1$, $L_\alpha^{(0)} \equiv 0 \pmod{5^\alpha}$.

What is $\sigma^{(\beta)}$?

We have:

$$\mathcal{R}^{(\beta)} := \mathbb{C}(t) + p_0^{(\beta)}\mathbb{C}(t) + p_1^{(\beta)}\mathbb{C}(t),$$

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where $\sigma^{(\beta)}$ fixes $\mathbb{C}(t)$ and sends $p_a^{(\beta)} \longmapsto p_a^{(1-\beta)}$.

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where $\sigma^{(\beta)}$ fixes $\mathbb{C}(t)$ and sends $p_a^{(\beta)} \longmapsto p_a^{(1-\beta)}$.

We can prove that for any $f \in \mathcal{R}^{(1)}$,

$$\sigma^{(1)}(f(\tau)) = f(\gamma_0\tau),$$

with

$$\gamma_0 = \begin{pmatrix} 27 & 7 \\ 50 & 13 \end{pmatrix} \in \Gamma_0(10).$$

Equivalence of Congruence Families

$$L_{\alpha}^{(1)} := \frac{1}{\mathrm{C}\Phi_2\left(q^{5-4a}\right)} \cdot \sum_{12n\equiv 1 \bmod 5^{\alpha}} c\phi_2(n)q^{\lfloor n/5^{\alpha}\rfloor+1},$$

$$L_{\alpha}^{(0)} := \frac{1}{\mathrm{C}\Psi_2\left(q^{5-4a}\right)} \cdot \sum_{6n\equiv -1 \bmod 5^{\alpha}} c\psi_2(n)q^{\lfloor n/5^{\alpha}\rfloor},$$

Theorem

For all $\alpha \geq 1$, $L_{\alpha}^{(0)} = L_{\alpha}^{(1)}(\gamma_0\tau)$.

Summary

- We have two congruence families associated with a sequence of functions $\left(L_{\alpha}^{(1)}\right)_{\alpha\geq 1}$, $\left(L_{\alpha}^{(0)}\right)_{\alpha\geq 1}$ over $\mathrm{X}_0(20)$.

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- $\sigma^{(1)}$ *fixes* the functions in the subfield $\mathcal{M}\left(\mathrm{X}_0(5)\right)$.

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- $\sigma^{(1)}$ *fixes* the functions in the subfield $\mathcal{M}\left(\mathrm{X}_0(5)\right)$.
- This can only happen when the families in question are associated with $\mathrm{X}_0(N)$ for *composite* N.
- This is all trivial for *prime* N.

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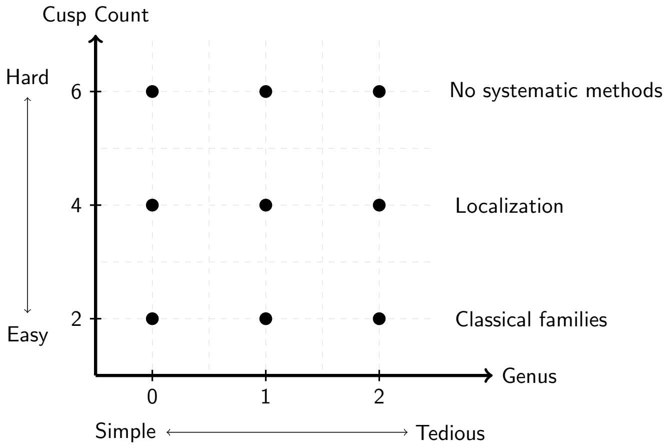
- $\sigma^{(1)}$ *fixes* the functions in the subfield $\mathcal{M}\left(\mathrm{X}_0(5)\right)$.
- This can only happen when the families in question are associated with $\mathrm{X}_0(N)$ for *composite* N.
- This is all trivial for *prime* N.
- $\mathrm{X}_0(N)$ has a large number of cusps when N has a lot of divisors.

Classification



Classifying congruence families by the topology of the associated modular curve $\mathrm{X}_0(N)$

(N.A. Smoot, "On the Classification of Modular Congruence Families," (2024), https://arxiv.org/abs/2403.10681)

Summary

- We have two congruence families associated with a sequence of functions $\left(L_{\alpha}^{(1)}\right)_{\alpha\geq 1}$, $\left(L_{\alpha}^{(0)}\right)_{\alpha\geq 1}$ over $\mathrm{X}_0(20)$.
- We can map $L_{\alpha}^{(1)}$ to $L_{\alpha}^{(0)}$ using an involution $\sigma^{(1)}$.
- $\sigma^{(1)}$ *fixes* the functions in the subfield $\mathcal{M}\left(\mathrm{X}_0(5)\right)$.
- This can only happen when the families in question are associated with $\mathrm{X}_0(N)$ for *composite* N.
- This is all trivial for *prime* N.
- $\mathrm{X}_0(N)$ has a large number of cusps when N has a lot of divisors.

What does this mean?

The Second Coincidence

$$\frac{1}{5^\alpha} L_\alpha^{(\beta)} \in \mathbb{Z}[t] + p_a^{(\beta)} \mathbb{Z}[t],$$

with

- $a \in \{0, 1\}$, $\alpha \equiv a \pmod 2$
- $t = \eta(5\tau)^6/\eta(\tau)^6$ is a modular function over $\mathrm{X}_0(5)$
- $p_a^{(1)}$ are modular functions over $\mathrm{X}_0(20)$
- $t, p_a^{(1)}$ have $\mathbb{Z}$ coefficients.

$$\mathcal{M}^0\left(\mathrm{X}_0(20)\right) = \mathbb{C}[x] + y\mathbb{C}[x].$$

$$x = q\frac{(q^2;q^2)_\infty(q^{10};q^{10})_\infty^3}{(q;q)_\infty^3(q^5;q^5)_\infty},$$

$$y = q^2\frac{(q^2;q^2)_\infty^2(q^4;q^4)_\infty(q^5;q^5)_\infty(q^{20};q^{20})_\infty^3}{(q;q)_\infty^5(q^{10};q^{10})_\infty^2}.$$

The Second Coincidence

$$\frac{1}{5^{\alpha}} L_{\alpha}^{(1)} \in \mathbb{Z}[t] + p_a^{(1)} \mathbb{Z}[t],$$

$$\begin{aligned} p_1^{(1)} =& \frac{1}{(1+5x)^2} \Big(x + 30x^2 + 352x^3 + 2032x^4 + 5760x^5 + 6400x^6 \\ & - 4y - 80xy - 608x^2y - 2048x^3y - 2560x^4y \Big), \\ p_0^{(1)} =& \frac{1}{(1+5x)^2} \Big(61x + 1426x^2 + 13520x^3 + 65200x^4 + 160000x^5 \\ & + 160000x^6 - 4y - 48xy - 160x^2y \Big). \end{aligned}$$

The Second Coincidence

$$\frac{1}{5^{\alpha}} L_{\alpha}^{(0)} \in \mathbb{Z}[t] + p_a^{(0)} \mathbb{Z}[t],$$

$$p_1^{(0)} = \frac{1}{(1+5x)^2}\Big(1 + 27x + 302x^2 + 1776x^3 + 5744x^4 + 9600x^5$$
$$+ 6400x^6 + 4y + 80xy + 608x^2y + 2048x^3y + 2560x^4y\Big),$$
$$p_0^{(0)} = \frac{1}{(1+5x)^2}\Big(1 + 79x + 1538x^2 + 13760x^3 + 65200x^4 + 160000x^5$$
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The Second Coincidence

$$p_a^{(1)} + p_a^{(0)} \in \mathbb{Z}[x]_{\mathcal{S}}.$$

$$x = q\frac{(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3}{(q;q)_\infty^3 (q^5;q^5)_\infty}.$$

$$L_\alpha^{(1)} + L_\alpha^{(0)} \in \mathbb{Z}[x]_{\mathcal{S}} \subseteq \mathcal{M}\left(\mathrm{X}_0(10)\right).$$

Conjecture

A proof of

$$L_\alpha^{(1)} + L_\alpha^{(0)} \equiv 0 \pmod{5^\alpha}$$

is accessible by the localization method.

The Third Coincidence

To prove the Andrews–Sellers congruences, Paule and Radu show that

$$\frac{1}{5^\alpha} L_\alpha^{(1)} \in \mathbb{Z}[t] + p_a^{(1)}\mathbb{Z}[t],$$

with

- $a \in \{0,1\}$, $\alpha \equiv a \pmod 2$
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How are $p_0^{(1)}, p_1^{(1)}$ found?

The Third Coincidence

$$W := \begin{pmatrix} 4 & -1 \\ 100 & -24 \end{pmatrix}.$$

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Paule and Radu discovered that

$$\frac{(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^2} L_{2\alpha-1}^{(1)}\left(W(\tau)\right) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].$$

$$\frac{(q^{2};q^{2})_\infty^5}{(q;q)_\infty^2} L_{2\alpha}^{(1)}\left(W(\tau)\right) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].$$

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They then construct $p_0^{(1)}, p_1^{(1)}$ with a similar behavior with respect to W.

The Third Coincidence

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$$\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} L_{2\alpha}^{(0)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].$$

This is one way to find $p_0^{(0)}, p_1^{(0)}$.

The Third Coincidence

$$\frac{(q^{10};q^{10})_\infty^5}{(q^5;q^5)_\infty^2}\cdot L_{2\alpha-1}^{(1)}(W(\tau))\in\mathbb{Z}[[q^4]]+q\mathbb{Z}[[q^4]],$$
$$\frac{(q^{2};q^{2})_\infty^5}{(q;q)_\infty^2}\cdot L_{2\alpha}^{(1)}(W(\tau))\in\mathbb{Z}[[q^4]]+q\mathbb{Z}[[q^4]],$$
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$$\frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2} \cdot L_{2\alpha}^{(1)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]],$$

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James discovered that

$$\varphi(q) \cdot L_{2\alpha}^{(1)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]],$$
$$\varphi(-q) \cdot L_{2\alpha}^{(0)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]].$$

$$\varphi(q) = \frac{(q^2;q^2)_\infty^5}{(q^4;q^4)_\infty^2 (q;q)_\infty^2}, \quad \varphi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}.$$

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$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

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$$\begin{aligned}\varphi(q^5)\cdot L_{2\alpha-1}^{(1)}\left(W(\tau)\right),\\ \varphi(q)\cdot L_{2\alpha}^{(1)}\left(W(\tau)\right),\\ \varphi(-q^5)\cdot L_{2\alpha-1}^{(0)}\left(W(\tau)\right),\\ \varphi(-q)\cdot L_{2\alpha}^{(0)}\left(W(\tau)\right)\\ \in \mathbb{Z}[[q^4]]+q\mathbb{Z}[[q^4]].\end{aligned}$$

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The Third Coincidence

Theorem

For $\alpha \in \mathbb{Z}_{\geq 1}$, $a, \beta \in \{0, 1\}$, $a \equiv \alpha \pmod 2$, *we have*

$$\varphi\left((-1)^{\beta+1} q^{1+4a}\right) \cdot L_{\alpha}^{(\beta)}(W(\tau)) \in \mathbb{Z}[[q^4]] + q\mathbb{Z}[[q^4]],$$

in which $\varphi(q)$ *is Ramanujan's phi function.*

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Conjecture

$$\sum_{\beta=0,1} \varphi\left((-1)^{\beta+1} q^{1+4a}\right) L_{\alpha}^{(\beta)}(W\tau) \in \mathbb{Z}[[q^4]].$$

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Thank you!