

# Joint Distributions of Hook Lengths in Integer Partitions

Philip Cuthbertson

February 26, 2026

# Definitions

- An *(integer) partition*,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ , is a weakly decreasing sequence of non-negative integers,
- $\lambda$  is a *partition of  $n$*  if  $\lambda_1 + \lambda_2 + \dots + \lambda_r = n$ ,
- For example,  $(7, 7, 4, 2, 1)$  is a partition of 21,
- Each  $\lambda_i$  is called a part of  $\lambda$ ,
- We also sometimes write  $\lambda \vdash n$ .

# Definitions

- An *(integer) partition*,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ , is a weakly decreasing sequence of non-negative integers,
- $\lambda$  is a *partition of  $n$*  if  $\lambda_1 + \lambda_2 + \dots + \lambda_r = n$ ,
- For example,  $(7, 7, 4, 2, 1)$  is a partition of 21,
- Each  $\lambda_i$  is called a part of  $\lambda$ ,
- We also sometimes write  $\lambda \vdash n$ .

# Definitions

- An *(integer) partition*,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ , is a weakly decreasing sequence of non-negative integers,
- $\lambda$  is a *partition of  $n$*  if  $\lambda_1 + \lambda_2 + \dots + \lambda_r = n$ ,
- For example,  $(7, 7, 4, 2, 1)$  is a partition of 21,
- Each  $\lambda_i$  is called a part of  $\lambda$ ,
- We also sometimes write  $\lambda \vdash n$ .

# Definitions

- An (*integer*) *partition*,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ , is a weakly decreasing sequence of non-negative integers,
- $\lambda$  is a *partition of  $n$*  if  $\lambda_1 + \lambda_2 + \dots + \lambda_r = n$ ,
- For example,  $(7, 7, 4, 2, 1)$  is a partition of 21,
- Each  $\lambda_i$  is called a part of  $\lambda$ ,
- We also sometimes write  $\lambda \vdash n$ .

# Definitions

- An (*integer*) *partition*,  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ , is a weakly decreasing sequence of non-negative integers,
- $\lambda$  is a *partition of  $n$*  if  $\lambda_1 + \lambda_2 + \dots + \lambda_r = n$ ,
- For example,  $(7, 7, 4, 2, 1)$  is a partition of 21,
- Each  $\lambda_i$  is called a part of  $\lambda$ ,
- We also sometimes write  $\lambda \vdash n$ .

We use the usual *q*-Pochhammer symbol shorthand,

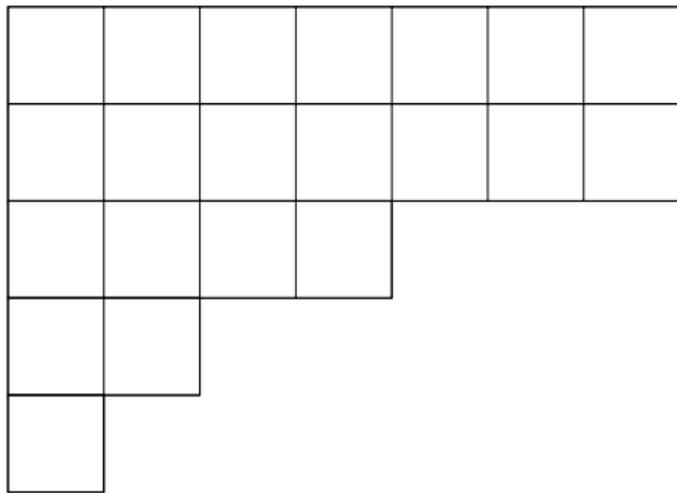
$$(a; b)_n := \prod_{t=0}^{n-1} (1 - ab^t),$$

and

$$(a; b)_\infty := \prod_{t=0}^{\infty} (1 - ab^t).$$

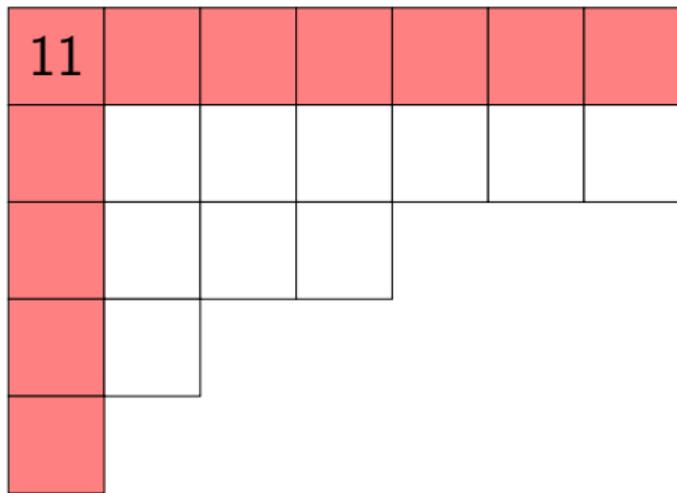
# Hook Lengths

$$\lambda = (7, 7, 4, 2, 1)$$



# Hook Lengths

$$\lambda = (7, 7, 4, 2, 1)$$



# Hook Lengths

$$\lambda = (7, 7, 4, 2, 1)$$

11						
		6				

# Hook Lengths

$$\lambda = (7, 7, 4, 2, 1)$$

11	9	7	6	4	3	2
10	8	6	5	3	2	1
6	4	2	1			
3	1					
1						

# Joint Distributions of Hook Lengths

Now, let  $\mathcal{H}_k(\lambda)$  denote the total number of hooks of length  $k$  in the partition  $\lambda$ . For example,

- $\mathcal{H}_1(\lambda) = 4$ ,
- $\mathcal{H}_6(\lambda) = 3$ ,
- $\mathcal{H}_6(\lambda) \cdot \mathcal{H}_1(\lambda) = 12$ .

11	9	7	6	4	3	2
10	8	6	5	3	2	1
6	4	2	1			
3	1					
1						

## Theorem (Han (2010))

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} z^{\mathcal{H}_k(\lambda)} q^n = \prod_{t=1}^{\infty} \frac{(1 - (1 - z)q^{kt})^k}{1 - q^t} = \frac{((1 - z)q^k; q^k)_{\infty}^k}{(q; q)_{\infty}}.$$

If we apply  $\frac{\partial}{\partial z}$  and then set  $z = 1$  we obtain the following:

## Corollary

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) q^n = \frac{kq^k}{(1 - q^k)(q; q)_{\infty}}.$$

## Theorem (Anible-Keith (2018))

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_2(\lambda) \cdot \mathcal{H}_1(\lambda) q^n = 2 \cdot \frac{q^2 + q^4 + q^5}{(1 - q^2)(1 - q^3)(q; q)_{\infty}}.$$

# Generalization

We want to generalize this to something of the form

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) \cdot \mathcal{H}_1(\lambda) q^n = ?$$

We will do this by taking a sum over all ways that a hook of size  $k$  can appear while keeping track of number of part sizes (hooks of size 1).

# Sketch of Proof (of a special case)

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) z^{\mathcal{H}_1(\lambda)} q^n$$

- For simplicity, we will focus on just the case  $k = 3$ ,
- There are only four ways that a hook of size 3 can appear in a partition,
- We will figure out the generating function in each case and sum all of them together.

## Sketch of Proof (of a special case)

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) z^{\mathcal{H}_1(\lambda)} q^n$$

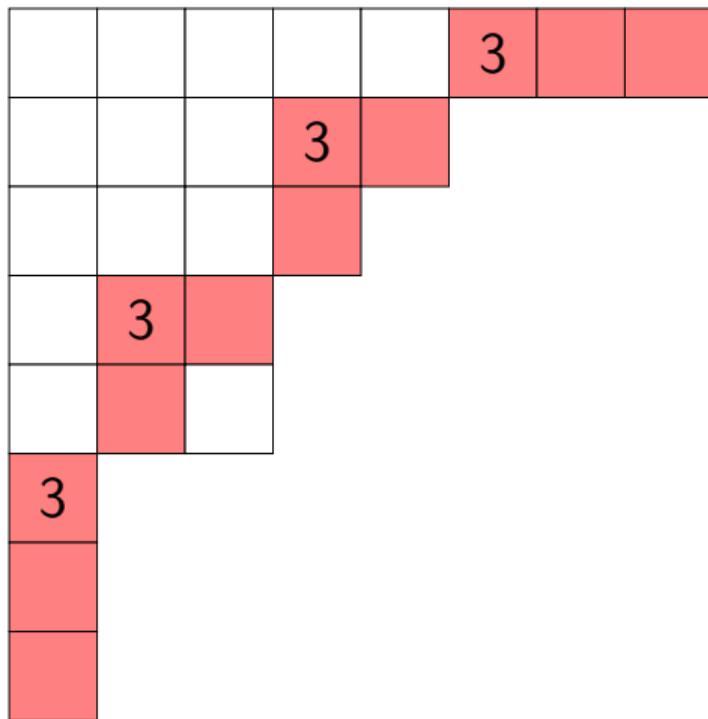
- For simplicity, we will focus on just the case  $k = 3$ ,
- There are only four ways that a hook of size 3 can appear in a partition,
- We will figure out the generating function in each case and sum all of them together.

## Sketch of Proof (of a special case)

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) z^{\mathcal{H}_1(\lambda)} q^n$$

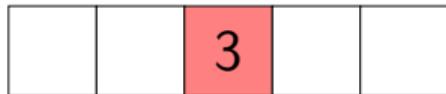
- For simplicity, we will focus on just the case  $k = 3$ ,
- There are only four ways that a hook of size 3 can appear in a partition,
- We will figure out the generating function in each case and sum all of them together.

Consider the partition  $(8, 5, 5, 3, 2, 1, 1, 1)$ :





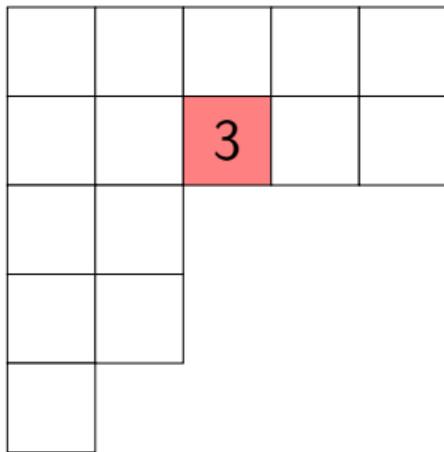
$$q^3$$



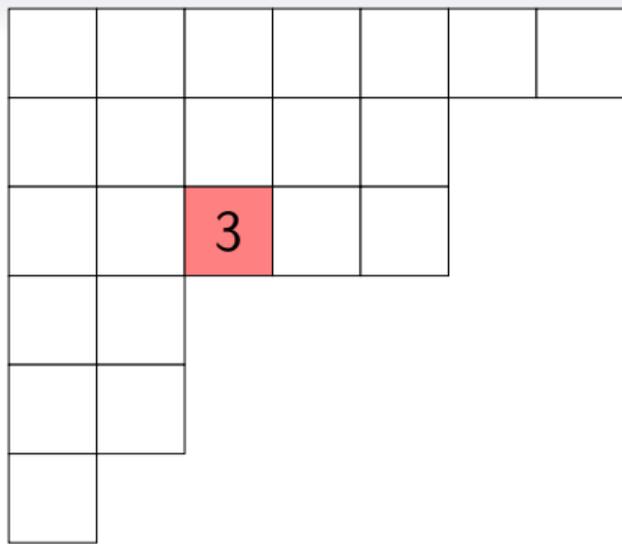
$$\sum_{t=0}^{\infty} q^{t+3}$$

		3		

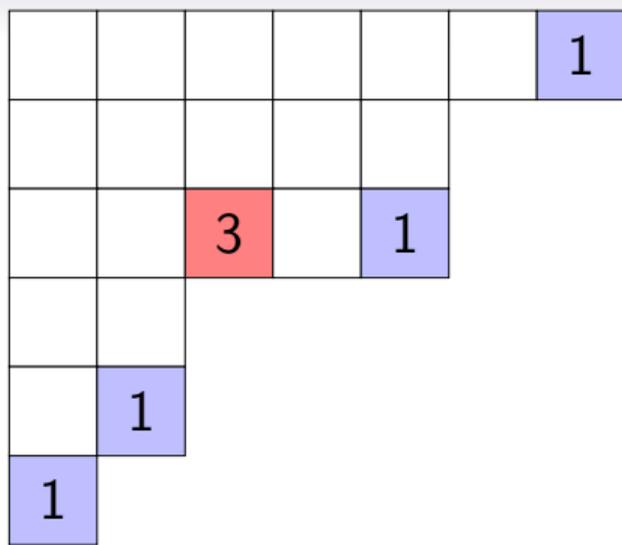
$$\sum_{t=0}^{\infty} \frac{q^{t+3}}{1 - q^{t+3}}$$



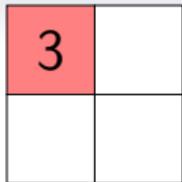
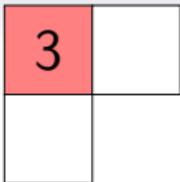
$$\sum_{t=0}^{\infty} \frac{q^{t+3}}{1 - q^{t+3}} \prod_{j=1}^t \left( 1 + \frac{q^j}{1 - q^j} \right)$$



$$\sum_{t=0}^{\infty} \frac{q^{t+3}}{1 - q^{t+3}} \prod_{j=1}^t \left( 1 + \frac{q^j}{1 - q^j} \right) \prod_{j=t+4}^{\infty} \left( 1 + \frac{q^j}{1 - q^j} \right)$$



$$\sum_{t=0}^{\infty} \frac{zq^{t+3}}{1-q^{t+3}} \prod_{j=1}^t \left(1 + \frac{zq^j}{1-q^j}\right) \prod_{j=t+4}^{\infty} \left(1 + \frac{zq^j}{1-q^j}\right)$$



$$\sum_{t=0}^{\infty} \frac{z^2 q^{2t+3}}{1-q^{t+2}} \prod_{j=1}^t \left(1 + \frac{zq^j}{1-q^j}\right) \prod_{j=t+3}^{\infty} \left(1 + \frac{zq^j}{1-q^j}\right)$$

$$\sum_{t=0}^{\infty} \frac{z^1 q^{2t+4}}{1-q^{t+2}} \prod_{j=1}^t \left(1 + \frac{zq^j}{1-q^j}\right) \prod_{j=t+3}^{\infty} \left(1 + \frac{zq^j}{1-q^j}\right)$$

$$\sum_{t=0}^{\infty} \frac{z^1 q^{3t+3}}{1-q^{t+1}} \prod_{j=1}^t \left(1 + \frac{zq^j}{1-q^j}\right) \prod_{j=t+2}^{\infty} \left(1 + \frac{zq^j}{1-q^j}\right)$$

The sum of these four terms gives the generating function for

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_3(\lambda) z^{\mathcal{H}_1(\lambda)} q^n.$$

Taking the derivative and setting  $z$  equal to 1 will give

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_3(\lambda) \cdot \mathcal{H}_1(\lambda) q^n.$$

Applying  $\frac{\partial}{\partial z}$ , setting  $z = 1$ , and combining each case gives:

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_3(\lambda) \cdot \mathcal{H}_1(\lambda) q^n = \frac{4q^3 - 5q^4 + 5q^6 - 4q^7 - 3q^8 + 3q^{10}}{(q; q)_4 (q; q)_{\infty}}.$$

## Theorem

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) z^{\mathcal{H}_1(\lambda)} q^n \\ &= \sum_{t=0}^{\infty} \sum_{\{\mu: h_{1,1}=k\}} \frac{z^{\mathcal{H}_1(\mu)} q^{\ell(\mu)t+|\mu|}}{1 - q^{t+\mu_1}} \prod_{\substack{j=1 \\ j \neq t+1, \dots, t+\mu_1}}^{\infty} \left( 1 + \frac{zq^j}{1 - q^j} \right). \end{aligned}$$

## Corollary

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) \cdot \mathcal{H}_1(\lambda) q^n \\ &= \frac{1}{(1-q)(q; q)_{\infty}} \sum_{t=0}^{\infty} \sum_{\{\mu: h_{1,1}=k\}} q^{\ell(\mu)t+|\mu|} (q^{t+1}; q)_{\mu_1-1} \\ & \quad \cdot (\mathcal{H}_1(\mu) + q - \mathcal{H}_1(\mu)q - q^{t+1} + q^{t+\mu_1+1}). \end{aligned}$$

For  $k = 1, 2, 3, 4$  we get:

$$\frac{q + q^3}{(q; q)_{1+1}(q; q)_{\infty}},$$

$$2 \cdot \frac{q^2 - q^3 + q^4 - q^6}{(q; q)_{2+1}(q; q)_{\infty}},$$

$$\frac{4q^3 - 5q^4 + 5q^6 - 4q^7 - 3q^8 + 3q^{10}}{(q; q)_{3+1}(q; q)_{\infty}},$$

$$2 \cdot \frac{3q^4 - 3q^5 - 3q^6 + 3q^7 + 3q^8 - 3q^9 - 5q^{10} + 3q^{11} + 2q^{12} + 2q^{13} - 2q^{15}}{(q; q)_{4+1}(q; q)_{\infty}}.$$

## Theorem

We have,

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) \cdot \mathcal{H}_1(\lambda) q^n = \frac{f(q)}{(q; q)_{k+1} (q; q)_{\infty}}$$

where  $f$  is a polynomial of degree  $T_{k+1}$  and order  $k$ . Specifically,  $f$  has leading term  $(-1)^{k+1} k q^{T_{k+1}}$ .

## Conjecture

For any multiset of natural numbers,  $\mathcal{I}$ , let  $\sum_{i \in \mathcal{I}} i = S$  and  $\prod_{i \in \mathcal{I}} i = P$ . We then have,

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \prod_{i \in \mathcal{I}} \mathcal{H}_i(\lambda) q^n = \frac{f(q)}{(q; q)_S (q; q)_{\infty}}$$

where  $f$  is a polynomial of degree  $T_S$  and order  $\max \mathcal{I}$ . Specifically,  $f$  has leading term  $(-1)^S P q^{T_S}$ .

# Example

We (conjecturally) have,

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_4(\lambda) \cdot \mathcal{H}_3(\lambda) \cdot \mathcal{H}_2(\lambda) \cdot \mathcal{H}_1(\lambda) q^n$$
$$= 2 \cdot \frac{q^4 + q^5 + 3q^7 + 5q^8 - 8q^9 + \dots - 3q^{51} + 8q^{52} - 12q^{53} + 12q^{55}}{(q; q)_{\infty} (q; q)_{4+3+2+1}}$$

We can also build an expression for

## Theorem

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) z_2^{\mathcal{H}_2(\lambda)} z_1^{\mathcal{H}_1(\lambda)} q^n \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\{\mu: h_{1,1}=\mu\}} z_2^{\mathcal{H}_2(\mu)} z_1^{\mathcal{H}_1(\mu)} q^{ij+\mu_1 i+\ell(\mu)j+|\mu|} d_i e_j. \end{aligned}$$

The  $d_i$  and  $e_j$  have some complicated recursive definition.

We construct this function in a similar manner:

- Sum over all ways that a hook of size  $k$  can appear and keep track of the hooks of size 1 and 2.
- There are two cases that we have to worry about: when adding an additional part removes a 2-hook and when it can't.

We construct this function in a similar manner:

- Sum over all ways that a hook of size  $k$  can appear and keep track of the hooks of size 1 and 2.
- There are two cases that we have to worry about: when adding an additional part removes a 2-hook and when it can't.

We construct this function in a similar manner:

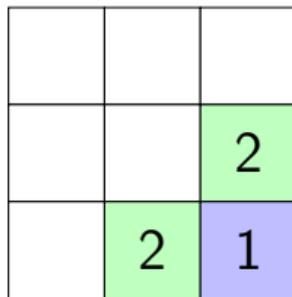
- Sum over all ways that a hook of size  $k$  can appear and keep track of the hooks of size 1 and 2.
- There are two cases that we have to worry about: when adding an additional part removes a 2-hook and when it can't.

- We use  $a_i$  as shorthand for  $z_1 q^i + \frac{z_1 z_2 q^{2i}}{1-q^i}$
- We use  $b_i$  to denote the generating function for the case that “adding” a part of size  $i$  removes a 2-hook,
- We use  $c_i$  to denote the generating function for the case that “adding” a part of size  $i$  does not remove a 2-hook.

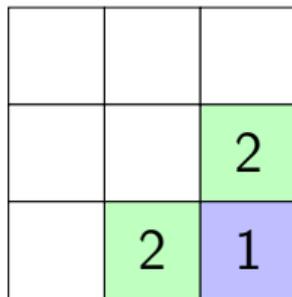
2
1

		2
	2	1

- We use  $a_i$  as shorthand for  $z_1 q^i + \frac{z_1 z_2 q^{2i}}{1-q^i}$
- We use  $b_i$  to denote the generating function for the case that “adding” a part of size  $i$  removes a 2-hook,
- We use  $c_i$  to denote the generating function for the case that “adding” a part of size  $i$  does not remove a 2-hook.



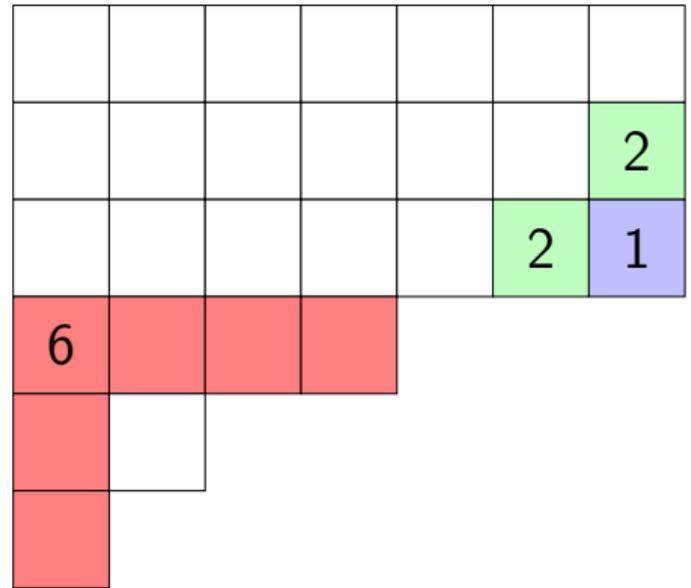
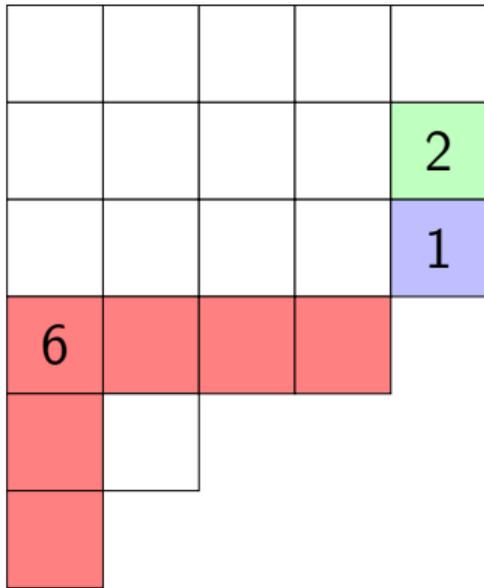
- We use  $a_i$  as shorthand for  $z_1 q^i + \frac{z_1 z_2 q^{2i}}{1-q^i}$
- We use  $b_i$  to denote the generating function for the case that “adding” a part of size  $i$  removes a 2-hook,
- We use  $c_i$  to denote the generating function for the case that “adding” a part of size  $i$  does not remove a 2-hook.



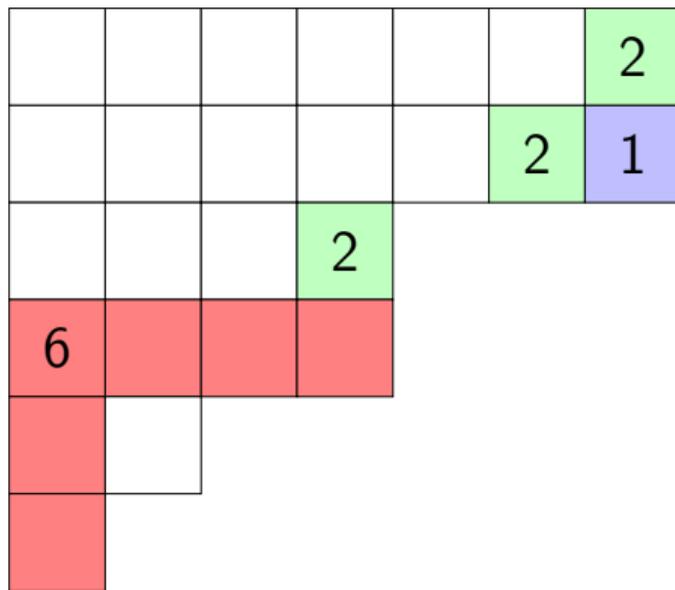
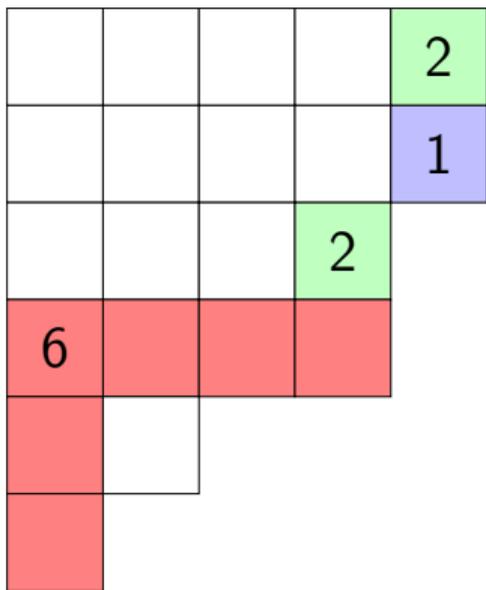
# Building $b_i$

			2
6			

# Building $b_i$



# Building $b_i$



Putting these together we get

$$b_k = a_k b_{k-1} + z_2 c_{k-1}$$

together with the initial condition

$$b_0 = 1.$$

# Building $c_i$

6			

# Building $c_j$

				2
				1
6				

						2
					2	1
6						

Putting these together we get

$$c_k = c_{k-1} + a_k b_{k-1}$$

together with the initial condition

$$c_0 = 1.$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_k(\lambda) z_2^{\mathcal{H}_2(\lambda)} z_1^{\mathcal{H}_1(\lambda)} q^n \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\{\mu: h_{1,1}=\mu\}} z_2^{\mathcal{H}_2(\mu)} z_1^{\mathcal{H}_1(\mu)} q^{ij+\mu_1 i + \ell(\mu)j + |\mu|} d_i e_j.
\end{aligned}$$

Where

$$d_i = \begin{cases} b_i, & \text{if } \mu_1 \neq \mu_2 \\ c_i, & \text{if } \mu_1 = \mu_2 \end{cases}$$

and

$$e_j = \begin{cases} b_j, & \text{if } \mu'_1 \neq \mu'_2 \\ c_j, & \text{if } \mu'_1 = \mu'_2 \end{cases}.$$

# Sketch of Proof (of a special case)

We now sketch a proof that the sum

$$\sum_{n=0}^{\infty} \sum_{\lambda \vdash n} \mathcal{H}_3(\lambda) \mathcal{H}_2(\lambda) q^n$$

is equal to

$$\frac{2q^3 - 2q^4 + 2q^6 - 2q^7 - q^8 + 3q^9 - 12q^{10} + 10q^{11} - q^{12} + q^{13} + 6q^{14} - 6q^{15}}{(q; q)_5 (q; q)_{\infty}}.$$

			3	
		2		
	3			
2				

				2
		3		
	3			
2				

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\{\mu: h_{1,1}=k\}} z_2^{\mathcal{H}_2(\mu)} z_1^{\mathcal{H}_1(\mu)} q^{ij+\mu_1 i+\ell(\mu)j+|\mu|} d_i e_j$$

We will also just focus on  $\mu = (3)$ . The corresponding summand is

$$S := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} z_2 z_1 q^{ij+3 \cdot i+1 \cdot j+3} b_i c_j$$

We set  $z_1 = 1$ , take the partial derivative with respect to  $z_2$ , then set  $z_2 = 1$ .

It can be shown that

$$b_m|_{z_1, z_2=1} = c_m|_{z_1, z_2=1} = \frac{1}{(q; q)_m}.$$

Also,

$$\frac{\partial}{\partial z_2} b_i|_{z_1, z_2=1} = \frac{1 + q^2 - q^i - q^{i+1} + q^{2i+1} - q^{2i+2}}{(1 - q^2)(q; q)_i}$$

and

$$\frac{\partial}{\partial z_2} c_j|_{z_1, z_2=1} = \frac{1}{1 - q^2} \sum_{\ell=0}^j \frac{q^\ell + q^{\ell+2} - q^{2\ell-1} - q^{2\ell+2} - q^{3\ell} + q^{3\ell-1}}{(q; q)_\ell}.$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial z_2} S|_{z_1, z_2=1} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{ij+3i+j+3}}{(q; q)_i (q; q)_j} \\ &+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{ij+3i+j+3}}{(q; q)_j} b'_i \\ &+ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q^{ij+3i+j+3}}{(q; q)_i} c'_j \\ &=: S_1 + S_2 + S_3. \end{aligned}$$

$$\begin{aligned} S_1 &= \sum_{i=0}^{\infty} \frac{q^{3i+3}}{(q; q)_i} \sum_{j=0}^{\infty} \frac{q^{j(i+1)}}{(q; q)_j} \\ &= \sum_{i=0}^{\infty} \frac{q^{3i+3}}{(q; q)_i} \cdot \frac{1}{(q^{i+1}; q)_{\infty}} \\ &= \frac{1}{(q; q)_{\infty}} \sum_{i=0}^{\infty} q^{3i+3} \\ &= \frac{q^3}{(1 - q^3)(q; q)_{\infty}}. \end{aligned}$$

$$\begin{aligned}
S_2 &= \sum_{i=0}^{\infty} \frac{q^{3i+3}(1+q^2-q^i-q^{i+1}+q^{2i+1}-q^{2i+2})}{(1-q^2)(q; q)_i} \sum_{j=0}^{\infty} \frac{q^{j(i+1)}}{(q; q)_j} \\
&= \frac{q^3}{(1-q^2)(q; q)_{\infty}} \sum_{i=0}^{\infty} q^{3i}(1+q^2-q^i-q^{i+1}+q^{2i+1}-q^{2i+2}) \\
&= \frac{q^3}{(1-q^2)(q; q)_{\infty}} \cdot \left( \frac{1+q^2}{1-q^3} - \frac{1+q}{1-q^4} + \frac{q-q^2}{1-q^5} \right)
\end{aligned}$$

The crux of evaluating  $S_3$  is to evaluate each of the following

$$\sum_{l=0}^j \frac{q^l}{(q; q)_l} = \frac{1}{(q; q)_j},$$

$$\sum_{l=0}^j \frac{q^{2l}}{(q; q)_l} = \frac{1 - q + q^{j+1}}{(q; q)_j},$$

$$\sum_{l=0}^j \frac{q^{3l}}{(q; q)_l} = \frac{1 - q - q^2 + q^3 + q^{j+1} - q^{j+3} + q^{2j+2}}{(q; q)_j}.$$

As a quick side note, if we define

$$f_{m,j} := \sum_{\ell=0}^j \frac{q^{m\ell}}{(q; q)_\ell} = \frac{(q; q)_{m-1} + q^{j+1}(\dots)}{(q; q)_j},$$

then  $f$  satisfies the following recurrence:

$$f_{m,j} = f_{m-1,j} - q^{m-1}f_{m-1,j-1}.$$

With

$$f_{1,j} = \frac{1}{(q; q)_j} \text{ and } f_{m,0} = 1.$$

An amount of algebra will then give:

$$S_3 = \frac{1}{(1 - q^2)(q; q)_\infty} \left( \frac{2q^5}{1 - q} - \frac{2q^7 + 2q^6 + q^5 + q^4}{1 - q^2} \right. \\ \left. + \frac{2q^8 + q^7 + 2q^6 + q^4}{1 - q^3} - \frac{q^8 + q^5}{1 - q^4} + \frac{q^7 - q^8}{1 - q^5} \right).$$

Now do the same thing for the other three hook shapes and combine to yield:

$$\frac{2q^3 - 2q^4 + 2q^6 - 2q^7 - q^8 + 3q^9 - 12q^{10} + 10q^{11} - q^{12} + q^{13} + 6q^{14} - 6q^{15}}{(q; q)_5(q; q)_\infty}.$$

# Theorem

$$\begin{aligned}
 \sum_{n=0}^{\infty} \sum_{\substack{\lambda \vdash n \\ \lambda \in \mathcal{O}}} \mathcal{H}_k(\lambda) \cdot \mathcal{H}_1(\lambda) q^n = \\
 \frac{1}{(1-q^2)(q; q^2)_{\infty}} \sum_{t=0}^{\infty} \sum_{\{\mu: h_{1,1}=k, \mu \in \mathcal{E}\}} q^{(2t+1)\ell(\mu)+|\mu|} (q^{2t+3}; q^2)_{\frac{\mu_1}{2}-1} \\
 \cdot (\mathcal{H}_1(\mu) - \mathcal{H}_1(\mu)q^2 + q - q^{2t+3} + q^{2t+3+\mu_1}) \\
 + \frac{1}{(1-q^2)(q; q^2)_{\infty}} \sum_{t=0}^{\infty} \sum_{\{\mu: h_{1,1}=k, \mu \in \mathcal{O}\}} q^{2t\ell(\mu)+|\mu|} (q^{2t+1}; q^2)_{\frac{\mu_1-1}{2}} \\
 \cdot (\mathcal{H}_1(\mu) - \mathcal{H}_1(\mu)q^2 + q - q^{2t+1} + q^{2t+2+\mu_1}).
 \end{aligned}$$

# Theorem

$$\sum_{n=0}^{\infty} \sum_{\substack{\lambda \vdash n \\ \lambda \in \mathcal{D}}} \mathcal{H}_k(\lambda) \cdot \mathcal{H}_1(\lambda) q^n =$$
$$(-q; q)_{\infty} \sum_{t=0}^{\infty} \sum_{\{\mu: h_{1,1}=\mu, \mu \in \mathcal{D}\}} \frac{q^{\ell(\mu)t+|\mu|}}{(-q^{t+1}; q)_{\mu_1}} \cdot \left( \ell(\mu) + \sum_{\substack{j=1 \\ j \neq t+1, \dots, t+\mu_1}}^{\infty} \frac{q^j}{1+q^j} \right).$$

# Future Work

- In principal, the same method would work for larger hooks, but the number of summands grows as  $2^{k-1}$  and would be tedious.
- Find a meaning behind those polynomials/rational functions and “skip” the algebra to the end.
- The only, apparent, congruences are when every coefficient in the series has a common factor. It appears that the coefficients are divisible by the gcd of the hook lengths being considered. Also, if any hook length is even then the series is even because of conjugation.

## Future Work

- In principal, the same method would work for larger hooks, but the number of summands grows as  $2^{k-1}$  and would be tedious.
- Find a meaning behind those polynomials/rational functions and “skip” the algebra to the end.
- The only, apparent, congruences are when every coefficient in the series has a common factor. It appears that the coefficients are divisible by the gcd of the hook lengths being considered. Also, if any hook length is even then the series is even because of conjugation.

## Future Work

- In principal, the same method would work for larger hooks, but the number of summands grows as  $2^{k-1}$  and would be tedious.
- Find a meaning behind those polynomials/rational functions and “skip” the algebra to the end.
- The only, apparent, congruences are when every coefficient in the series has a common factor. It appears that the coefficients are divisible by the gcd of the hook lengths being considered. Also, if any hook length is even then the series is even because of conjugation.

Philip Cuthbertson  
pecuthbe@mtu.edu