Fixed hooks in arbitrary columns of partitions

Philip Cuthbertson

November 7, 2024

Definitions

- A *partition*, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, is a weakly decreasing sequence of non-negative integers,
- λ is a *partition of n* if $\lambda_1 + \lambda_2 + \ldots + \lambda_r = n$,
- *Hook length*: $h_{i,j}(\lambda) = \lambda_i + \lambda'_j - i - j + 1$,
- A sequence, $\{s_i\}$, has a *fixed point* if $s_j = j$,
- A sequence, $\{s_i\}$, has an *h-fixed point* if $s_j = j + h$,
- If $\{h_{i,m}\}$ has a h-fixed point, we call it a *h-fixed hook*.

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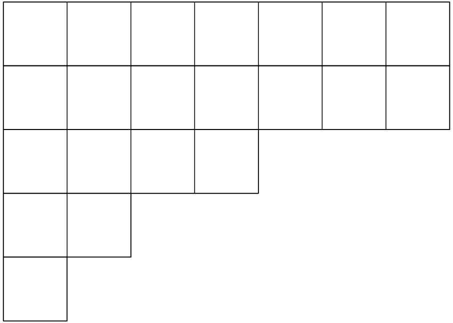
$\lambda = (7, 7, 4, 2, 1)$



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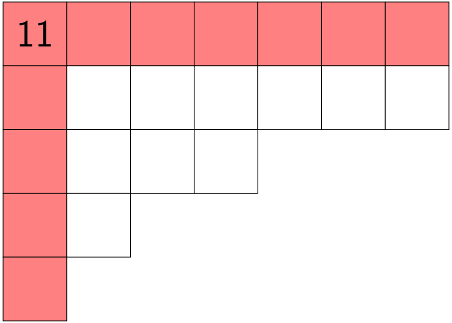

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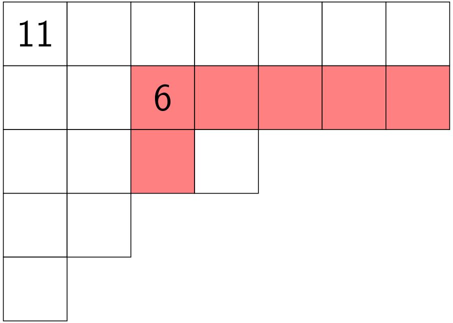

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11	9	7	6	4	3	2
10	8	6	5	3	2	1
6	4	2	1			
3	1					
1						

We consider the sequences of hook lengths in a given column:

- $\{11, 10, 6, 3, 1\}$ has a -1-fixed hook,
- $\{7, 6, 2\}$ has a 4-fixed hook,
- $\{3, 2\}$ has a 0-fixed hook.

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Additional Notation

We will also need the *q-Pochhammer* and

$$(a;b)_n := \prod_{k=0}^{n-1}(1-ab^k),$$

and the *Gaussian binomial*

$$\begin{bmatrix} A+B \\ B \end{bmatrix}_q := \frac{(q;q)_{A+B}}{(q;q)_A (q;q)_B}.$$

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Theorem (C.-Hemmer-Hopkins-Keith)

The generating function for the number of partitions of n with an h-fixed hook (in the first column) arising from a part of size k is

$$\sum_{s=0}^{\infty} \frac{q^{s(k+1)+k(k-h)}}{(q;q)_{s+k-h-1}} \begin{bmatrix} s+k-1 \\ k-1 \end{bmatrix}_q .$$

Theorem

The generating function for the number of partitions of n with an h-fixed hook in the mth column arising from a part of size $k \geq m$ is given by,

$$\sum_{s=0}^{\infty} \frac{q^{s(k+m)+k(k-h-m+1)}}{(q;q)_{s+k-h-m}(q;q)_{m-1}} \begin{bmatrix} s+k-m \\ k-m \end{bmatrix}_q .$$

Theorem (C.-Hemmer-Hopkins-Keith)

The generating function for the number of partitions of n with an h-fixed hook in the first column that arises from a hook of size k is

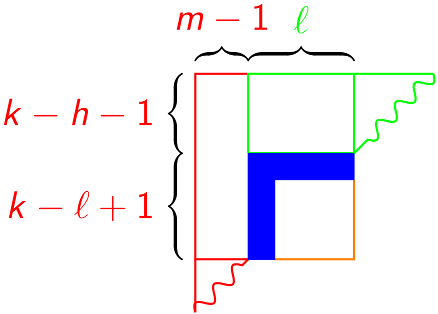
$$\sum_{\ell=1}^{k} \frac{q^{k+\ell(k-h-1)}}{(q;q)_{k-h-1}} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_q .$$

Theorem

The generating function for the number of partitions of n with an h-fixed hook in the mth column that arises from a hook of size k is

$$\sum_{\ell=1}^{k} \frac{q^{(m-1)(2k-h-\ell)+k+\ell(k-h-1)}}{(q;q)_{k-h-1}(q;q)_{m-1}} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_q .$$

Proof



$$\sum_{\ell=1}^{k} \frac{q^k}{}$$

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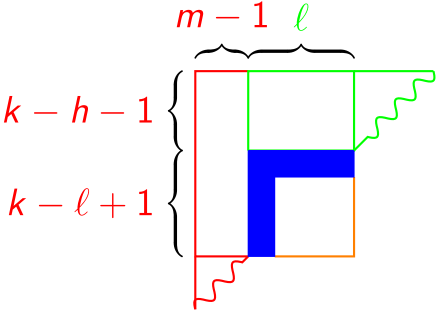

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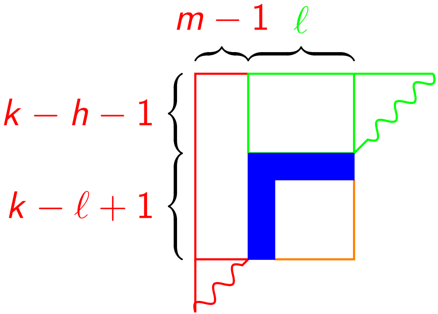

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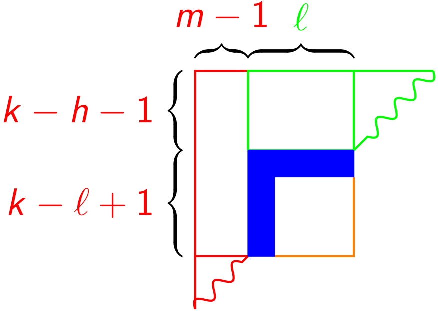

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Interesting Corollaries

Setting $h = 0$, fixing m, and summing over all k yields,

$$\sum_{k=1}^{\infty} \sum_{\ell=1}^{k} \frac{q^{(m-1)(2k-\ell)+k+\ell(k-1)}}{(q;q)_{m-1}(q;q)_{k-1}} \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}_q$$

$$= \frac{1}{(q;q)_{m-1}(q;q)_\infty} \sum_{\ell=1}^{\infty} q^{\ell(\ell+m-1)} (q^\ell;q)_{2m-1}.$$

Theorem

The number of partitions of n having a 0-fixed hook in the mth column is equal to the sum over L of the number of times across all partitions of n, with two colors of parts $1, 2, \ldots, m-1$*, that a part of size L appears exactly* $L+m-1$ *times in the first color, but* $L+1, L+2, \ldots, L+2m-2$ *are not parts in the first color.*

Using $m = 3$ and looking at partitions of 10 we find the two sets,

Description 1	Description 2
$(6,4)$	$(7,1^3)$
$(5,4,1)$	$(6,1,1^3)$
$(4,4,2)$	$(2,2^4)$
$(4,4,1,1)$	$(2^4,1^2)$
$(4,3,3)$	$(2^4,1,1)$
$(3,3,3,1)$	$(2^4,1^2)$
$(3,2,2,2,1)$	$(2^3,1^1,1^3)$
$(3,2,2,1,1,1)$	$(2^2,1^3,1^3)$
$(3,2,1,1,1,1,1)$	$(2,1^5,1^3)$
$(3,1,1,1,1,1,1,1)$	$(1^7,1^3)$

Summing, instead, over all h yields

Theorem

The generating function for the number of mth column hooks of size k in all partitions of n is

$$\frac{q^{m+k-1}}{(q^k;q)_\infty}\sum_{\ell=1}^{k}\frac{q^{(m-1)(k-\ell)}(q^m;q)_{\ell-1}}{(q;q)_{\ell-1}(q;q)_{k-\ell}}$$

For a fixed m, this stabilizes to q^k times the generating function for the number parts of size $m-1$ appearing in all partitions of n.

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Formally for $m \geq 2$ we have,

$$\lim_{k\to\infty} \frac{q^{m-1}}{(q^k;q)_\infty} \sum_{\ell=1}^{k} \frac{q^{(m-1)(k-\ell)}(q^m;q)_{\ell-1}}{(q;q)_{\ell-1}(q;q)_{k-\ell}} = \frac{q^{m-1}}{(1-q^{m-1})(q;q)_\infty}.$$

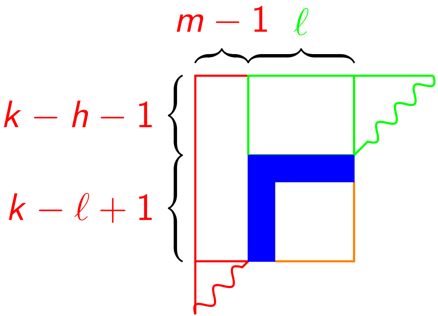
Restricted Partitions

Theorem

The generating function for the number of distinct partitions of n with an h-fixed hook in the mth column that arises from a hook of size k is

$$\sum_{\ell=\lceil (k+1)/2\rceil}^{k} \frac{q^{k+\ell(k-h-1)+(m-1)(2k-h-\ell)+\binom{k-h}{2}+\binom{k-\ell}{2}}(-q;q)_{m-1}}{(q;q)_{k-h-1}} \begin{bmatrix} \ell-1 \\ k-\ell \end{bmatrix}_q .$$

Proof



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Theorem

The generating function for the number of odd partitions of n with an h-fixed hook in the mth column that arises from a hook of size k is

$$\sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{k} \frac{q^{k+\ell(k-h-1)+(m-1)(2k-h-\ell)}}{(q^2;q^2)_{k-h-1}(q;q^2)_{(m-1)/2}} \binom{k-\ell+(\ell-1)/2}{k-\ell}_{q^2}$$

if m is odd and

$$\sum_{\substack{\ell=1 \\ \ell \text{ even}}}^{k} \frac{q^{k+\ell(k-h-1)+(m-1)(2k-h-\ell)+(k-\ell)}}{(q^2;q^2)_{k-h-1}(q;q^2)_{m/2}} \binom{k-\ell+(\ell-2)/2}{k-\ell}_{q^2}$$

if m is even.

Future Work

- Similar functions for other equinumerous families of partitions,
- Asymptotics/congruences,
- Connections to the Truncated Pentagonal Number Theorem.

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