

# Neighborly Partitions, Hypergraphs and Gordon's identities

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# Introduction

## Gordon's identities

Given integers  $r \geq 2$  and  $1 \leq i \leq r$ , let

- $A_{r,i}(n) = \#\{(\lambda_1, \dots, \lambda_s) : n \mid \lambda_k \not\equiv 0, \pm i \pmod{2r+1}\}.$
- $B_{r,i}(n) = \#\{(\lambda_1, \dots, \lambda_m) : n \mid \lambda_k - \lambda_{k+r-1} \geq 2, \lambda_i \geq 2\}.$

Then  $A_{r,i}(n) = B_{r,i}(n)$  for all integers  $n$ .

## analytic form of Gordon's identities

For the integers  $2 \leq r$ ,  $1 \leq i \leq r$ , we have

$$\sum_{n_1, n_2, \dots, n_{r-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \dots + N_{r-1}^2 + N_i + N_{i+1} + \dots + N_{r-1}}}{(q)_{n_1} (q)_{n_2} \dots (q)_{n_{r-1}}}$$

$$= \prod_{\substack{n \geq 1 \\ n \not\equiv 0, \pm i \pmod{2r+1}}} \frac{1}{1 - q^n} = \frac{\prod_{\substack{n \geq 1, \\ n \equiv 0, \pm i \pmod{2r+1}}} (1 - q^n)}{\prod_{n \geq 1} (1 - q^n)}.$$

Where  $q$  is a variable and  $N_j = n_j + n_{j+1} + \dots + n_{r-1}$  for all  $1 \leq j \leq r-1$  and  $(q)_n = (1-q)(1-q^2)\dots(1-q^n)$ .

# Neighborly partitions

Definition for the case  $i = r$  (P.A., H. Mourtada)

Let  $\lambda = (\lambda_1 \geq \dots \geq \lambda_m)$  be an integer partition and let  $r \geq 2$  be an integer. We say that  $\lambda$  is a  $(r, r)$ -Neighborly partition and we write  $\lambda \in \mathcal{N}_{r,r}$  if each part  $\lambda_j$  of  $\lambda$  satisfies the following conditions:

- $1 \leq m_\lambda(\lambda_j) \leq r$ .
- $\exists B_j = (\lambda_k \geq \dots \geq \lambda_{k+r-1}), \lambda_j \in B_j, \lambda_k - \lambda_{k+r-1} \leq 1$ .

- $B_j = (\overbrace{(\ell+1), \dots, (\ell+1)}^{(r-s)\text{-times}}, \overbrace{\ell, \dots, \ell}^{s\text{-times}}),$  where  $\lambda_j \in \{\ell, \ell+1\}$  and  $s \in [|1, r|]$ .

# Hypergraphs

## Definition (Hypergraph)

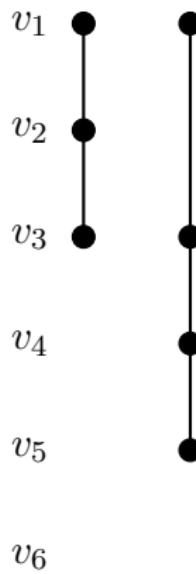
A hypergraph  $\mathcal{H}$  is a pair  $(V(\mathcal{H}), E(\mathcal{H}))$  where  $V(\mathcal{H})$  is a set of elements called the vertices of  $\mathcal{H}$  and  $E(\mathcal{H})$  is a set of subsets of  $V(\mathcal{H})$  called the edges of  $\mathcal{H}$ .

### ■ PAOH representation of a hypergraph

- **The vertices of  $\mathcal{H}$ :** parallel horizontal rows,
- **The edges of  $\mathcal{H}$ :** vertical lines in which a point represents a vertex of the edge.

# Example

- $V(\mathcal{H}) = \{v_1, \dots, v_6\}$ .
- $E(\mathcal{H}) = \{(v_1, v_2, v_3), (v_1, v_3, v_4, v_5)\}$ .



# Hypergraph associated with a neighborly partition

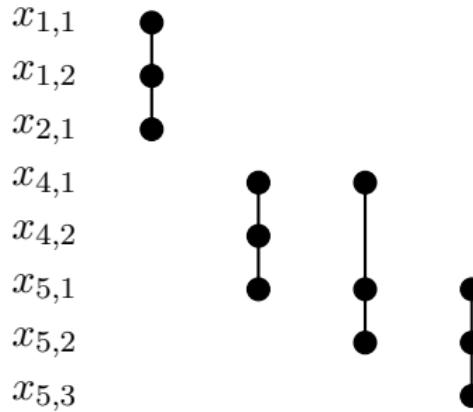
Definition for the case  $i = r$  (P.A., H. Mourtada)

Let  $\lambda \in \mathcal{N}_{r,r}$ . We define its associated hypergraph  $\mathcal{H}_\lambda$ :

- $V(\mathcal{H}_\lambda) = \{x_{j,k} \mid j \text{ is a part of } \lambda \text{ and } 1 \leq k \leq m_\lambda(j)\}.$
- $E(\mathcal{H}_\lambda) = \{(x_{\ell,1}, \dots, x_{\ell,s}, x_{(\ell+1),1}, \dots, x_{(\ell+1),(r-s)})\}, \text{ for any}$   
sub-partition  $B_j = (\underbrace{(\ell+1), \dots, (\ell+1)}_{(r-s)\text{-times}}, \underbrace{\ell, \dots, \ell}_{s\text{-times}})$  of  $\lambda$ .

# Example

- $\lambda = (5, 5, 5, 4, 4, 2, 1, 1) \in \mathcal{N}_{3,3}$ .
- $V(\mathcal{H}_\lambda) = \{x_{1,1}, x_{1,2}, x_{2,1}, x_{4,1}, x_{4,2}, x_{5,1}, x_{5,2}, x_{5,3}\}$ .



# Signature of a Neighborly partition

- $\mathcal{L} \in Sub_e(\mathcal{H}) :$ 
  - $E(\mathcal{L}) \subset E(\mathcal{H}).$
  - $V(\mathcal{L}) = \{v \in V(\mathcal{H}) \mid v \in E(\mathcal{L})\}.$

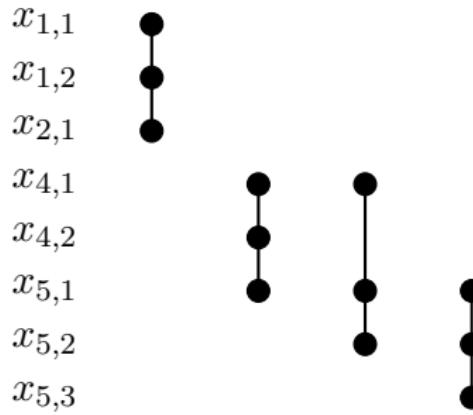
Definition (P.A., H. Mourtada)

Let  $\lambda \in \mathcal{N}_{r,i}$ . We define the signature of  $\lambda$  as follows:

$$\delta(\lambda) = \sum_{\substack{\mathcal{L} \in Sub_e(\mathcal{H}_\lambda) \\ V(\mathcal{L}) = V(\mathcal{H}_\lambda)}} (-1)^{|E(\mathcal{L})|}.$$

# Example

- $\lambda = (5, 5, 5, 4, 4, 2, 1, 1) \in \mathcal{N}_{3,3}$ ,



- $\delta(\lambda) = \sum_{\substack{\mathcal{L} \in Sub_e(\mathcal{H}_\lambda) \\ V(\mathcal{L})=V(\mathcal{H}_\lambda)}} (-1)^{|E(\mathcal{L})|} = (-1)^4 + (-1)^3 = 0$ .

## Gordon's "dual" identities

$$\mathcal{R}_{r,i}(n) := \{(\lambda_1, \dots, \lambda_s) : n \mid \lambda_k - \lambda_{k+1} \geq 1, \lambda_k \equiv 0, \pm i \pmod{2r+1}\}.$$

### Theorem (P.A., H. Mourtada)

Let  $1 \leq i \leq r$  be the integers. Then:

$$\sum_{\lambda \in \mathcal{N}_{r,i}} \delta(\lambda) q^{|\lambda|} = \sum_{n \in \mathbb{N}} \mathcal{R}_{r,i}(n) q^n = \prod_{j=0, \pm j[2r+1]} (1 - q^j),$$

where  $|\lambda|$  is the sum of the parts of  $\lambda$ .

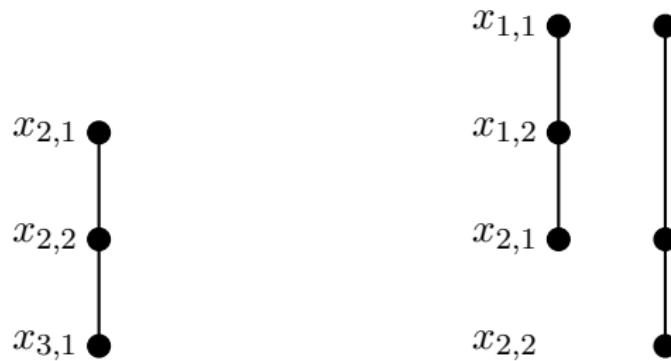
## Theorem (P.A., H. Mourtada)

Let  $1 \leq i \leq r$  and  $n$  be the positive integers. Then:

$$\sum_{\lambda \in \mathcal{N}_{r,i}(n)} \delta(\lambda) = \sum_{\lambda \in \mathcal{R}_{r,i}(n)} (-1)^{\ell(\lambda)}.$$

# Example

- $\delta(\lambda) = \sum_{\substack{\mathcal{L} \in \text{Sub}_e(\mathcal{H}_\lambda) \\ V(\mathcal{L}) = V(\mathcal{H}_\lambda)}} (-1)^{|E(\mathcal{L})|}$
- $\mathcal{N}_{3,3}(6) = \{\underbrace{2 + 2 + 2}_{\lambda_1}, \underbrace{2 + 2 + 1 + 1}_{\lambda_2}\},$



$$\Rightarrow \sum_{\lambda \in \mathcal{N}_{3,3}(6)} \delta(\lambda) = (-1)^1 + (-1)^2 = 0.$$

# Example

- $\mathcal{R}_{3,3}(6) = \emptyset \implies \sum_{\lambda \in \mathcal{R}_{3,3}(6)} (-1)^{\ell(\lambda)} = 0.$

- Thus:

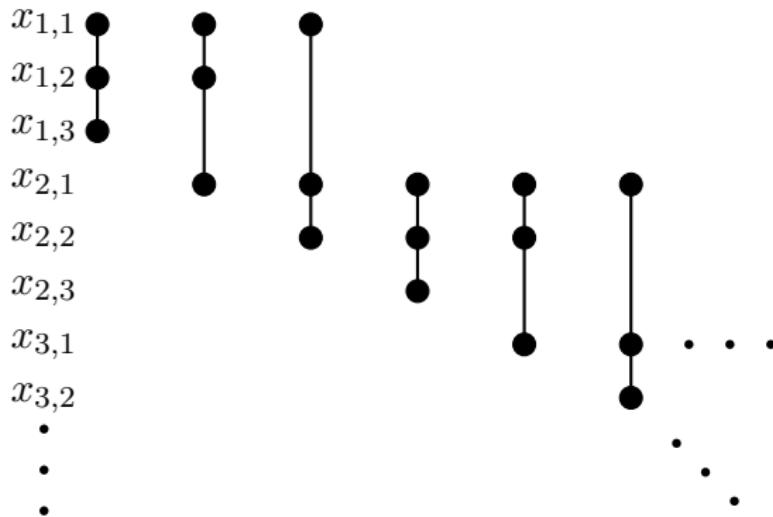
$$\sum_{\lambda \in \mathcal{R}_{3,3}(6)} (-1)^{\ell(\lambda)} = \sum_{\lambda \in \mathcal{N}_{3,3}(6)} \delta(\lambda) = 0.$$

## Sketch of proof for the case $i = r$

### ■ Infinite hypergraph $\mathcal{H}_{r,r}^\infty$ :

- $V(\mathcal{H}_{r,r}^\infty) = \{x_{j,k} \mid k \in [|1, r|], j \in \mathbb{N}^*\}$ .
- $E(\mathcal{H}_{r,r}^\infty) = \{(x_{\ell,1}, \dots, x_{\ell,s}, x_{(\ell+1),1}, \dots, x_{(\ell+1),(r-s)}) \mid \ell \in \mathbb{N}^*\}$

### ■ $\mathcal{H}_{3,3}^\infty$ :



## Sketch of proof for the case $i = r$

- Let  $\mathbb{K}$  be a field of characteristic zero. **The edge ideal of  $\mathcal{H}_{r,r}^\infty$ :**

$$\mathcal{I}(\mathcal{H}_{r,r}^\infty) = \langle x_{\ell,1} \cdots x_{\ell,s} x_{(\ell+1),1} \cdots x_{(\ell+1),(r-s)} \mid \ell \in \mathbb{N}^* \rangle$$

In the ring  $S = \mathbb{K}[x_{j,k} \mid k \in [|1, r|], j \in \mathbb{N}^*]$ .

- $J_{r,r} = \langle x_\ell^s x_{\ell+1}^{r-s} \mid 1 \leq s \leq r \text{ and } \ell \geq 1 \rangle \subset R := \mathbb{K}[x_j \mid j \in \mathbb{N}^*]$

- Polarization:**

$$x_\ell^s x_{\ell+1}^{r-s} \longrightarrow x_{\ell,1} x_{\ell,2} \cdots x_{\ell,s} x_{(\ell+1),1} x_{(\ell+1),2} \cdots x_{(\ell+1),(r-s)}$$

The ideal  $\mathcal{I}(\mathcal{H}_{r,r}^\infty)$ , is the polarization of the ideal  $J_{r,r}$ .

## Sketch of proof for the case $i = r$

- The algebra  $S/\mathcal{I}(\mathcal{H}_{r,r}^\infty)$  is graded by  $\text{wt}.x_{j,k} = j$ .
- The algebra  $R/J_{r,r}$  is graded by  $\text{wt}.x_j = j$ .
- **The Hilbert-Poincaré series** of a graded  $\mathbb{K}$ -algebra  $A = \bigoplus_{n \in \mathbb{N}} A_n$  such that  $\dim_{\mathbb{K}}(A_n) < \infty$  is by definition the following  $q$ -series:

$$HP_A(q) = \sum_{n \in \mathbb{N}} \dim_{\mathbb{K}}(A_n) q^n.$$

## Sketch of proof for the case $i = r$



$$HP_{S/\mathcal{I}(\mathcal{H}_{r,r}^\infty)}(q) = \frac{HP_{R/J_{r,r}}(q)}{\prod_{j \in \mathbb{N}^*} (1 - q^j)^{r-1}}.$$

- C.Bruschek, H.Mourtada and J.Schepers In "Arc Spaces and Rogers-Ramanujan Identities":

$$\begin{aligned}
 HP_{R/J_{r,r}}(q) &= \sum_{n \in \mathbb{N}} B_{r,r}(n) q^n \\
 &= \sum_{n \in \mathbb{N}} A_{r,r}(n) q^n \\
 &= \frac{\prod_{\substack{j \equiv 0, \pm i \pmod{2r+1}}} (1 - q^j)}{\prod_{j \in \mathbb{N}^*} (1 - q^j)}.
 \end{aligned}$$

## Sketch of proof for the case $i = r$



$$HP_{S/\mathcal{I}(\mathcal{H}_{r,r}^{\infty})}(q) = \frac{\prod_{j \equiv 0, \pm i \pmod{2r+1}} (1 - q^j)}{\prod_{j \in \mathbb{N}^*} (1 - q^j)^r}.$$

Proposition (P.A., H. Mourtada)

For the integer  $r \geq 2$  we have:

$$HP_{S/\mathcal{I}(\mathcal{H}_{r,r}^{\infty})}(q) = \frac{\sum_{\lambda \in \mathcal{N}_{r,r}} \delta(\lambda) q^{|\lambda|}}{\prod_{j \in \mathbb{N}^*} (1 - q^j)^r}.$$

## Sketch of proof for the case $i = r$

■ Thus:

$$\begin{aligned} \sum_{\lambda \in \mathcal{N}_{r,r}} \delta(\lambda) q^{|\lambda|} &= \prod_{j \equiv 0, \pm r \pmod{2r+1}} (1 - q^j) \\ &= \sum_{n \in \mathbb{N}} \mathcal{R}_{r,r}(n) q^n. \quad \square \end{aligned}$$

## References

-  P. Afsharijoo and H. Mourtada, *Neighborly partitions, hypergraphs and Gordon's identities*, Submitted.
-  Z. Mohsen, H. Mourtada, *Neighborly partitions and the numerators of Rogers-Ramanujan identities*. Int. J. Number Theory 19, 4, 859-872 (2023).
-  P. Afsharijoo, *Looking for a New Version of Gordon's Identities*, Annals of Combinatorics volume. 25, 543–571 (2021). <https://doi.org/10.1007/s00026-021-00530-x>. Comm. Algebra 34 (2006), no. 5, 1591-1598.
-  C. Bruschek, H. Mourtada, J. Schepers, *Arc spaces and Rogers-Ramanujan identities*, The Ramanujan Journal: Volume 30, Issue 1 (2013), Page 9-38.

# Thank you for your attention!