Rademacher-type exact formulas for ranks of partitions with a new proof of Ramanujan congruences

Seminar in Partition Theory, q-Series and Related Topics
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03 March 2026

Funded by the European Union

European Research Council
Established by the European Commission

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Integer partitions

A partition of n: $\Lambda = \{\Lambda_1 \geqslant \Lambda_2 \geqslant \cdots \geqslant \Lambda_\kappa\}$, $\sum \Lambda_j = n$.

$p(n)$: number of all partitions of n. $p(0) := 1$

e.g. $p(4) = 5$: $\{4\}, \{3,1\}, \{2,2\}, \{2,1,1\}, \{1,1,1,1\}$.

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Generating function:

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{j=1}^{\infty} \sum_{k=0}^{\infty} q^{jk} = \prod_{j=1}^{\infty} \frac{1}{1-q^j} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}$$

Growth rate by Hardy and Ramanujan (1919):

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right).$$

Hardy, Ramanujan, and Rademacher

Kronecker symbol $\left(\frac{\cdot}{\cdot}\right)$; $e(z) := e^{2\pi i z}$; $s(d,c)$: Dedekind sum.

$$A_c(n) := \frac{1}{2}\sqrt{\frac{c}{12}} \sum_{\substack{x \pmod{24c} \\ x^2 \equiv -24n+1 \pmod{24c}}} \left(\frac{12}{x}\right) e\left(\frac{x}{12}\right)$$

$$= \sum_{d \pmod{c}^*} e^{-\pi i s(d,c)} e\left(\frac{nd}{c}\right).$$

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Hardy and Ramanujan (1919):

$$p(n) \sim \frac{1}{2\pi\sqrt{2}} \sum_{c \leqslant \alpha\sqrt{n}} A_c(n)\sqrt{n} \cdot \frac{d}{dn}\left(\frac{\sinh\left(\pi\sqrt{\frac{2}{3}}\sqrt{n-\frac{1}{24}}/c\right)}{\sqrt{n-\frac{1}{24}}}\right).$$

Rademacher (1938): $p(n) = \uparrow$ summing c to ∞.

Why Dedekind sum?

Dedekind eta function:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n), \quad q = e(z) = e^{2\pi i z}, \ z \in \mathbb{H}.$$

$$\sum_{n=0}^{\infty} p(n) q^{n-\frac{1}{24}} = \frac{1}{\eta(z)}.$$

Transformation law:

$$\eta\left(\frac{az+b}{cz+d}\right) = \nu_\eta\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(cz+d)^{\frac{1}{2}}\eta(z), \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

$$\nu_\eta\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = e(-\tfrac{1}{8}) e^{-\pi i s(d,c)} e\left(\frac{a+d}{24c}\right).$$

Contents

Dyson's conjectures

$p(5n+4) \equiv 0 \pmod 5$, $p(7n+5) \equiv 0 \pmod 7$, $p(11n+6) \equiv 0 \pmod{11}$.

$\Lambda = \{\Lambda_1 \geqslant \Lambda_2 \geqslant \cdots \Lambda_\kappa\}$, $\mathrm{rank}(\Lambda) := \Lambda_1 - \kappa$.

$N(m,n) := \#\{\Lambda \text{ of } n : \quad \mathrm{rank}(\Lambda) = m\}$

$N(a,b;n) := \#\{\Lambda \text{ of } n : \quad \mathrm{rank}(\Lambda) \equiv a \pmod b\}$

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Dyson (1944) conjectured (proved by Atkin and Swinnerton-Dyer (1953)):

$$5N(a,5;5n+4) = p(5n+4), \quad 7N(a,7;7n+5) \equiv p(7n+5), \quad \text{for all } a.$$

Generating function: $\zeta_u = e(1/u)$, $q = e(z) = e^{2\pi i z}$,

$$\mathcal{R}(\zeta_u^\ell; q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n)\zeta_u^{\ell m} q^n =: 1 + \sum_{n=1}^{\infty} A\left(\frac{\ell}{u}; n\right) q^n.$$

Ranks of partitions modulo 1 and 2

$u = 1$, $\mathcal{R}(1;q) = 1 + \sum p(n)q^n$.

$$A(1;n) = p(n) = \frac{2\pi e(\frac{1}{8})}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^{\infty} \frac{S(0,n,c,\overline{\nu_\eta})}{c} I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right).$$

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$u = 2$, $\mathcal{R}(-1; q) = f(q)$. Bringmann and Ono (2006):

$$A\left(\frac{1}{2}; n\right) = \alpha(n) = \frac{2\pi\, e(-\frac{1}{8})}{(24n-1)^{\frac{1}{4}}} \sum_{2|c>0} \frac{S(0, n, c, \overline{\psi})}{c} I_{\frac{1}{2}} \left(\frac{\pi\sqrt{24n-1}}{6c} \right).$$

- Ramanujan's last letter: $\tilde{n} = n - 1/24$,

$$\alpha(n) = (-1)^{n-1} \frac{\exp(\pi\sqrt{\tilde{n}/6})}{2\sqrt{\tilde{n}}} + O\left(\frac{\exp(\frac{\pi}{2}\sqrt{\tilde{n}/6})}{2\sqrt{\tilde{n}}} \right).$$

- Leila Dragonette (1952): $\sum_{c \leqslant \sqrt{n}} \cdots + O_\varepsilon(n^{1/2+\varepsilon})$.
- George E. Andrews (1966): $\sum_{c \leqslant \sqrt{n}} \cdots + O_\varepsilon(n^\varepsilon)$.

Ranks of partitions modulo 3

$u = 3$, $\mathcal{R}(\zeta_3; q) = \gamma(q)$. Bringmann (2009):

$$A(\tfrac{1}{3}; n) = A(\tfrac{2}{3}; n) = \frac{2\pi\, e(-\frac{1}{8})}{(24n-1)^{\frac{1}{4}}} \sum_{3|c\leqslant\sqrt{n}} \frac{S(0,n,c,(\frac{\cdot}{3})\overline{\nu_\eta})}{c} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right) + O_\varepsilon(n^\varepsilon).$$

Theorem 2.1 (Andrews & Lewis (2000) conjectured, Bringmann (2009) proved)

If $n \in \mathbb{Z} \setminus \{3, 9, 21\}$, we have

$$N(0,3;n) < N(1,3;n), \quad \text{if } n \equiv 0, 2 \pmod 3;$$
$$N(0,3;n) > N(1,3;n), \quad \text{if } n \equiv 1 \pmod 3.$$

Bringmann & Ono (2006): harmonic Maass form

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right).$$

Definition 2.2

Smooth $f:\mathbb{H}\to\mathbb{C}$ is a weight k harmonic Maass form on $\Gamma_0(N)$ with character χ if:
(1) $f(\gamma z)=\chi(d)\nu_\theta(\gamma)^{2k}(cz+d)^k f(z)$, $\gamma\in\Gamma_0(N)$;
(2) $\Delta_k f=0$;
(3) There exists a polynomial $\mathcal{P}(z)=\sum_{n\leqslant 0}a^+(n)q^n$ with coefficients in $\mathbb{C}$ such that

$$f(z)-\mathcal{P}(z)=O(e^{-Cy})$$

for some $C>0$. Analogous conditions are required for all cusps.

$H_k(\Gamma_0(N),\chi\nu_\theta^{2k})$, or $H_k(\Gamma_0(N),\nu)$ for weight $k\in\mathbb{Z}+\frac{1}{2}$ multiplier ν.

Examples of harmonic Maass forms

e.g. holomorphic theta functions $\theta_{\chi,t}(z) := \sum_{n\in\mathbb{Z}} \chi(n) q^{tn^2}$.

(Serre-Stark basis theorem: basis of weight $\frac{1}{2}$ modular forms)

e.g. Maass-Poincaré series. Bringmann & Ono (2006) defined

$$P(s,m,N;z) := \frac{1}{\Gamma(3/2)} \sum_{\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty\backslash\Gamma_0(N)} \overline{\psi}(\gamma)(cz+d)^{-\frac{1}{2}} \varphi_{s,\frac{1}{2}}(\tilde{m}\gamma z).$$

e.g. This time we define

$$P_{\mathfrak{a}}(z) := \frac{1}{\Gamma(2s)} \sum_{\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathfrak{a}}\backslash\Gamma} \mu(\gamma)^{-1}\overline{w(\sigma_{\mathfrak{a}}^{-1},\gamma)} j(\sigma_{\mathfrak{a}}^{-1}\gamma,z)^{-k} \varphi_{s,k}(\tilde{m}\sigma_{\mathfrak{a}}^{-1}\gamma z).$$

"Principal part" of harmonic Maass forms:

(Bruinier & Funke, 2004)

$$M(z) = \sum_{n>0} c^+(n)q^n + \sum_{n_0 \leqslant n \leqslant 0} c^+(n)q^n + \sum_{n<0} c^-(n)\Gamma(1-k, 4\pi|n|y)q^n.$$

Uniqueness: either holomorphic

or with principal part & non-holomorphic part

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$P_{\mathfrak{a}}$ has Fourier coefficient of form $\sum \frac{S(\cdots)}{c} \text{Bessel}(\frac{4\pi\sqrt{mn}}{c})$

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Let's go to the mod 3 case! It's on $\Gamma_0(3)$.

- $q^{-\frac{1}{24}}\mathcal{R}(\zeta_3; q)$: constant at cusp 0.
- Multiplier system: $\left(\frac{\cdot}{3}\right)\overline{\nu_\eta}$.
- Fourier expansion: similar methods.

Same idea as in Bringmann and Ono (2006, 2012).

Convergence matters

"Pattern" with Whittaker function to construct Maass-Poincaré series as harmonic Maass forms:

$$\varphi_{s,k}(z) := |4\pi y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn} y,\, s-\frac{1}{2}}(|4\pi y|)e(x)$$

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- Construct $P_{\mathfrak{a}}(z; s, k)$ using $\varphi_{s,k}$, $k \in \mathbb{Z} + \frac{1}{2}$, $\operatorname{Re} s > 1$.

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- Construct $P_{\mathfrak{a}}(z; s, k)$ using $\varphi_{s,k}$, $k \in \mathbb{Z} + \frac{1}{2}$, $\operatorname{Re} s > 1$.
- "Harmonic point" at $s = 1 - \frac{k}{2}$. Rank generating functions: $k = \frac{1}{2}$, so $s = \frac{3}{4} < 1$.

$$\Delta_k \varphi_{s,k}(z) = \left(s(1-s) - \frac{k}{2}\left(1 - \frac{k}{2}\right)\right)\varphi_{s,k}(z)$$

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$P_{\mathfrak{a}}(z; s, k)$ needs to be convergent at $s = \frac{3}{4}$.

Contents

Bringmann's asymptotic formula

Bringmann (2009): not only for mod 3, but for modulus odd $u \geqslant 3$.

$$A\left(\frac{\ell}{u}; n\right) = \frac{4\sqrt{3}\, i}{\sqrt{24n-1}} \sum_{c:\ u|c \leqslant \sqrt{n}} \frac{B_{\ell,u,c}(-n,0)}{\sqrt{c}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6c}\right) + O_{u,\varepsilon}(n^{\varepsilon})$$

$$+ \frac{8\sqrt{3}\sin(\frac{\pi\ell}{u})}{\sqrt{24n-1}} \sum_{r \geqslant 0} \sum_{\substack{a \leqslant \sqrt{n}: \\ v = u \nmid a, \\ \delta_{\ell,v,a,r} > 0}} \frac{D_{\ell,v,a}(-n, m_{\ell,v,a,r})}{\sqrt{a}} \sinh\left(\frac{\pi\sqrt{2\delta_{\ell,v,a,r}(24n-1)}}{a\sqrt{3}}\right)$$

$B_{\ell,u,c}$, $D_{\ell,v,a}$: Kloosterman-type exponential sums.

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$B_{\ell,u,c}$, $D_{\ell,v,a}$: Kloosterman-type exponential sums.

We consider $u = v = p \geqslant 5$. Why?

- $\Gamma_0(p)$ only has two cusps, ∞ and 0;
- $B_{\ell,u,c}$, $D_{\ell,v,a}$, $m_{\ell,v,a,r}$, $\delta_{\ell,v,a,r}$ are a little bit simpler;
- We have transformation laws by Garvan (2019).

Idea: transformation law $\to$ multiplier system

Believe: Bringmann's formula is exact.
Try: Garvan's transformation law μ_p to build Maass-Poincaré series.

$$\mathcal{G}_1\left(\frac{\ell}{p}; z\right) := \mathcal{N}\left(\frac{\ell}{p}; z\right) + \cdots = \csc\left(\frac{\pi\ell}{p}\right) q^{-\frac{1}{24}}\mathcal{R}(\zeta_p^\ell; q) + \text{non-holo},$$
$$\mathcal{G}_2\left(\frac{\ell}{p}; z\right) := \mathcal{M}\left(\frac{\ell}{p}; z\right) + \varepsilon_2\left(\frac{\ell}{p}; z\right) - T_2\left(\frac{\ell}{p}; z\right),$$
$$\mathcal{G}_1\left(a, b, p; z\right) := \cdots, \quad \mathcal{G}_2\left(a, b, p; z\right) := \cdots$$

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$$\mathcal{G}_1(a,b,p;z) := \cdots, \quad \mathcal{G}_2(a,b,p;z) := \cdots$$

Theorem 3.1 (Theorem 3.4 in Bringmann and Ono (2010))

$$\left\{\mathcal{G}_1\left(\frac{\ell}{p};z\right), \mathcal{G}_2\left(\frac{\ell}{p};z\right) : 1 \leqslant \ell < p\right\} \cup \{\mathcal{G}_1(a,b,p;z), \mathcal{G}_2(a,b,p;z) : 0 \leqslant a <$$

is a vector valued Maass form of weight $\frac{1}{2}$ *for* $\mathrm{SL}_2(\mathbb{Z})$.

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$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$. $0 \leqslant [A] < p$: $A \equiv [A] \pmod{p}$.

Garvan (2017):

$$\mathcal{G}_1\left(\frac{\ell}{p}; \gamma z\right) = \mu(c, d, \ell, p)\overline{\nu_\eta}(\gamma)(cz+d)^{\frac{1}{2}}\mathcal{G}_1\left(\frac{[d\ell]}{p}; z\right)$$

$$\text{vector: } \mathbf{G}_1(z) = \left(\mathcal{G}_1\left(\frac{1}{p}; z\right), \mathcal{G}_1\left(\frac{2}{p}; z\right), \cdots, \mathcal{G}_1\left(\frac{p-1}{p}; z\right)\right)^{\mathrm{T}}$$

We do: $M_p : \Gamma_0(p) \to \mathrm{M}_{p-1}(\mathbb{C})$ by

$$M_p(\gamma) := \sum_{\ell=1}^{p-1} \mu(c, d, \ell, p) E_{\ell, [d\ell]} \quad \text{and} \quad \mu_p(\gamma) := \overline{\nu_\eta}(\gamma) M_p(\gamma).$$

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Recall weight k multiplier system:

$$|\nu| = 1; \quad \nu(-I) = e^{-\pi i k}; \quad \nu(\gamma_1\gamma_2) = w_k(\gamma_1, \gamma_2)\nu(\gamma_1)\nu(\gamma_2).$$

$$w_k(\gamma_1, \gamma_2) := j(\gamma_2, z)^k j(\gamma_1, \gamma_2 z)^k j(\gamma_1\gamma_2, z)^{-k}$$

Vector-valued "multiplier system"

Definition 3.2

Congruence subgroup Γ *of* $\mathrm{SL}_2(\mathbb{Z})$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in \Gamma$. *We say* ξ *is a* D*-dimensional multiplier system if it satisfies:*

- ξ *is unitary:* $\xi(\gamma)^{-1} = \xi(\gamma)^{\mathrm{H}}$;
- $\xi(-I) = e^{-\pi i k} I_D$;
- $\xi(\gamma_1\gamma_2) = w_k(\gamma_1,\gamma_2)\xi(\gamma_1)\xi(\gamma_2)$.
- *For every cusp* $\mathfrak{a}$ *of* Γ, *we have* $\alpha_{\xi,\mathfrak{a}}^{(\ell)} \in [0,1)$ *such that*

$$\xi\left(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}\right) = \mathrm{diag}\left\{e(-\alpha_{\xi,\mathfrak{a}}^{(1)}), \cdots, e(-\alpha_{\xi,\mathfrak{a}}^{(D)})\right\}$$

We want vector-valued (harmonic) (Maass) forms on (Γ,ξ) have good Fourier expansions on cusp $\mathfrak{a}$ like

$$(\mathbf{V}|_k\sigma_{\mathfrak{a}})(z) = \sum_{\ell=1}^{D}\sum_{n\in\mathbb{Z}} a_{\mathbf{V}}^{(\ell)}(y,n)e\big((n-\alpha_{\xi,\mathfrak{a}})x\big)\,\mathfrak{e}_\ell\,.$$

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$\alpha_\infty = \frac{1}{24}$, $\alpha_0^{(\ell)} \in [0,1)$ is decided by

$$e(-\alpha_0^{(\ell)}) = e\left(-\frac{3\ell^2}{2p} - \frac{p}{24}\right)(-1)^\ell.$$

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In Bringmann (2009): let $t = \frac{a\ell - [a\ell]}{p} \in \mathbb{Z}$, for $0 < \frac{[a\ell]}{p} < \frac{1}{6}$,

$$\delta_{\ell,p,a,r} = \frac{3}{2}\left(\frac{[a\ell]}{p}\right)^2 - (\tfrac{1}{2}+r)\frac{[a\ell]}{p} + \frac{1}{24}, \quad -m_{\ell,p,a,r} = \frac{3}{2}t^2 + (\tfrac{1}{2}+r)t.$$

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Magic equation: $\frac{3}{2}x^2 - (\frac{1}{2}+r)x + \frac{1}{24} = 0$.

$x_r \in (0, \frac{1}{2})$ the only solution in this range.

$\triangleright\, r \triangleleft := \{1 \leqslant \ell \leqslant p-1 : \frac{\ell}{p} \in (0, x_r) \cup (1-x_r, 1)\}$

Behavior of $\mathcal{G}_1$ at cusps ∞ and 0

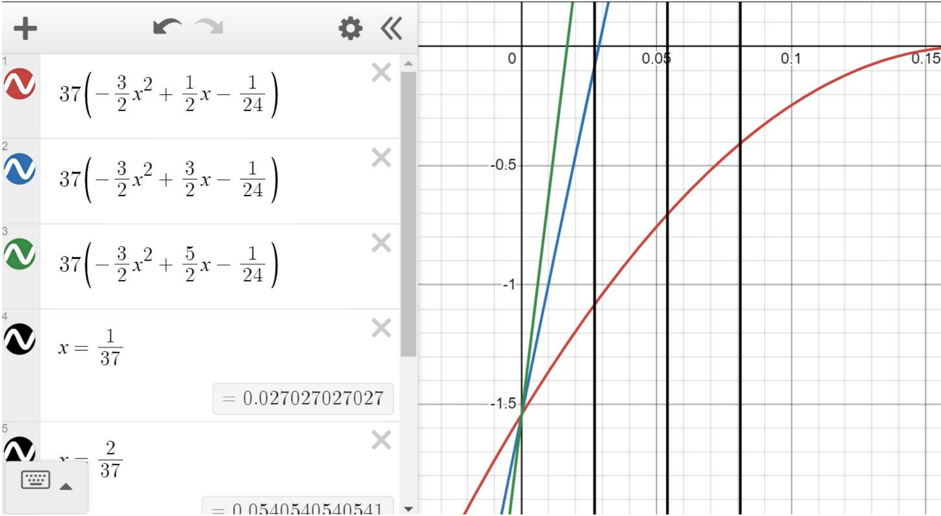
Recall: $\mathcal{G}_1(\frac{\ell}{p}; z) = \csc(\frac{\pi\ell}{p}) q^{-\frac{1}{24}} \mathcal{R}(\zeta_p^{\ell}; q)$+non-holo.

$\mathcal{G}_1(\frac{\ell}{p}; \cdot)|_{\frac{1}{2}} \sigma_0 = e(-\frac{1}{8}) p^{\frac{1}{4}} \mathcal{G}_2(\frac{\ell}{p}; pz)$.

$$\mathcal{G}_2\left(\frac{\ell}{p}; z\right) = 2q^{-\frac{3}{2}\left(\frac{\ell}{p}\right)^2 + \frac{\ell}{2p} - \frac{1}{24}} \left(1 + q^{\frac{\ell}{p}} + q^{\frac{2\ell}{p}} + \cdots\right)$$

Order for $\mathcal{G}_2(\frac{\ell}{p}; pz)$: $X_r^{(\ell)} \leqslant 0$

$$X_r^{(\ell)} := \begin{cases} \left\lceil -\frac{3\ell^2}{2p} + (\frac{1}{2} + r)\ell - \frac{p}{24} \right\rceil, & \text{when } 0 < \frac{\ell}{p} < x_r, \\ \left\lceil -\frac{3p}{2}(1 - \frac{\ell}{p})^2 + (\frac{1}{2} + r)p(1 - \frac{\ell}{p}) - \frac{p}{24} \right\rceil, & \text{when } 1 - x_r < \frac{\ell}{p} < 1, \\ 0, & \text{otherwise.} \end{cases}$$


37(−3/2 x^2 + 1/2 x − 1/24)
37(−3/2 x^2 + 3/2 x − 1/24)
37(−3/2 x^2 + 5/2 x − 1/24)
x = 1/37
= 0.027027027027
x = 2/37
= 0.0540540540541
0
0.05
0.1
0.15
-0.5
-1
-1.5

Maass-Poincaré series at ∞ and 0

$$\mathbf{P}_\infty(z; p, s, k, m, \mu_p) := \frac{2}{\sqrt{\pi}} \sum_{\ell=1}^{p-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(p)} \mu_p(\gamma)^{-1} \frac{\varphi_{s,k}(m_\infty \gamma z)}{(cz+d)^{\frac{1}{2}} \sin(\frac{\pi \ell}{p})} \mathfrak{e}_\ell .$$

Principal part of $\mathbf{P}_\infty$ at ∞: $\displaystyle\sum_{\ell=1}^{p-1} \csc(\tfrac{\pi\ell}{p}) q^{-\frac{1}{24}} \mathfrak{e}_\ell$.

Maass-Poincaré series at ∞ and 0

$$\mathbf{P}_\infty(z;p,s,k,m,\mu_p) := \frac{2}{\sqrt{\pi}} \sum_{\ell=1}^{p-1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \mu_p(\gamma)^{-1} \frac{\varphi_{s,k}(m_\infty \gamma z)}{(cz+d)^{\frac{1}{2}} \sin(\frac{\pi\ell}{p})} \mathfrak{e}_\ell \,.$$

Principal part of $\mathbf{P}_\infty$ at ∞: $\displaystyle\sum_{\ell=1}^{p-1} \csc(\frac{\pi\ell}{p}) q^{-\frac{1}{24}} \mathfrak{e}_\ell$.

$$\mathbf{P}_0(z;p,s,k,\mathbf{X}_r,\mu_p)$$

$$:= \frac{2e(-\frac{1}{8})p^{\frac{1}{4}}}{\sqrt{\pi}} \sum_{\ell\in\triangleright r\triangleleft} \sum_{\substack{\gamma\in\Gamma_0\backslash\Gamma_0(p)\\ \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}}} \mu_p(\gamma)^{-1} \overline{w_{\frac{1}{2}}(\sigma_0^{-1},\gamma)} \frac{\varphi_{s,k}\left(X_{r,0}^{(\ell)} \sigma_0^{-1}\gamma z\right)}{(-a\sqrt{p}z - b\sqrt{p})^{\frac{1}{2}}} \mathfrak{e}_\ell,$$

Principal part of $\mathbf{P}_0$ at 0: $\displaystyle e(-\frac{1}{8})p^{\frac{1}{4}} \sum_{\ell\in\triangleright r\triangleleft} q^{X_{r,0}^{(\ell)}} \mathfrak{e}_\ell$

Final proof of the exact formula

Lemma 3.3

For $\mathbf{X}_r$ defined above, the function

$$\mathbf{G}(z) := \mathbf{G}_1(z;p) - \mathbf{P}_\infty(z;p,\tfrac{3}{4},\tfrac{1}{2},0,\mu_p) - 2\sum_{\substack{r\geqslant 0\\ x_r^{-1}<p}} \mathbf{P}_0(z;p,\tfrac{3}{4},\tfrac{1}{2},\mathbf{X}_r,\mu_p)$$

has constant principal parts at both ∞ and 0, i.e. $\mathbf{G}(z) \in M_{\frac{1}{2}}(\Gamma_0(p),\mu_p)$.

Lemma 3.4

$\mathbf{G}(z) = \mathbf{0}$.

Reason: Serre-Stark basis theorem and μ_p.

Two ingredients needed: 1. Convergence

"Naturally" convergence at $\mathrm{Re}\, s > 1$, but we need expansion at $s = \frac{3}{4}$.

Estimate sums of *vector-valued* Kloosterman sums?

Thanks Goldfeld and Sarnak (1983) $\rightarrow$ generalize

$$\sum_{\substack{a \leqslant x:\, p \nmid a, \\ [a\ell] = L}} \frac{S_{0\infty}^{(\ell)}(m^{(L)}, n, a, \mu_p)}{a\sqrt{p}} = \sum_{\frac{1}{2} < s_j \leqslant \frac{3}{4}} \tau_{j,0,(L)}^{(\ell)}(m^{(L)}, n) \frac{x^{2s_j - 1}}{2s_j - 1} + O_{p,\varepsilon}\left(|m_{+0}^{(L)} n|^3 x^{\frac{1}{3}+\varepsilon}\right)$$

The following is then absolutely convergent:

$$\sum_{n_{+\infty} > 0} \left| \sum_{\substack{a > 0:\, p \nmid a, \\ \ell \in \rhd a, r \lhd}} \left| \frac{m_{+0}^{([a\ell])}}{n_{+\infty}} \right|^{\frac{1}{4}} \frac{S_{0\infty}^{(\ell)}(m^{([a\ell])}, n, a, \mu_p)}{a\sqrt{p}} I_{\frac{1}{2}}\left(\frac{4\pi |m_{+0}^{([a\ell])} n_{+\infty}|^{\frac{1}{2}}}{c} \right) \right| q^{n_{+\infty}}$$

Two ingredients needed: 2. KL sums match

The Fourier expansion of $\mathbf{P}_\infty$ at ∞ gives $S^{(\ell)}_{\infty\infty}(m, n, c, \mu_p)$.

The Fourier expansion of $\mathbf{P}_0$ at ∞ gives $S^{(\ell)}_{0\infty}(X_r^{([a\ell])}, n, a, \mu_p)$.

By the convergence of the expansions at $s = \frac{3}{4}$, we have

Theorem 3.5 (S. (2026))

$$A\left(\frac{\ell}{p}; n\right) = \frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{p})}{(24n-1)^{\frac{1}{4}}} \sum_{c>0:\, p|c} \frac{S^{(\ell)}_{\infty\infty}(0, n, c, \mu_p)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right)$$

$$+ \frac{4\pi \sin(\frac{\pi\ell}{p})}{(n-\frac{1}{24})^{\frac{1}{4}}} \sum_{\substack{r\geqslant 0\\ x_r^{-1}<p}} \sum_{\substack{a>0:\, p\nmid a,\\ \frac{[a\ell]}{p}\in(0,x_r)\\ \text{or } \frac{[a\ell]}{p}\in(1-x_r,1)}} \frac{S^{(\ell)}_{0\infty}(\lceil -p\delta_{\ell,p,a,r}\rceil, n, a, \mu_p)}{a\cdot \delta_{\ell,p,a,r}^{-\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{\delta_{\ell,p,a,r}(n -}}{a}\right.$$

Two ingredients needed: 2. KL sums match

Bringmann (2009):

$$A\left(\frac{\ell}{p}; n\right) = \frac{4\sqrt{3}\, i}{\sqrt{24n-1}} \sum_{c:\ p|c\leqslant\sqrt{n}} \frac{B_{\ell,u,c}(-n,0)}{\sqrt{c}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6c}\right) + O_{u,\varepsilon}(n^{\varepsilon})$$

$$+ \frac{8\sqrt{3}\sin(\frac{\pi\ell}{p})}{\sqrt{24n-1}} \sum_{r\geqslant 0} \sum_{\substack{a\leqslant\sqrt{n}:\\ p\nmid a,\\ \delta_{\ell,p,a,r}>0}} \frac{D_{\ell,p,a}(-n, m_{\ell,p,a,r})}{\sqrt{a}} \sinh\left(\frac{\pi\sqrt{2\delta_{\ell,p,a,r}(24n-1)}}{a\sqrt{3}}\right)$$

We prove:

$$e(-\tfrac{1}{8})\overline{B_{\ell,p,c}(-n,0)} = \sin(\tfrac{\pi\ell}{p}) S^{(\ell)}_{\infty\infty}(0,n,c,\mu_p),$$

$$\overline{D_{\ell,p,a,r}(-n, m_{\ell,p,a,r})} = S^{(\ell)}_{0\infty}\left(\lceil -p\delta_{\ell,a,p,r}\rceil, n, a, \mu_p\right).$$

Theorem 3.6 (For prime $p \geqslant 5$)

Bringmann's asymptotic formula converges to be the exact formula.

Contents

Dyson's rank conjectures

$$5N(\ell,5;5n+4)=p(5n+4),\quad 7N(\ell,7;7n+5)=p(7n+5).$$

Relation:

$$u\cdot N(\ell,u;n)=p(n)+\sum_{j=1}^{u-1}\zeta_u^{-\ell j}A\left(\frac{\ell}{u};n\right).$$

$\left(\zeta_p^{-\ell j}+\zeta_p^{\ell j}\right)_{1\leqslant \ell\leqslant \frac{p-1}{2},\ \ 1\leqslant j\leqslant \frac{p-1}{2}}$ is an invertible matrix.

$$5N(\ell,5;5n+4)=p(5n+4),\ \forall\ell \quad\Leftrightarrow\quad A\left(\frac{\ell}{5};5n+4\right)=0,\forall\ell.$$

Can we show $A(\frac{\ell}{5};5n+4)=0$ and $A(\frac{\ell}{7};7n+5)=0$ for all ℓ and $n\geqslant 0$?

$$A\left(\frac{\ell}{5};n\right)=\frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{5})}{(24n-1)^{\frac{1}{4}}}\sum_{c>0:\,5|c}\frac{S^{(\ell)}_{\infty\infty}(0,n,c,\mu_5)}{c}I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right);$$

$$A\left(\frac{\ell}{7};n\right)=\frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{7})}{(24n-1)^{\frac{1}{4}}}\sum_{c>0:\,7|c}\frac{S^{(\ell)}_{\infty\infty}(0,n,c,\mu_7)}{c}I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right)$$
$$+\frac{4\pi\sin(\frac{\pi\ell}{7})}{(24n-1)^{\frac{1}{4}}}\sum_{\substack{a>0:\,p\nmid a,\\ [a\ell]=1\text{ or }6}}\frac{S^{(\ell)}_{0\infty}(0,n,a,\mu_7;0)}{\sqrt{7}\,a}I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24\times 7a}\right).$$

$$A\left(\frac{\ell}{5};n\right)=\frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{5})}{(24n-1)^{\frac{1}{4}}}\sum_{c>0:\,5|c}\frac{S^{(\ell)}_{\infty\infty}(0,n,c,\mu_5)}{c}I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right);$$

$$A\left(\frac{\ell}{5};n\right)=\frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{5})}{(24n-1)^{\frac{1}{4}}}\sum_{c>0:\,5|c}\frac{S^{(\ell)}_{\infty\infty}(0,n,c,\mu_5)}{c}I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right);$$

$$S^{(\ell)}_{\infty\infty}(0,5n+4,c,\mu_5)=\sum_{d\pmod{c}^*}\frac{e\left(-\frac{3\pi ic'a\ell^2}{10}\right)}{\sin(\frac{\pi a\ell}{5})}e^{-\pi is(d,c)}e\left(\frac{(5n+4)d}{c}\right)$$

$5|c$. What happens if $n\to n+1$?

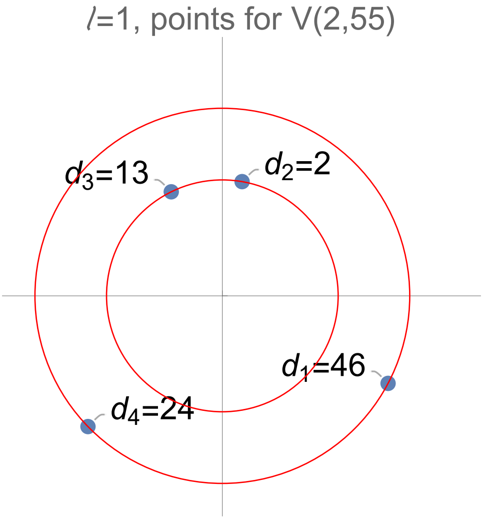
$$A\left(\frac{\ell}{5};n\right)=\frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{5})}{(24n-1)^{\frac{1}{4}}}\sum_{c>0:\,5|c}\frac{S^{(\ell)}_{\infty\infty}(0,n,c,\mu_5)}{c}I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right);$$

$$S^{(\ell)}_{\infty\infty}(0,5n+4,c,\mu_5)=\sum_{d \pmod{c}^*}\frac{e\left(-\frac{3\pi i c' a\ell^2}{10}\right)}{\sin(\frac{\pi a\ell}{5})}e^{-\pi i s(d,c)}e\left(\frac{(5n+4)d}{c}\right)$$

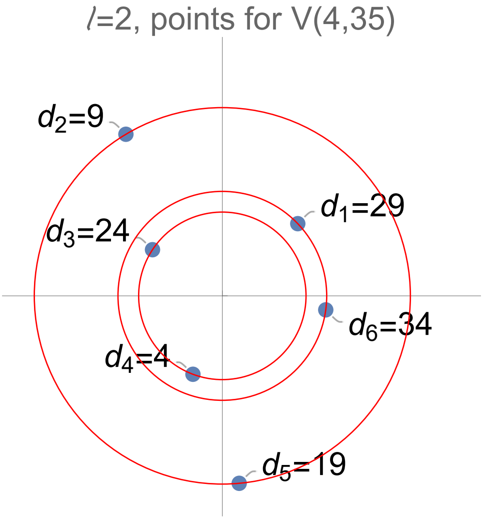
$5|c$. What happens if $n\to n+1$?

$(r,\frac{c}{5})=1,\ V(r,c):=\{d(c)^*: d\equiv r \pmod{\frac{c}{5}}\}$; $|V(r,c)|=4$ or 5.

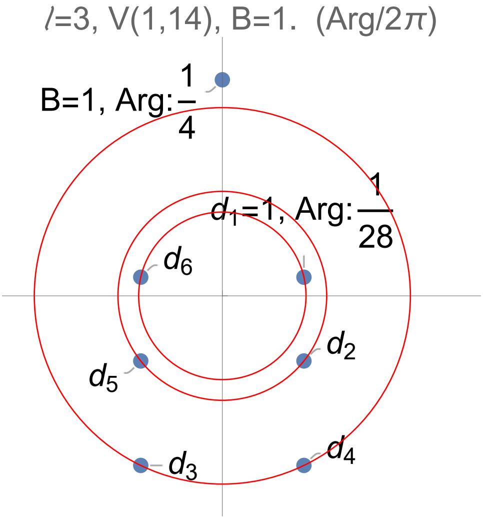
Vanishing KL sums: $p = 5$



Vanishing KL sums: $p = 7$, case 1



Vanishing KL sums: $p = 7$, case $a\ell \equiv \pm 1 \pmod 7$



Properties of KL sums

Theorem 4.1 (S. (2025))

For all $n \geqslant 0$ *and* $1 \leqslant \ell \leqslant p-1$ *when* $p = 5, 7$, *we have the following vanishing conditions for the Kloosterman sums appearing at my exact formula:*

1. *If* $5|c$, *we have* $S^{(\ell)}_{\infty\infty}(0, 5n+4, c, \mu_5) = 0$.
2. *If* $7|c$ *and* $\frac{c}{7} \cdot \ell \not\equiv \pm 1 \pmod 7$, *then* $S^{(\ell)}_{\infty\infty}(0, 7n+5, c, \mu_7) = 0$.
3. *If* $7|c$, $7 \nmid a$, $a\ell \equiv \pm 1 \pmod 7$, *and* $c = 7a$,

$$e(-\tfrac{1}{8}) S^{(\ell)}_{\infty\infty}(0, 7n+5, c, \mu_7) + 2\sqrt{7} S^{(\ell)}_{0\infty}(0, 7n+5, a, \mu_7) = 0.$$

$$u \cdot N(\ell, u; n) = p(n) + \sum_{j=1}^{u-1} \zeta_u^{-\ell j} A\left(\frac{\ell}{u}; n\right).$$

Proves $N(\ell, 5; 5n+4) = \frac{1}{5} p(5n+4)$ & $N(\ell, 7; 7n+5) = \frac{1}{7} p(7n+5)$.

Dyson's conjectures: other rank equalities

$$
\begin{aligned}
N(1,5;5n+1) &= N(2,5;5n+1);\\
N(0,5;5n+2) &= N(2,5;5n+2);\\
N(2,7;7n) &= N(3,7;7n);\\
N(1,7;7n+1) = N(2,7;7n+1) &= N(3,7;7n+1);\\
N(0,7;7n+2) &= N(3,7;7n+2);\\
N(0,7;7n+3) = N(2,7;7n+3), &\quad N(1,7;7n+3) = N(3,7;7n+3);\\
N(0,7;7n+4) = N(1,7;7n+4) &= N(3,7;7n+4);\\
N(0,7;7n+6) + N(1,7;7n+6) &= N(2,7;7n+6) + N(3,7;7n+6).
\end{aligned}
$$

- Setting n as $pn+k$ in our KL sums
- checking $A(\frac{\ell}{p}; pn+k)$ for all the cases...

Thank you!

References:

1 Qihang Sun. Exact formulae for ranks of partitions. *Trans. Amer. Math. Soc.* **379**(3), 2026. arXiv:2406.06294.

2 Qihang Sun. Vanishing properties of Kloosterman sums and Dyson's conjectures. *Ramanujan J.* **66**(50), 2025. arXiv:2406.07469.