Bailey pairs and radial limits of q-hypergeometric false theta functions

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Joint work with Jeremy Lovejoy (CNRS, Paris)

Seminar in Partition Theory, q-Series and Related Topics
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Outline

Background

Bailey pair machinery

Bailey pairs of Lovejoy

Work of Hikami

Proof of Hikami's conjecture

$\widetilde{\Phi}_m^{(a)}(q) = q^{\frac{(m-1-a)^2}{4m}} Y_{m,N}^{(a)}(q)$

Further results

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Q-SERIES

q-series (Eulerian series, basic hypergeometric series, q-hypergeometric series) are constructed from the q-rising factorials (q-Pochhammer symbols).

The usual notation for the conventional q-Pochammer symbols are

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k),\ (a)_\infty = (a;q)_\infty := \lim_{n\to\infty}(a;q)_n,$$
$$(a_1,a_2,\ldots,a_j;q)_n = (a_1;q)_n(a_2;q)_n\ldots(a_j;q)_n,$$
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Generically, q-series take the form ($|q|<1$)

$$\sum_{n\geq 0}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n z^n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}.$$

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The fact that q-series are everywhere is well exemplified by the Rogers-Ramanujan identities :

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}$$

How does one prove q-series identities?

In fact, there is a great deal of structure behind many q-series identities!

Perhaps one of the most important structural elements in q-series are Bailey pairs.

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Definition and illustration

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$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}.$$

Building on the work of L.J. Rogers, W.N. Bailey (1949) showed that the identity

$$\sum_{n\geq 0}(b)_n(c)_n(aq/bc)^n\beta_n = \frac{(aq/b)_\infty(aq/c)_\infty}{(aq)_\infty(aq/bc)_\infty}\sum_{n\geq 0}\frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n$$

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An example

It can be shown that the Bailey pair relation is satisfied for $a = 1$ by

$$\alpha_n = \begin{cases} 1, & \text{if } n = 0 \\ (-1)^n q^{n(3n-1)/2} \left(1 + q^n\right), & \text{if } n > 0 \end{cases}$$

and

$$\beta_n = \frac{1}{(q)_n}.$$

Inserting this into Bailey's general identity, taking $b, c \to \infty$, and using the fact that

$$\lim_{x\to\infty} (x)_n (1/x)^n = (-1)^n q^{\binom{n}{2}}$$

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First Rogers-Ramanujan identity

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n}=\frac{1}{(q)_\infty}\left(1+\sum_{n\geq 1}(-1)^n q^{n(5n-1)/2}(1+q^n)\right)$$
$$=\sum_{n\in\mathbb{Z}}(-1)^n q^{n(5n-1)/2}$$
$$=\frac{\left(q^2;q^5\right)_\infty\left(q^3;q^5\right)_\infty\left(q^5;q^5\right)_\infty}{(q)_\infty}$$

which is the first Rogers-Ramanujan identity!

We can similarly obtain the second Rogers-Ramanujan identity as well.

There's more!

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Bailey's lemma

If (α_n, β_n) is a Bailey pair relative to (a, q) then so is (α'_n, β'_n), where

$$\alpha'_n = \frac{(b)_n(c)_n}{(aq/b)_n(aq/c)_n}(aq/bc)^n\alpha_n$$

and

$$\beta'_n = \frac{(aq/bc)_n}{(q)_n(aq/b)_n(aq/c)_n}\sum_{k=0}^{n}\frac{(b)_k(c)_k(q^{-n})_k q^k}{(bcq^{-n}/a)_k}\beta_k.$$

- This may be iterated, giving rise to what is called the Bailey chain.
- One Bailey pair gives infinitely many.
- So one identity gives infinitely many.
- A simple yet powerful illustration of this technique is to start with the unit Bailey pair and iterating k times along the Bailey chain and inserting this Bailey pair into the original definition and letting $n \to \infty$ to obtain subfamilies of Andrews-Gordon identities.

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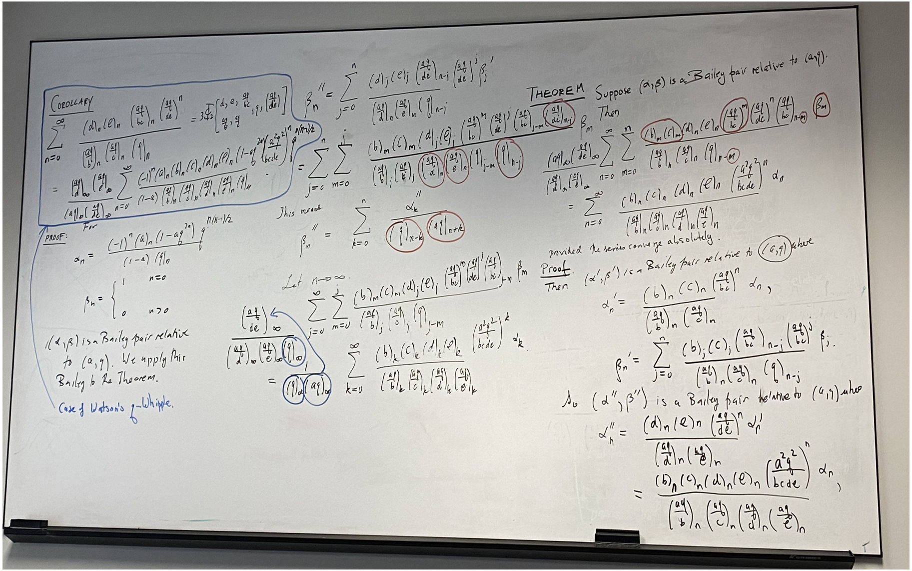
Subfamily of Andrews-Gordon identities

$$\sum_{n_{k-1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_1^2 + \cdots + n_{k-1}^2}}{(q)_{n_{k-1}-n_{k-2}} \cdots (q)_{n_2-n_1} (q)_{n_1}}$$
$$= \frac{1}{(q)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n q^{n((2k+1)n-1)/2}$$
$$= \frac{\left(q^k; q^{2k+1}\right)_\infty \left(q^{k+1}; q^{2k+1}\right)_\infty \left(q^{2k+1}; q^{2k+1}\right)_\infty}{(q)_\infty}.$$

Generating function of the crank



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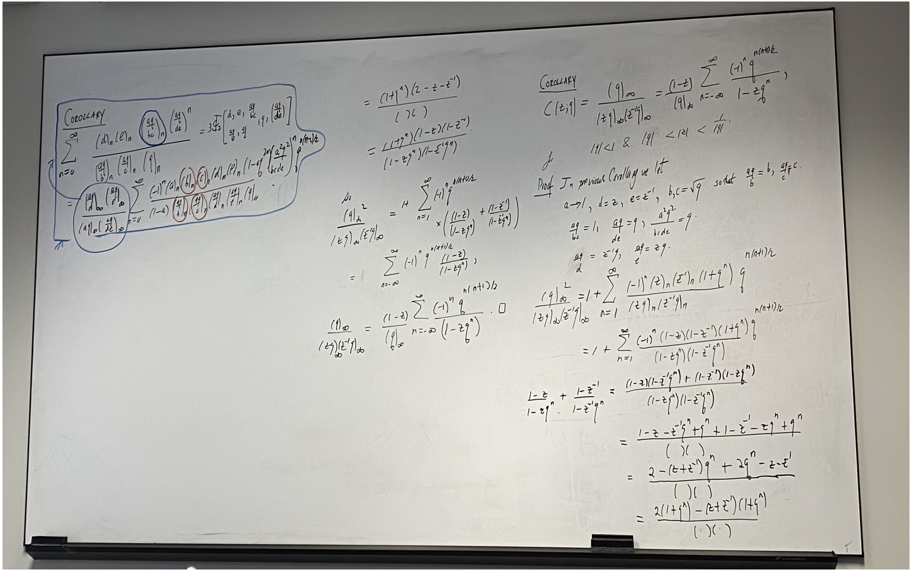

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Jeremy Lovejoy

Abstract. Zagier introduced the term "strange identity" to describe an asymptotic relation between a certain q-hypergeometric series and a partial theta function at roots of unity. We show that behind Zagier's strange identity lies a statement about Bailey pairs. Using the iterative machinery of Bailey pairs then leads to many families of multisum strange identities, including Hikami's generalization of Zagier's identity.

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$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{1}$$

Here we have used the standard q-hypergeometric notation,

$$(a)_n = (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}),$$

defined for integers $n \geq 0$ and in the limit as $n \to \infty$. Bailey pairs are one of the principal structural elements in the theory of q-hypergeometric series. Much of their power comes from the fact that Bailey pairs give rise to new Bailey pairs, and they do so in many different ways. This leads to an iterative "machinery" that produces infinite families of identities starting from a single identity. For example, once one understands how to use Bailey pairs to prove the Rogers-Ramanujan identities,

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The strange function and strange identities

The Kontsevich-Zagier "strange" function

$$\mathcal{F}(q) := \sum_{n\geq 0}(q;q)_n.$$

This series does not converge on any open subset of $\mathbb{C}$, but it is well-defined both at roots of unity and as a power series when q is replaced by $1-q$.

The Kontsevich-Zagier function satisfies the "strange identity" as recorded by **Zagier (2001)**

$$\mathcal{F}(q) \text{``} = \text{''} -\frac{1}{2}\sum_{n\geq 1} n\left(\frac{12}{n}\right) q^{(n^2-1)/24}.$$

Here the symbol " $=$ " means that the two sides agree to all orders at every root of unity.

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Hikami's generalization of Zagier's strange identity

$$\sum_{n_1,\ldots,n_k\geq 0}(q)_{n_k}q^{n_1^2+\cdots+n_{k-1}^2+n_{a+1}+\cdots+n_{k-1}}\prod_{i=1}^{k-1}\begin{bmatrix}n_{i+1}+\delta_{i,a}\\ n_i\end{bmatrix}$$

$$\text{“}=\text{''}-\frac{1}{2}\sum_{n\geq 0}n\chi_{8k+4}^{(a)}(n)q^{\frac{n^2-(2k-2a-1)^2}{8(2k+1)}}$$

where $\chi_{8k+4}^{(a)}(n)$ is the even periodic function defined by

$$\chi_{2m}^{(a)}(n)=\begin{cases}1 & \text{if } n\equiv 2k-2a-1 \text{ or } 6k+2a+5 \pmod{8k+4},\\ -1 & \text{if } n\equiv 2k+2a+3 \text{ or } 6k-2a+1 \pmod{8k+4},\\ 0, & \text{otherwise.}\end{cases}$$

- Hikami's proof of the strange identities involves long and impressive computations using q-difference equations.
- Lovejoy considers the problem in the context of Bailey pairs.

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Lovejoy's proof of Hikami's strange identity

Lovejoy's Bailey pair relative to (q, q)

$$\alpha_n = \frac{(1-q^{(a+1)(2n+1)})(-1)^n q^{\binom{n+1}{2}+(a+1)n^2+(m-a-1)(n^2+n)}}{1-q}$$

and

$$\beta_n = \beta_{n_m} = \sum_{n_1,\ldots,n_{m-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{m-1}^2+n_{a+1}+\cdots+n_{m-1}}}{(q)_{n_m}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}.$$

Obtained by repetitive iterations of the Bailey pair

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using Bailey's Lemma and appropriate substitutions. The proof then involves inserting the Bailey pair into a variation of Bailey's Lemma, subtracting $(x)_\infty$ multiplied by the multisum obtained in the previous step and then differentiating with respect to x and lastly setting $x = 1$.

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using Bailey's Lemma and appropriate substitutions. The proof then involves inserting the Bailey pair into a variation of Bailey's Lemma, subtracting $(x)_\infty$ multiplied by the multisum obtained in the previous step and then differentiating with respect to x and lastly setting $x = 1$.

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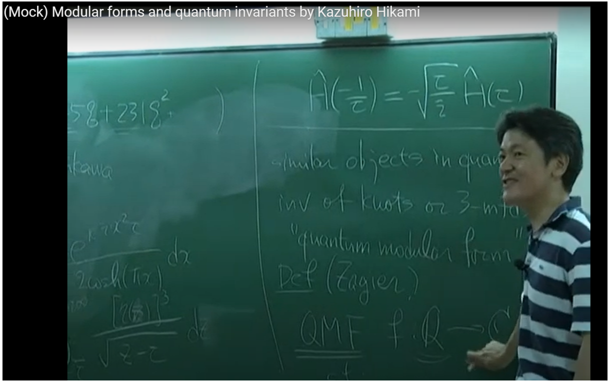
Kazuhiro Hikami, Kyushu University, Japan



- K. Hikami, Quantum invariant for torus link and modular forms, *Comm. Math. Phys.* **246** (2004), no. 2, 403–426.

Knot theory link

- Computes the Kashaev's invariant or the colored Jones function for the torus link $T(2,2m)$.
- Considers q-series identities related to these invariants.
- Gives an asymptotic expansion of the invariant and shows that the invariant for $T(2,2m)$ has a nearly modular property.
- Kashaev's invariant κ_N for the torus link $\kappa = T(2,2m)$ is explicitly given by

$$T(2,2m)_N = N \sum_{N-1 \geq c_{m-1} \geq \cdots \geq c_2 \geq c_1 \geq 0} (-1)^{c_{m-1}} \omega^{\frac{1}{2} c_{m-1}(c_{m-1}+1)} \prod_{i=1}^{m-2} \omega^{c_i(c_i+1)} \begin{bmatrix} c_{i+1} \\ c_i \end{bmatrix}$$

where $\omega = \exp(\frac{2\pi i}{N})$.

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A q-series closely related to $T(2,2m)_N$

- For $n \geq 0$, define the q-binomial coefficient (or Gaussian polynomial)

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

- **Hikami (2004)** considers the q-hypergeometric series $\widetilde{\Phi}^{(a)}_m(q)$, where $m \geq 2$ and $0 \leq a \leq m-2$, defined as

$$\widetilde{\Phi}^{(a)}_m(q) = mq^{\frac{(m-1-a)^2}{4m}} \sum_{n_1,\ldots,n_{m-1} \geq 0} (-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2}+n_1^2+\cdots+n_{m-2}^2+n_{a+1}+\cdots+n_{m-2}} \times \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1}+\delta_{i,a} \\ n_i \end{bmatrix}.$$

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Radial limits of false theta functions

- Hikami shows that the series $\widetilde{\Phi}_m^{(a)}(q)$ is the false theta function

$$\widetilde{\Phi}_m^{(a)}(q) = m \sum_{n \geq 0} \chi_{2m}^{(a)}(n) q^{\frac{n^2}{4m}},$$

where $\chi_{2m}^{(a)}(n)$ is the odd periodic function defined by

$$\chi_{2m}^{(a)}(n) = \begin{cases} 1 & \text{if } n \equiv m-a-1 \pmod{2m}, \\ -1 & \text{if } n \equiv m+a+1 \pmod{2m}, \\ 0, & \text{otherwise.} \end{cases}$$

- Such false theta functions have well-defined limiting values as q approaches a root of unity radially from inside the unit disk and that the resulting function is a quantum modular form (**Goswami-Osburn, 2021**).

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Theorem 1

Consider the partial theta series

$$\theta_f(z) := \sum_{n\geq 0} f(n)\, q^{\frac{n^2}{2M}}, \qquad \Theta_f(z) := \sum_{n\geq 0} n f(n)\, q^{\frac{n^2}{2M}}$$

where $q = e^{2\pi i z}$, $z \in \mathbb{H}$, f *is a function with period* $M \geq 2$ *and certain support. Let* $\alpha \in \mathbb{Q}$. *If* f *is even, then* $\Theta_f(\alpha)$ *is a quantum modular form of weight* $3/2$ *with respect to* Γ_M. *If* f *is odd, then* $\theta_f(\alpha)$ *is a "strong" quantum modular form of weight* $1/2$ *on* $\mathbb{Q}$ *with respect to* Γ_M *and is a quantum modular form of weight* $1/2$ *with certain support conditions and with respect to* Γ_M.

Hikami's conjecture

- Hikami further conjectures that in the case of $\widetilde{\Phi}_m^{(a)}(q)$ these radial limits are given by evaluating a truncated version of the q-series.
- Define the polynomial $Y_{m,N}^{(a)}(q)$ by

$$Y_{m,N}^{(a)}(q) = \sum_{n_1,\dots,n_{m-1}=0}^{N-1} (-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2}+n_1^2+\cdots+n_{m-2}^2+n_{a+1}+\cdots+n_{m-2}} \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1}+\delta_{i,a} \\ n_i \end{bmatrix}$$

Conjecture 2 (Hikami, 2004)

Let $q = e^{2\pi i/N}$. For any $m \geq 2$ and $0 \leq a \leq m-2$ we have

$$\widetilde{\Phi}_m^{(a)}(q) = q^{\frac{(m-1-a)^2}{4m}} Y_{m,N}^{(a)}(q).$$

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Proof of Hikami's conjecture using Bailey pair machinery

- Using Bailey pairs due to Lovejoy, we prove Hikami's conjecture and construct other families of q-hypergeometric false theta functions whose radial limits at roots of unity are obtained by evaluating the truncated series.
- The following constitutes a Bailey pair relative to (q, q)

$$\alpha_n = \frac{(1-q^{(a+1)(2n+1)})(-1)^n q^{\binom{n+1}{2}+(a+1)n^2+(m-a-1)(n^2+n)}}{1-q}$$

and

$$\beta_n = \beta_{n_m} = \sum_{n_1,\dots,n_{m-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{m-1}^2+n_{a+1}+\cdots+n_{m-1}}}{(q)_{n_m}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix}.$$

Hikami's conjecture is true

General idea : Setting up the Bailey pair framework by using the previous Bailey pair and inserting in the definition of a Bailey pair, we prove Hikami's conjecture.

Theorem 3 (Lovejoy-S, Kyushu J. Math, 2024)

Let $q = e^{2\pi i M/N}$ be a primitive Nth root of unity with $N > 0$. Then for any $m \geq 2$ and $0 \leq a \leq m-2$ we have

$$\widetilde{\Phi}_m^{(a)}(q) = q^{\frac{(m-1-a)^2}{4m}} Y_{m,N}^{(a)}(q)$$

where

$$\widetilde{\Phi}_m^{(a)}(q) = m \sum_{n \geq 0} \chi_{2m}^{(a)}(n) q^{\frac{n^2}{4m}},$$

and

$$Y_{m,N}^{(a)}(q) = \sum_{n_1,\dots,n_{m-1}=0}^{N-1} (-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2} + n_1^2 + \cdots + n_{m-2}^2 + n_{a+1} + \cdots + n_{m-2}} \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1} + \delta_{i,a} \\ n_i \end{bmatrix}.$$

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Two key lemmas

Lemma 4 (Hikami, 2004)

For coprime integers M and N with $N > 0$ we have

$$\widetilde{\Phi}_m^{(a)}(\zeta_N^M) = m \sum_{n=0}^{mN} \chi_{2m}^{(a)}(n) \left(1 - \frac{n}{mN}\right) \zeta_N^{Mn^2/4m}.$$

Remark : Used to manipulate the false theta function side of Hikami's conjecture.

Lemma 5

If (α_n, β_n) is a Bailey pair relative to (q, q), then

$$(q^2)_n \sum_{k=0}^{n} (q^{-n})_k (-1)^k q^{\binom{k+1}{2}+(n+1)k} \beta_k = \sum_{k=0}^{n} \frac{(q^{-n})_k}{(q^{2+n})_k} (-1)^k q^{\binom{k+1}{2}+(n+1)k} \alpha_k.$$

Remark : Starting point for obtaining the identity in Hikami's conjecture.

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We obtain this identity by inserting the pair from Bailey Lemma into the definition of a Bailey pair with $b, c \to \infty$ and use

$$(q)_{n-k} = \frac{(q)_n}{(q^{-n})_k}(-1)^k q^{\binom{k}{2}-nk}.$$

The Bailey lemma

If (α_n, β_n) is a Bailey pair relative to (a, q) then so is (α'_n, β'_n), where

$$\alpha'_n = \frac{(b)_n(c)_n}{(aq/b)_n(aq/c)_n}(aq/bc)^n\alpha_n$$

and

$$\beta'_n = \frac{(aq/bc)_n}{(q)_n(aq/b)_n(aq/c)_n}\sum_{k=0}^{n}\frac{(b)_k(c)_k(q^{-n})_k q^k}{(bcq^{-n}/a)_k}\beta_k.$$

Summary of proof of Hikami's conjecture

Step I

Make appropriate substitutions in Bailey's Lemma and insert in the definition of Bailey pair to obtain the identity stated in Lemma 5

$$(q^2)_n \sum_{k=0}^{n} (q^{-n})_k (-1)^k q^{\binom{k+1}{2}+(n+1)k} \beta_k = \sum_{k=0}^{n} \frac{(q^{-n})_k}{(q^{2+n})_k} (-1)^k q^{\binom{k+1}{2}+(n+1)k} \alpha_k .$$

Step II

Recall the Bailey pair of Lovejoy that he used in the proof of Hikami's generalization of Zagier's strange identity

$$\alpha_n = \frac{(1-q^{(a+1)(2n+1)})(-1)^n q^{\binom{n+1}{2}+(a+1)n^2+(m-a-1)(n^2+n)}}{1-q}$$

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Step III

Insert the Bailey pair of Step II into the identity found in Step I with $m = m-1$ and $n = N-1$ to obtain

$$(q)_N \sum_{n_1,\ldots,n_{m-1}=0}^{N-1} \frac{(q^{1-N})_{n_{m-1}}(-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2}+Nn_{m-1}+n_1^2+\cdots+n_{m-2}^2+n_{a+1}+\cdots+n_{m-2}}}{(q)_{n_{m-1}}} \times \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1}+\delta_{a,i} \\ n_i \end{bmatrix} = \sum_{k=0}^{N-1} \frac{(q^{1-N})_k}{(q^{1+N})_k}(1-q^{(a+1)(2k+1)})q^{mk^2+(m-a-1)k+Nk}.$$

Remark : Note that the LHS closely resembles the multisum q-series side of Hikami's conjecture

$$Y_{m,N}^{(a)}(q) = \sum_{n_1,\ldots,n_{m-1}=0}^{N-1} (-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2}+n_1^2+\cdots+n_{m-2}^2+n_{a+1}+\cdots+n_{m-2}} \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1}+\delta_{i,a} \\ n_i \end{bmatrix}.$$

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Step IV

Dividing both sides of the equation in Step III by $(q)_N$ and taking $\lim_{q \to \zeta_N^M}$ we then have

$$Y_{m,N}^{(a)}(\zeta_N^M) = \lim_{q \to \zeta_N^M} \frac{1}{(q)_N} \sum_{k=0}^{N-1} \frac{(q^{1-N})_k}{(q^{1+N})_k} (1 - q^{(a+1)(2k+1)}) q^{mk^2+(m-a-1)k+Nk}$$

$$= \frac{1}{N} \lim_{q \to \zeta_N^M} \frac{1}{1-q^N} q^{\frac{-(m-a-1)^2}{4m}} \sum_{k=0}^{2mN} \chi_{2m}^{(a)}(k) q^{k^2/4m}$$

$$= -\frac{1}{4mN^2} \zeta_N^{\frac{-M(m-a-1)^2}{4m}} \sum_{k=0}^{2mN} k^2 \chi_{2m}^{(a)}(k) \zeta_N^{Mk^2/4m}.$$

Note I : The second equality follows from a short computation involving completing the square.

Note II : $\prod_{i=1}^{N-1}(1-q^i x) = \frac{1-x^N}{1-x}$ which gives $(q;q)_{N-1} = N$.

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Step V

Simplify the RHS to obtain the desired false theta function at a root of unity using Lemma 4 of Hikami.

$$\zeta_N^{\frac{M(m-a-1)^2}{4m}} Y_{m,N}^{(a)}(\zeta_N^M) = -\frac{1}{4mN^2}\sum_{k=0}^{2mN} k^2\chi_{2m}^{(a)}(k)\zeta_N^{Mk^2/4m}$$
$$= -\frac{1}{4mN^2}\left(\sum_{k=0}^{mN} k^2\chi_{2m}^{(a)}(k)\zeta_N^{Mk^2/4m} + \sum_{k=mN}^{2mN} k^2\chi_{2m}^{(a)}(k)\zeta_n^{Mk^2/4m}\right)$$
$$= -\frac{1}{4mN^2}\left(\sum_{k=0}^{mN} k^2\chi_{2m}^{(a)}(k)\zeta_N^{Mk^2/4m} + \sum_{k=0}^{mN}(2mN-k)^2\chi_{2m}^{(a)}(2mN-k)\zeta_N^{M(2mN-k)^2/4m}\right)$$
$$= -\frac{1}{4mN^2}\left(\sum_{k=0}^{mN} k^2\chi_{2m}^{(a)}(k)\zeta_N^{Mk^2/4m} - \sum_{k=0}^{mN}(2mN-k)^2\chi_{2m}^{(a)}(k)\zeta_N^{Mk^2/4m}\right)$$
$$= \sum_{k=0}^{mN}\chi_{2m}^{(a)}(k)\left(m-\frac{k}{N}\right)\zeta_N^{Mk^2/4m}$$
$$= \widetilde{\Phi}_m^{(a)}(\zeta_N^M).$$

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Setup for further examples

We have found further examples of families of q-hypergeometric false theta functions whose radial limits are given by evaluating a truncated version of the q-series. We consider a specialization of the Bailey lemma different from the lemma previously used. Setting $a = b = q$ and $c \to \infty$ in

The Bailey lemma

If (α_n, β_n) is a Bailey pair relative to (a, q) then so is (α'_n, β'_n), where

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and then using the definition of a Bailey pair with $n \to \infty$ we have the following, which is well-known.

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We have found further examples of families of q-hypergeometric false theta functions whose radial limits are given by evaluating a truncated version of the q-series. We consider a specialization of the Bailey lemma different from the lemma previously used. Setting $a = b = q$ and $c \to \infty$ in

The Bailey lemma

If (α_n, β_n) is a Bailey pair relative to (a, q) then so is (α'_n, β'_n), where

$$\alpha'_n = \frac{(b)_n(c)_n}{(aq/b)_n(aq/c)_n}(aq/bc)^n\alpha_n$$

and

$$\beta'_n = \frac{(aq/bc)_n}{(q)_n(aq/b)_n(aq/c)_n}\sum_{k=0}^{n}\frac{(b)_k(c)_k(q^{-n})_k q^k}{(bcq^{-n}/a)_k}\beta_k.$$

and then using the definition of a Bailey pair with $n \to \infty$ we have the following, which is well-known.

Two key lemmas for the new identities

Lemma 6

If (α_n, β_n) is a Bailey pair relative to (q, q), then

$$\sum_{n\geq 0}(q)_n(-1)^n q^{\binom{n+1}{2}}\beta_n = (1-q)\sum_{n\geq 0}(-1)^n q^{\binom{n+1}{2}}\alpha_n.$$

Remark : Starting point for obtaining the new identities.

Lemma 7

Let $C_f(n)$ be a periodic function with mean value 0 and modulus f. Then as $t \searrow 0$ we have the asymptotic expansion

$$\sum_{n\geq 1} C_f(n)e^{-n^2 t} \sim \sum_{k\geq 0} L(-2k, C_f)\frac{(-t)^k}{k!}, \text{ where}$$

$$L(-k, C_f) = -\frac{f^k}{k+1}\sum_{n=1}^{f} C_f(n)B_{k+1}\left(\frac{n}{f}\right),$$

with $B_k(x)$ being the kth Bernouilli polynomial.

Remark : Used to manipulate the false theta function side of our new identities.

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Sketch of proofs of the new identities

- We first use a Bailey pair of Lovejoy together with Lemma 6 above to produce a family of q-hypergeometric false theta functions.
- Next we use Lemma 7 above to compute the radial limits of the false theta functions.
- Finally, we use the same Bailey pair in Lemma 5 to produce a truncated version of the q-series and whose values at roots of unity coincide with the radial limits of the infinite series.

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Step I

Start with the Bailey pair relative to (q, q) due to Lovejoy,

$$\alpha_n = \frac{1-q^{2n+1}}{1-q}(-1)^n q^{mn^2+(m-1)n}$$

and

$$\beta_n = \beta_{n_m} = \sum_{n_1,n_2,\ldots,n_{m-1}\geq 0} \frac{q^{n_1^2+\cdots+n_{m-1}^2+n_1+\cdots+n_{m-1}}}{(q)_{n_m}(-q)_{n_1}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}.$$

Step II

Insert this pair into the Lemma 6 to obtain the "multisum q-series = false theta function" identity,

$$\frac{2m-1}{2}\sum_{k=0}^{\infty}\chi_{4m-2}(k)\, q^{\frac{k^2}{8(2m-1)}} = \frac{2m-1}{2} q^{\frac{(2m-3)^2}{8(2m-1)}} \sum_{n_1,n_2,\ldots,n_{m-1}\geq 0} \frac{(-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2}+n_1^2+\cdots+n_{m-2}^2+n_1+\cdots+n_{m-2}}}{(-q)_{n_1}} \times \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$

where $\chi_{4m-2}(k)$ is an odd periodic function.

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Step III

Employ Hikami's lemma to calculate the radial limits of $\widetilde{\Psi}_m(q) = \frac{2m-1}{2}\sum_{k=0}^{\infty}\chi_{4m-2}(k)\, q^{\frac{k^2}{8(2m-1)}}$ as q approaches a root of unity.

For coprime integers M and N with N odd and positive we have

$$\widetilde{\Psi}_m(\zeta_N^M) = \frac{-1}{8(2m-1)N^2}\sum_{k=0}^{(4m-2)N} k^2\chi_{4m-2}(k)\zeta_N^{\frac{k^2}{8(2m-1)}M}.$$

Step IV

Finally we determine the value of the truncated version of our q-series

$$Z_{m,N}(q) = \sum_{n_1,\dots,n_{m-1}=0}^{N-1}\frac{(-1)^{n_{m-1}}q^{\binom{n_{m-1}+1}{2}+n_1^2+\cdots+n_{m-2}^2+n_1+\cdots+n_{m-2}}}{(-q)_{n_1}}\prod_{i=1}^{m-2}\begin{bmatrix}n_{i+1}\\ n_i\end{bmatrix}.$$

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at primitive odd Nth roots of unity and find that it coincides with the radial limits of the infinite series.

Theorem (Lovejoy-S, 2024)

- Let $q = e^{2\pi i M/N}$ be a primitive odd Nth root of unity. Then

$$\widetilde{\Psi}_m(q) = q^{\frac{(2m-3)^2}{8(2m-1)}} Z_{m,N}(q),$$

where

$$\widetilde{\Psi}_m(q) = \frac{2m-1}{2} q^{\frac{(2m-3)^2}{8(2m-1)}} \sum_{n_1,n_2,\dots,n_{m-1}\geq 0} \frac{(-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2}+n_1^2+\cdots+n_{m-2}^2+n_1+\cdots+n_{m-2}}}{(-q)_{n_1}} \times \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}$$

and its truncated version

$$Z_{m,N}(q) = \sum_{n_1,\dots,n_{m-1}=0}^{N-1} \frac{(-1)^{n_{m-1}} q^{\binom{n_{m-1}+1}{2}+n_1^2+\cdots+n_{m-2}^2+n_1+\cdots+n_{m-2}}}{(-q)_{n_1}} \prod_{i=1}^{m-2} \begin{bmatrix} n_{i+1} \\ n_i \end{bmatrix}.$$

Related and future work

Hikami's observations on unified WRT invariants and false theta functions

Toshiki Matsusaka

Dedicated to the memory of Toshie Takata.

Abstract. The object of this article is a family of q-series originating from Habiro's work on the Witten–Reshetikhin–Turaev invariants. The q-series usually make sense only when q is a root of unity, but for some instances, it also determines a holomorphic function on the open unit disc. Such an example is Habiro's unified WRT invariant $H(q)$ for the Poincaré homology sphere. In 2007, Hikami observed its discontinuity at roots of unity. More precisely, the value of $H(\zeta)$ at a root of unity is 1/2 times the limit value of $H(q)$ as q tends towards ζ radially within the unit disc. In this article, we explain the appearance of the 1/2-factor and generalize Hikami's observations by using Bailey's lemma and the theory of false theta functions.

Contents

1. Introduction

The WRT invariants are derived from the work of Witten [36] and Reshetikhin–Turaev [30]. Witten answered Atiyah's question on a 3-dimensional definition of the Jones polynomials of knot theory and introduced certain invariants of 3-manifolds using quantum field theory. Its rigorous mathematical definition was subsequently given by Reshetikhin and Turaev using the quantum group $U_q(sl_2)$ at roots of unity and has been extensively investigated.

Related and future work

Here is one example. The WRT-invariant $\tau_N(\Sigma(2,3,5))$ associated to the Poincaré homology sphere $M = \Sigma(2,3,5)$ is computed as

$$e^{\frac{2\pi i}{N}\frac{121}{120}}(e^{\frac{2\pi i}{N}} - 1)\tau_N(\Sigma(2,3,5)) = \frac{e^{\pi i/4}}{2\sqrt{60N}} \sum_{\substack{0\le n\le 60N-1 \\ N\nmid n}} e^{-\frac{\pi i n^2}{60N}} \frac{\prod_{j=1}^{3}(e^{\frac{\pi i n}{Np_j}} - e^{-\frac{\pi i n}{Np_j}})}{e^{\frac{\pi i n}{N}} - e^{-\frac{\pi i n}{N}}}$$

for $N \in \mathbb{Z}_{>1}$, where $(p_1, p_2, p_3) = (2,3,5)$ (see Lawrence–Rozansky [25] and Hikami [21]). One of the topics of research on the WRT invariants is to find a "unified" function that can capture the values for all N. More precisely, we find a function $\tau(M) : \mathcal{Z} \to \mathbb{C}$ defined on the set of all roots of unity $\mathcal{Z}$ such that the value $\tau(M)(e^{2\pi i/N})$ coincides with the WRT invariant $\tau_N(M)$. A number-theoretic (or analytic) approach was given by Lawrence–Zagier [26] using false theta functions. They considered the q-series defined by

$$\widetilde{\Phi}^{(1,1,1)}_{(2,3,5)}(\tau) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)\chi^{(1,1,1)}_{(2,3,5)}(n) q^{\frac{n^2}{120}} \qquad (q = e^{2\pi i \tau}),$$

where $q^r = e^{2\pi i r\tau}$ for $r \in \mathbb{Q}$ and $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$, and

$$\chi^{(1,1,1)}_{(2,3,5)}(n) = \begin{cases} 1 & \text{if } n \equiv 31, 41, 49, 59 \pmod{60}, \\ -1 & \text{if } n \equiv 1, 11, 19, 29 \pmod{60}, \\ 0 & \text{if otherwise.} \end{cases}$$

Then they showed that

$$\lim_{t\to 0} \widetilde{\Phi}^{(1,1,1)}_{(2,3,5)}\left(\frac{1}{N} + it\right) = -\frac{1}{60N}\sum_{n=1}^{60N} n\chi^{(1,1,1)}_{(2,3,5)}(n) e^{\pi i \frac{n^2}{60N}}$$

and

$$-\frac{e^{-\frac{\pi i}{60N}}}{2} \lim_{t\to 0} \widetilde{\Phi}^{(1,1,1)}_{(2,3,5)}\left(\frac{1}{N} + it\right) = 1 + e^{\frac{2\pi i}{N}}(1 - e^{\frac{2\pi i}{N}})\tau_N(\Sigma(2,3,5)).$$

In this sense, the q-series $\widetilde{\Phi}^{(1,1,1)}_{(2,3,5)}(\tau)$ unifies the WRT-invariants via the limits to the roots of unity.

Related and future work

Definition 1.2. For any positive integer $p \geq 1$, we define five Habiro-type series by

$$H_p^{(1)}(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p} (q^{s_p+1})_{s_p+1} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q,$$

$$H_p^{(2)}(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p} (q^{s_p})_{s_p} \prod_{i=1}^{p-1} q^{s_i^2} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q,$$

$$H_p^{(3)}(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{2s_p} (q^{s_p+1})_{s_p} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q,$$

$$H_p^{(4)}(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p} (q^{s_p+1})_{s_p} \prod_{i=1}^{p-1} q^{s_i(s_i+1)} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q,$$

$$H_p^{(5)}(q) = \sum_{s_p \geq \cdots \geq s_1 \geq 0} q^{s_p} (q^{s_p+1})_{s_p} \prod_{i=1}^{p-1} q^{s_i^2} \begin{bmatrix} s_{i+1} \\ s_i \end{bmatrix}_q.$$

If the notations are to match those adapted by the spirit of Hikami [23], then the above series should be named $H_p^{(1)}(q) = M_p^{(1)}(q)$, $H_p^{(2)}(q) = M_p^{(p)}(q)$, $H_p^{(3)}(q) = M_p^{(2p-1)}(q)$, $H_p^{(4)}(q) = M_p^{(2p)}(q)$, and $H_p^{(5)}(q) = M_p^{(3p-1)}(q)$. However, since the superscripts overlap when $p = 1$, the notations here are purposely changed. These five series are infinite families that extend each of Hikami's $M_2^{(k)}(q)$ for $k = 1, 2, 3, 4, 5$. Moreover, $H_1^{(2)}(q) = M_1^{(1)}(q)$ and $H_1^{(4)}(q) = H_1^{(5)}(q) = M_1^{(2)}(q)$ hold in Hikami's notations.

Our main theorems stated in Theorem 3.15 and Theorem 3.21 give similar expressions in terms of false theta functions and limit formulas of these five families as in Theorem 1.1. For

instance, we have

$$\lim_{\tau \to 1/N} \text{convergent part of } H_2^{(2)}(q) = -\frac{1}{2} \lim_{\tau \to 1/N} q^{-\frac{1}{264}} \widetilde{\Phi}_{(2,3,11)}^{(1,1,2)}(\tau)$$
$$= \frac{e^{-\frac{2\pi i}{264N}}}{264N} \sum_{n=1}^{132N} n \chi_{(2,3,11)}^{(1,1,2)}(n) e^{\pi i \frac{n^2}{132N}},$$

where

$$\chi_{(2,3,11)}^{(1,1,2)}(n) = \begin{cases} 1 & \text{if } n \equiv 67, 89, 109, 131 \pmod{132}, \\ -1 & \text{if } n \equiv 1, 23, 43, 65 \pmod{132}, \\ 0 & \text{if otherwise.} \end{cases}$$

Moreover, numerical calculations suggest that the above limit value coincides with the value $H_2^{(2)}(e^{2\pi i/N})$, that is,

$$H_2^{(2)}(e^{2\pi i/N}) = \frac{e^{-\frac{2\pi i}{264N}}}{264N} \sum_{n=1}^{132N} n \chi_{(2,3,11)}^{(1,1,2)}(n) e^{\pi i \frac{n^2}{132N}} \tag{1.5}$$

holds. The similarity with Theorem 1.1 leads us to expect the coincidence to hold, but it is a conjecture. For other cases, too, Hikami [23, Conjectures 1–3] conjectured the coincidence between the limits of $\widetilde{\Phi}_{(2,3,6p-1)}^{(\ell_1,\ell_2,\ell_3)}(\tau)$ and the values of Habiro-type series through numerical calculations, but they are still open problems.

Related and future work

BAILEY PAIRS AND AN IDENTITY OF CHERN-LI-STANTON-XUE-YEE

SHASHANK KANADE AND JEREMY LOVEJOY

Dedicated to James Lepowsky on the occasion of his 80th birthday and Stephen Milne on the occasion of his 75th birthday

Abstract. We show how Bailey pairs can be used to give a simple proof of an identity of Chern, Li, Stanton, Xue, and Yee. The same method yields a number of related identities as well as false theta companions.

1. Introduction

Recall the usual q-series notation,

$$(a_1, a_2, \ldots, a_k)_\infty = (a_1, a_2, \ldots, a_k; q)_\infty = \prod_{j=0}^{\infty}(1 - a_1 q^j)(1 - a_2 q^j)\cdots(1 - a_k q^j)$$

and

$$(a_1, a_2, \ldots, a_k)_n = (a_1, a_2, \ldots, a_k; q)_n = \prod_{j=0}^{n-1}(1 - a_1 q^j)(1 - a_2 q^j)\cdots(1 - a_k q^j),$$

valid for $n \geq 0$, along with the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q)_n}{(q)_{n-k}(q)_k}, & \text{if } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

In a recent study of q-series and partitions related to Ariki-Koike algebras, Chern, Li, Stanton, Xue, and Yee [9] established the following family of q-multisum identities:

Theorem 1.1. *Let $m \geq 1$ and $0 \leq a \leq m-1$. Then we have*

$$\sum_{n_m, \ldots, n_1 \geq 0} \frac{q^{\binom{n_m+1}{2} + \cdots + \binom{n_1+1}{2}}}{(q)_{n_m}} \prod_{i=1}^{m-1} \begin{bmatrix} n_{i+1} + \delta_{a,i} \\ n_i \end{bmatrix} = \frac{(q^{a+1}, q^{m+1-a}, q^{m+2}; q^{m+2})_\infty}{(q)_\infty (q; q^2)_\infty}. \tag{1.2}$$

This generalizes a classical identity in the theory of partitions [5, Eq. (2.26), $t = q$],

$$\sum_{n \geq 0} \frac{q^{\binom{n+1}{2}}}{(q)_n} = \frac{1}{(q; q^2)_\infty} = (-q)_\infty.$$

The proof of (1.1) in [9] is lengthy and impressive, involving a symmetry property, a q-binomial coefficient multisum transformation formula, and two identities of Andrews [6] and Kim–Yee [14].

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THANK YOU!